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Adaptive control of multiple chaotic systems with unknown parameters in two different synchronization modes

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Abstract

This paper investigates the synchronization of multiple chaotic systems with unknown parameters using adaptive control method, and two kinds of different synchronization modes are considered here. One is that more response systems synchronize with one drive system, and the other is the ring transmission synchronization, which guarantees that all chaotic systems can synchronize with each other. The definition of adaptive synchronization of multiple chaotic systems with unknown parameters is given, and then based on the idea of adaptive control method, adaptive laws are derived to estimate the unknown parameters, and nonlinear adaptive controllers are developed to ensure the asymptotical stability of two classes of error systems. Finally, simulation results are presented to verify the effectiveness of proposed synchronization schemes.

Keywords: multiple chaotic systems; unknown parameters; adaptive control; synchronization modes; stability analysis

1 Introduction

In recent years, chaos synchronization [1] of multiple chaotic systems has became a much-studied topic in nonlinear research area. It is in favor of its potential applications in multilateral communications, secret signaling, and many other engineering fields [2, 3]. Various kinds of synchronization of multiple chaotic systems have been discussed by using some advanced control techniques. Lü and Liu addressed complete synchronization of N coupled chaotic systems with chain and ring connection based on feedback control methods [4]. Tang and Fang extended the work of [4] to multiple fractional-order chaotic systems [5]. Chen *et al.* used the direct design method to study complete synchronization, anti-synchronization and hybrid synchronization among multiple chaotic systems [6, 7]. Sun *et al.* and Jiang *et al.* investigated generalized combination synchronization of multiple chaotic systems in [8, 9], respectively. Sun *et al.* discussed compound synchronization among four memristor chaotic oscillator systems based on the adaptive techniques [10]. For a special ring connection structure of multiple chaotic systems, a new transmission synchronization mode of multiple chaotic systems was proposed in [11]. Xi *et al.* proposed adaptive function projective combination synchronization of three different fractional-order chaotic systems [12]. It should be pointed out that most of aforementioned results are only concerned with the synchronization of multiple systems based on knowing exactly the system parameters, and the influences of unknown parameters for such systems are not considered. In fact, chaotic systems are unavoidably affected by unknown parameters, and it is hard to exactly know the values of the systems parameters in a priori. Thus, it is a challenging problem to study the synchronization among multiple chaotic systems with unknown parameters, which is the main motivation of this work.

On another research frontier, adaptive control method [13] is an effective way to estimate the unknown parameters due to its advantages on witnessed rapid and impressive developments leading to global stability and tracking results for nonlinear systems. It has been successfully applied to synchronize chaotic systems with unknown parameters, and many important results have been presented. For example, Park studied adaptive synchronization of a unified chaotic systems with an unknown parameter [14, 15]. Zhang *et al.* proposed the adaptive controllers and adaptive laws to synchronize two different chaotic systems with unknown parameters [16]. In [17], the adaptive complete synchronization between chaotic systems with fully uncertain parameters were realized. Li *et al.* gave a deeply research on adaptive impulsive synchronization for fractional-order chaotic systems with unknown parameters [18]. In [19], adaptive synchronization of two different chaotic systems was addressed by considering the time varying unknown parameters. Adaptive added-order and reduced-order anti-synchronization of chaotic systems were investigated in [20, 21], respectively. He *et al.* made a thorough inquiry about synchronization of hyperchaotic systems with multiple unknown parameters [22]. Zhao *et al.* presented a discussion of chaos synchronization between the coupled systems on network with unknown parameters based on adaptive control method [23]. Liu developed adaptive anti-synchronization of chaotic complex nonlinear systems with unknown parameters [24]. Wu and Yang achieved the adaptive synchronization of coupled nonidentical chaotic systems with complex variables and stochastic perturbations [25]. However, all of these works only deal with the synchronization problems between two chaotic systems with unknown parameters. Up to now, no related results have been established for the synchronization of multiple chaotic systems with unknown parameters, which is another motivation of this paper.

At present, there are two different synchronization modes for synchronizing multiple chaotic systems. One is that more response systems synchronize with one drive system, which can be considered as a previous synchronization model. For this synchronization mode in Figure 1(a), it is widely used to realize the synchronization of multiple chaotic systems or coupled complex networks. Lu and Cao considered the adaptive synchronization problems of three same dynamic networks [26]. Chen *et al.* investigated complete synchronization of N different chaotic systems in the above synchronization mode [27]. Tang *et al.* proposed the adaptive control problem for cluster synchronization of coupled complex networks in [28, 29]. Yang *et al.* established finite-time synchronization of coupled discontinuous neural networks with mixed delays and nonidentical perturbations in [30]. And the other mode is the ring transmission synchronization among multiple systems in Figure 1(b), which is constructed by utilizing the general benefits of the ring control approach and cluster synchronization scheme for the drive-response dynamical networks and extending it to multi-systems. It make the first system synchronize with the second system, and the second system synchronize with the third system. In the same way, the ring transmission synchronization among multi-systems in transmission method is realized.

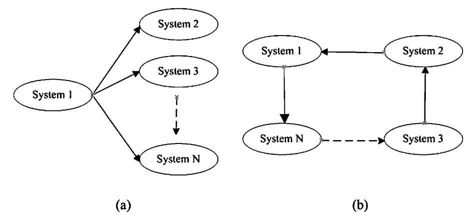

Figure 1 The diagram of two kinds of synchronization modes among multi-systems, one is that more systems synchronize with one system, and the other is the transmission synchronization.

This synchronization mode is quite different from the first mode, and it can overcome the trouble of the occurrence of a fault without affecting multiple system's synchronization. Sun *et al.* adopted an impulsive control technique to deal with the transmission synchronization problem for multi-systems with delayed coupling [11]. Chen *et al.* addressed the transmission synchronization in an array of nonidentical coupled chaotic systems based on a special antisymmetric structure [31]. Meanwhile, the transmission synchronization mode has good applications in multilateral communications [32] and signal transmission on complex networks with a ring connection with N nodes [33–35]. Hence, it is highly desirable to discuss the adaptive control problem of multiple chaotic systems with unknown parameters by considering the above synchronization modes.

In response to the above discussions, in this paper, the adaptive control method is adopted to investigate the synchronization of multiple chaotic systems with unknown parameters by considering two different synchronization modes. By designing the adaptive controllers and the adaptive laws, the sufficient conditions are derived to guarantee the asymptotical stability of the error dynamic systems. Simulation results show the effectiveness of the presented schemes. The main contributions of this paper lie in the following. (1) The results in [11, 15, 16, 27, 31] are extended to multiple chaotic system with unknown parameters. (2) Transmission synchronization of multiple chaotic systems with unknown parameters is first discussed with the special connection structure for such systems. (3) The proposed controllers and adaptive laws effectively realize the synchronization of multiple uncertain chaotic systems and estimate precisely unknown parameters.

2 Adaptive synchronization of multiple chaotic systems with unknown parameters

In this section, the synchronization problems of multiple chaotic systems with unknown parameters are taken into account, which has two kinds of different synchronization modes. The controllers and adaptive laws are designed using adaptive control method, respectively. Furthermore, two examples are give to demonstrate the effectiveness of the synchronization schemes.

2.1 Adaptive synchronization between more response systems and one drive system

Considering the following multiple chaotic systems with unknown parameters, one drive system is described by

$$\begin{cases} \dot{x}_{11}(t) = f_{11}(x_{11}(t),\dots,x_{1n}(t)) + F_{11}(x_{11}(t),\dots,x_{1n}(t))\hat{\theta}_{11}, \\ \dot{x}_{12}(t) = f_{12}(x_{11}(t),\dots,x_{1n}(t)) + F_{12}(x_{11}(t),\dots,x_{1n}(t))\hat{\theta}_{12}, \\ \vdots \\ \dot{x}_{1n}(t) = f_{1n}(x_{11}(t),\dots,x_{1n}(t)) + F_{1n}(x_{11}(t),\dots,x_{1n}(t))\hat{\theta}_{1n}, \end{cases} \tag{2.1}$$

where $x_1(t) = [x_{11}, x_{12}, \ldots, x_{1n}]^T$ is the state of the drive system (2.1), $f_{1i}(x_{11}(t), \ldots, x_{1n}(t))$ $(i = 1, \ldots, n)$ is a continuous function and $f_1(x_1(t)) = [f_{11}, f_{12}, \ldots, f_{1n}]^T$; $F_{1i}(x_{11}(t), \ldots, x_{1n}(t))$ $(i = 1, \ldots, n)$ is the matrices function and $F_1(x_1(t)) = [F_{11}, F_{12}, \ldots, F_{1n}]^T$; $\hat{\theta}_{1i}$ $(i = 1, \ldots, n)$ is the unknown parameter and $\hat{\theta}_1 = [\hat{\theta}_{11}, \hat{\theta}_{12}, \ldots, \hat{\theta}_{1n}]^T$.

The other $N-1$ response systems can be given by

$$\begin{cases} \dot{x}_{j1}(t) = f_{j1}(x_{j1}(t), \ldots, x_{jn}(t)) + F_{j1}(x_{j1}(t), \ldots, x_{jn}(t))\hat{\theta}_{j1} + u_{j-1,1}, \\ \dot{x}_{j2}(t) = f_{j2}(x_{j1}(t), \ldots, x_{jn}(t)) + F_{j2}(x_{j1}(t), \ldots, x_{jn}(t))\hat{\theta}_{j2} + u_{j-1,2}, \\ \vdots \\ \dot{x}_{jn}(t) = f_{jn}(x_{j1}(t), \ldots, x_{jn}(t)) + F_{jn}(x_{j1}(t), \ldots, x_{jn}(t))\hat{\theta}_{jn} + u_{j-1,n}, \end{cases} \tag{2.2}$$

where $j = 2, \ldots, N$, and $x_j(t) = [x_{j1}, x_{j2}, \ldots, x_{jn}]^T$ is the state of the response system (2.2), $f_{ji}(x_{j1}(t), \ldots, x_{jn}(t))$ $(i = 1, \ldots, n)$ is a continuous function and $f_j(x_j(t)) = [f_{j1}, f_{j2}, \ldots, f_{jn}]^T$; $F_{ji}(x_{j1}(t), \ldots, x_{jn}(t))$ $(i = 1, \ldots, n)$ is the matrices function and $F_j(x_j(t)) = [F_{j1}, F_{j2}, \ldots, F_{jn}]^T$; $\hat{\theta}_{ji}$ $(i = 1, \ldots, n)$ is the unknown parameter and $\hat{\theta}_j = [\hat{\theta}_{j1}, \hat{\theta}_{j2}, \ldots, \hat{\theta}_{jn}]^T$, the control input is $u_{j-1} = [u_{j-1,1}, u_{j-1,2}, \ldots, u_{j-1,n}]^T$. If $f_s(x_s) \neq f_t(x_t)$ $(s, t = 1, \ldots, N,\ s \neq t)$ and $F_s(x_s) \neq F_t(x_t)$ $(s, t = 1, \ldots, N, s \neq t)$, then (2.1) and (2.2) are multiple nonidentical chaotic systems.

Then the above multiple chaotic systems can be rewritten into the following form:

$$\begin{cases} \dot{x}_1 = f_1(x_1(t)) + F_1(x_1(t))\hat{\theta}_1, \\ \dot{x}_2 = f_2(x_2(t)) + F_2(x_2(t))\hat{\theta}_2 + u_1, \\ \cdots \\ \dot{x}_N = f_N(x_N(t)) + F_N(x_N(t))\hat{\theta}_N + u_{N-1}, \end{cases} \tag{2.3}$$

and the definition of the adaptive synchronization of multiple chaotic systems is first given, which has $N-1$ response systems and one drive system with unknown parameters.

Definition 1 For N chaotic systems (2.3), if there exist controllers $u_1(t), \ldots, u_{N-1}(t)$ such that $x_1(t)$ and all other trajectories vectors $x_2(t), \ldots, x_N(t)$ in (2.3) with any initial condition $(x_1(0), \ldots, x_N(0))$ satisfy the following conditions:

$$\lim_{t\to\infty} \|e_i(t)\| = \lim_{t\to\infty} \|x_{i+1}(t) - x_1(t)\| = 0, \quad i = 1, \ldots, N-1, \tag{2.4}$$

then it is said that they are the adaptive synchronization among N systems.

Remark 1 According to Figure 1(a) and Definition 1, the errors dynamic systems are obtained as follows:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \vdots \\ \dot{e}_{N-1} \end{bmatrix} = \begin{bmatrix} \dot{x}_2 - \dot{x}_1 \\ \dot{x}_3 - \dot{x}_1 \\ \dot{x}_4 - \dot{x}_1 \\ \vdots \\ \dot{x}_N - \dot{x}_1 \end{bmatrix} = \begin{bmatrix} f_2(x_2) - f_1(x_1) + F_2(x_2)\hat{\theta}_2 - F_1(x_1)\hat{\theta}_1 + u_1 \\ f_3(x_3) - f_1(x_1) + F_3(x_3)\hat{\theta}_3 - F_1(x_1)\hat{\theta}_1 + u_2 \\ \vdots \\ f_N(x_N) - f_1(x_1) + F_n(x_N)\hat{\theta}_N - F_1(x_1)\hat{\theta}_1 + u_{N-1} \end{bmatrix}; \tag{2.5}$$

then it is easy to see that if the error dynamic systems (2.5) are asymptotically stable, the adaptive synchronization for the systems (2.3) with unknown parameters will be realized.

The adaptive control method is used to ensure the asymptotical stability of the error dynamic systems (2.5), the control laws $u_1(t),\dots,u_{N-1}(t)$ can be proposed as

$$u_i = K_i e_i + f_1(x_1) - f_i(x_i) + F_1(x_1)\theta_1 - F_i(x_i)\theta_i, \quad i = 2,\dots,N, \tag{2.6}$$

where K_i is the coefficient matrix, and $K_i = K_{i1} + K_{i2}$, where $K_{i1}^T = -K_{i1}$ and $K_{i2} = \operatorname{diag}(k_{i1},\dots,k_{in}), k_{ij} < 0$ $(j = 1,\dots,n)$. The estimations for the unknown parameters $\hat{\theta}_1$ and $\hat{\theta}_i$ $(i = 2,\dots,N)$ can be defined as θ_1 and θ_i $(i = 2,\dots,N)$. In order to deal with the unknown parameters, the appropriate adaptive laws are given as follows:

$$\begin{cases} \dot{\theta}_1 = -F_1^T(x_1)e_{i-1}, & \dot{\theta}_i = F_i^T(x_i)e_{i-1}, \quad i = 2,\dots,N, \\ \theta_1(0) = \theta_{10}, & \theta_i(0) = \theta_{i0}, \end{cases} \tag{2.7}$$

where θ_{10} and θ_{i0} are the initial values of the adaptive laws θ_1 and θ_i $(i = 2,\dots,N)$.

Theorem 1 *If the error systems* (2.5) *can be controlled by the controllers* (2.6) *and the adaptive laws* (2.7), *then the error system* (2.5) *is asymptotically stable, which means the adaptive synchronization is reached between multiple controlled response systems and one drive system.*

Proof The error systems (2.5) can be rewritten as

$$\dot{e}_{i-1} = f_i(x_i) - f_1(x_1) + F_i(x_i)\hat{\theta}_i - F_1(x_1)\hat{\theta}_1 + u_i. \tag{2.8}$$

Choose the following Lyapunov function:

$$V_{i-1} = \frac{1}{2}\left(e_{i-1}^T e_{i-1} + \bar{\theta}_1^T\bar{\theta}_1 + \bar{\theta}_i^T\bar{\theta}_i\right), \tag{2.9}$$

where $\bar{\theta}_1 = \theta_1 - \hat{\theta}_1$ and $\bar{\theta}_i = \theta_i - \hat{\theta}_i$ $(i = 2,\dots,N)$ are the parameter errors, and it is clear that $\dot{\bar{\theta}}_1 = \dot{\theta}_1, \dot{\bar{\theta}}_i = \dot{\theta}_i$.

Taking the derivative of V_i along the trajectory of (2.9), one has

$$\begin{aligned} \dot{V}_i &= \frac{1}{2}\left(\dot{e}_{i-1}^T e_{i-1} + e_{i-1}^T\dot{e}_{i-1} + \dot{\bar{\theta}}_1^T\bar{\theta}_1 + \bar{\theta}_1^T\dot{\bar{\theta}}_1 + \dot{\bar{\theta}}_i^T\bar{\theta}_i + \bar{\theta}_i^T\dot{\bar{\theta}}_i\right) \\ &= \frac{1}{2}\left[f_i(x_i) - f_1(x_1) + F_i(x_i)\hat{\theta}_i - F_1(x_1)\hat{\theta}_1 + u_i\right]^T e_{i-1} \\ &\quad + \frac{1}{2}e_{i-1}{}^T\left[f_i(x_i) - f_1(x_1) + F_i(x_i)\hat{\theta}_i - F_1(x_1)\hat{\theta}_1 + u_i\right] \\ &\quad + \frac{1}{2}\left(\dot{\theta}_1^T\bar{\theta}_1 + \bar{\theta}_1^T\dot{\theta}_1 + \dot{\theta}_i^T\bar{\theta}_i + \bar{\theta}_i^T\dot{\theta}_i\right). \end{aligned} \tag{2.10}$$

Substituting (2.6) into equation (2.10), one can easily get

$$\begin{aligned} \dot{V}_{i-1} &= \frac{1}{2}\left[K_i e_{i-1} + F_1(x_1)(\theta_1 - \hat{\theta}_1) - F_i(x_i)(\theta_i - \hat{\theta}_i)\right]^T e_{i-1} \\ &\quad + \frac{1}{2}e_{i-1}^T\left[K_i e_{i-1} + F_1(x_1)(\theta_1 - \hat{\theta}_1) - F_i(x_i)(\theta_i - \hat{\theta}_i)\right] \\ &\quad + \frac{1}{2}\left(\dot{\theta}_1^T\bar{\theta}_1 + \bar{\theta}_1^T\dot{\theta}_1 + \dot{\theta}_i^T\bar{\theta}_i + \bar{\theta}_i^T\dot{\theta}_i\right) \end{aligned}$$

$$= \frac{1}{2}\big[F_1(x_1)\bar{\theta}_1 - F_i(x_i)\bar{\theta}_i\big]^T + \frac{1}{2}e_{i-1}^T\big[e_{i-1} + F_1(x_1)\bar{\theta}_1 - F_i(x_i)\bar{\theta}_i\big] + \frac{1}{2}\big(\dot{\theta}_1^T\bar{\theta}_1 + \bar{\theta}_1^T\dot{\theta}_1 + \dot{\theta}_i^T\bar{\theta}_i + \bar{\theta}_i^T\dot{\theta}_i\big). \tag{2.11}$$

Inserting the adaptive laws (2.7) into equation (2.11), one obtains

$$\dot{V}_{i-1} = \frac{1}{2}e_{i-1}^T\big(K_i^T + K_i\big)e_{i-1}.$$

Therefore, it is easy to see that

$$\dot{V}_{i-1} = \frac{1}{2}e_{i-1}^T\big(K_{i1}^T + K_{i2}^T + K_{i1} + K_{i2}\big)e_{i-1} = \frac{1}{2}e_{i-1}^T\big(K_{i2}^T + K_{i2}\big)e_{i-1} < 0;$$

then the error system (2.5) is asymptotically stable, that is, adaptive synchronization between $N-1$ respond systems and one drive system is achieved. The proof is completed. □

Remark 2 According to Theorem 1, the adaptive control method is used to estimate unknown parameters, which are directly used in the adaptive synchronization controllers. When adaptive controllers are designed, special consideration is necessary of convergence and robustness issues. Lyapunov stability theory is typically used to derive adaptive control laws and show convergence.

2.2 Adaptive transmission synchronization of multiple chaotic systems with unknown parameters

In this subsection, the first chaotic system is described as follows:

$$\begin{cases} \dot{y}_{11} = A_{11}y_1(t) + g_{11}(y_{11}(t),\dots,y_{1n}(t)) + G_{11}(y_{11}(t),\dots,y_{1n}(t))\hat{\phi}_{11}, \\ \dot{y}_{12} = A_{12}y_2(t) + g_{12}(y_{11}(t),\dots,y_{1n}(t)) + G_{12}(y_{11}(t),\dots,y_{1n}(t))\hat{\phi}_{12}, \\ \cdots \\ \dot{y}_{1n} = A_{1n}y_n(t) + g_{1n}(y_{11}(t),\dots,y_{1n}(t)) + G_{1n}(y_{11}(t),\dots,y_{1n}(t))\hat{\phi}_{1n}, \end{cases} \tag{2.12}$$

where $A_1(t) = [A_{11}, A_{12},\dots,A_{1n}]^T$ is the coefficient matrix, $y_1(t) = [y_{11}, y_{12},\dots,y_{1n}]^T$ is the state of the system (2.12), $g_{1i}(y_{11}(t),\dots,y_{1n}(t))$ $(i = 1,\dots,n)$ is a continuous function and $g_1(y_1(t)) = [g_{11}, g_{12},\dots,g_{1n}]^T$; $G_{1i}(y_{11}(t),\dots,y_{1n}(t))$ $(i = 1,\dots,n)$ is the matrix function and $G_1(y_1(t)) = [G_{11}, G_{12},\dots,G_{1n}]^T$; $\hat{\phi}_{1i}$ $(i = 1,\dots,n)$ is for the unknown parameters and $\hat{\phi}_1 = [\hat{\phi}_{11}, \hat{\phi}_{12},\dots,\hat{\phi}_{1n}]^T$.

The other $N-1$ chaotic systems are given by

$$\begin{cases} \dot{y}_{j1} = A_{j1}y_1(t) + g_{j1}(y_{j1}(t),\dots,y_{jn}(t)) + G_{j1}(y_{j1}(t),\dots,y_{jn}(t))\hat{\phi}_{j1} + v_{j-1,1}, \\ \dot{y}_{j2} = A_{j2}y_2(t) + g_{j2}(y_{j1}(t),\dots,y_{jn}(t)) + G_{j2}(y_{j1}(t),\dots,y_{jn}(t))\hat{\phi}_{j2} + v_{j-1,2}, \\ \cdots \\ \dot{y}_{jn} = A_{jn}y_n(t) + g_{jn}(y_{j1}(t),\dots,y_{jn}(t)) + G_{jn}(y_{j1}(t),\dots,y_{jn}(t))\hat{\phi}_{jn} + v_{j-1,n}, \end{cases} \tag{2.13}$$

where $j = 2,\dots,N$ and $A_j(t) = [A_{j1}, A_{j2},\dots,A_{jn}]^T$ is the coefficient matrix, the state of the system (2.13) is $y_j(t) = [y_{j1}, y_{j2},\dots,y_{jn}]^T$, $g_{ji}(y_{j1}(t),\dots,y_{jn}(t))$ $(i = 1,\dots,n)$ is a continuous function and $g_j(y_j(t)) = [g_{j1}, g_{j2},\dots,g_{jn}]^T$; $G_{ji}(y_{j1}(t),\dots,y_{jn}(t))$ $(i = 1,\dots,n)$ is the matrix function and $G_j(y_j(t)) = [G_{j1}, G_{j2},\dots,G_{jn}]^T$; $\hat{\phi}_{ji}$ $(i = 1,\dots,n)$ is for the unknown parameters and

$\hat{\phi}_j = [\hat{\phi}_{j1}, \hat{\phi}_{j2}, \ldots, \hat{\phi}_{jn}]^T$, and the control input is $v_{j-1} = [v_{j-1,1}, v_{j-1,2}, \ldots, v_{j-1,n}]^T$. If $A_i \neq A_j$ $(i,j = 1,\ldots,N, i \neq j)$, $g_i(y_i) \neq g_j(y_j)$ $(i,j = 1,\ldots,N, i \neq j)$ and $G_i(y_i) \neq G_j(y_j)$ $(i,j = 1,\ldots,N, i \neq j)$, then the systems (2.12) and (2.13) are an array of nonidentical chaotic systems.

In accordance with (2.12) and (2.13), the above N chaotic systems can be rewritten as

$$\begin{cases} \dot{y}_1 = A_1 y_1 + g_1(y_1) + G_1(y_1)\hat{\phi}_1, \\ \dot{y}_2 = A_2 y_2 + g_2(y_2) + G_2(y_2)\hat{\phi}_2 + v_1, \\ \cdots \\ \dot{y}_N = A_N y_N + g_N(y_N) + G_N(y_N)\hat{\phi}_N + v_{N-1}. \end{cases} \tag{2.14}$$

In the framework of the transmission synchronization mode, the state error can be defined as $e_j(t) = y_{j+1}(t) - y_j(t)$, $j = 1,\ldots,N-1$, then the definition of adaptive transmissions synchronization of N systems is obtained.

Definition 2 For N chaotic systems described by (2.14), if there exist adaptive controllers $v_1(t),\ldots,v_{N-1}(t)$ such that the error dynamic systems

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \vdots \\ \dot{e}_{N-1} \end{bmatrix} = \begin{bmatrix} \dot{y}_2 - \dot{y}_1 \\ \dot{y}_3 - \dot{y}_2 \\ \dot{y}_4 - \dot{y}_3 \\ \vdots \\ \dot{y}_N - \dot{y}_{N-1} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} A_2 y_2 - A_1 y_1 + g_2(y_2) - g_1(y_1) \\ +G_2(y_2)\hat{\phi}_2 - G_1(y_1)\hat{\phi}_1 + v_1 \end{pmatrix} \\ \begin{pmatrix} A_3 y_3 - A_2 y_2 + g_3(y_3) - g_2(y_2) \\ +G_3(y_3)\hat{\phi}_3 - G_2(y_2)\hat{\phi}_2 + v_2 - v_1 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} A_N y_N - A_{N-1} y_{N-1} + g_N(y_N) - g_{N-1}(y_{N-1}) \\ +G_N(x_N)\hat{\phi}_N - G_{N-1}(x_{N-1})\hat{\phi}_{N-1} + v_N - v_{N-1} \end{pmatrix} \end{bmatrix}$$

satisfy the following condition:

$$\lim_{t\to\infty} \|e_j(t)\| = \lim_{t\to\infty} \|y_{j+1}(t) - y_j(t)\| = 0, \quad j = 1,\ldots,N-1, \tag{2.15}$$

it is said that the adaptive transmission synchronization are realized among N systems with unknown parameters.

Now, adaptive control method is used to design the controllers and adaptive laws to achieve $\lim_{t\to\infty} \|e_j(t)\| = 0$, and the synchronization among (2.12) and (2.13) are realized under the transmissions synchronization mode. The control inputs $v_1(t),\ldots,v_{N-1}(t)$ can be designed as

$$\begin{cases} v_1 = -(A_2 - A_1)y_1 - g_2(y_2) + g_1(y_1) - G_2(y_2)\phi_2 + G_1(y_1)\phi_1 + H_1 e_1, \\ v_j - v_{j-1} = (-(A_{j+1} - A_j)y_j - g_{j+1}(y_{j+1}) + g_j(y_j) - G_{j+1}(y_{j+1})\phi_{j+1} + G_j(y_j)\phi_j + H_j e_j), \\ \quad j = 2,\ldots,N-1, \end{cases} \tag{2.16}$$

where H_j is the coefficient matrix, which can be constructed in order to ensure that system (2.15) is asymptotically stable. ϕ_1 and ϕ_j $(j = 2,\ldots,N)$ are the estimations of the unknown parameters $\hat{\phi}_1$ and $\hat{\phi}_j$ $(j = 2,\ldots,N)$. And the proper adaptive laws can be obtained as follows:

$$\begin{cases} \dot{\phi}_1 = -G_1^T(y_1)e_1, \\ \dot{\phi}_j = G_j^T(y_j)e_{j-1} = -G_j^T(y_j)e_j, \quad j = 2, \dots, N-1, \\ \dot{\phi}_N = G_N^T(y_N)e_{N-1}, \\ \phi_1(0) = \phi_{10}, \qquad \phi_j(0) = \theta_{j0}, \qquad \phi_N(0) = \theta_{N0}. \end{cases} \tag{2.17}$$

Theorem 2 *Considering the error dynamic systems $e_j(t)$ with the controllers in* (2.16) *and adaptive laws in* (2.17), *it is easy to ensure that the error dynamic systems $e_j(t)$ are asymptotically stable, and adaptive transmission synchronization among N systems with unknown parameters is realized.*

Proof First of all, the first error dynamic system $\dot{e}_1(t) = \dot{y}_2(t) - \dot{y}_1(t)$ is considered. Choose the following Lyapunov function for $e_1(t)$,

$$V_1 = \frac{1}{2}\left(e_1^T e_1 + \bar{\phi}_1^T \bar{\phi}_1 + \bar{\phi}_2^T \bar{\phi}_2\right),$$

where $\bar{\phi}_1 = \phi_1 - \hat{\phi}_1$ and $\bar{\phi}_2 = \phi_2 - \hat{\phi}_2$ are the parameters errors.

Similarly to Theorem 1, substituting the adaptive laws $\dot{\phi}_1 = -G_1^T(y_1)e_1$ and $\dot{\phi}_2 = G_2^T(y_2)e_1$ and the controller v_1 into the derivative of V_1, one has

$$\dot{V}_1 = \frac{1}{2}e_1^T\left(A_2^T + A_2 + H_1 + H_1^T\right)e_1.$$

By constructing the appropriate coefficient matrix H_1, one can guarantee $\dot{V}_1 < 0$, then the error dynamic systems $e_1(t)$ is asymptotically stable, that is, the adaptive synchronization between the first system and the second system is achieved.

Second, for the error dynamic system $e_j(t)$ ($j = 2, \dots, N-1$) between the jth system and the $(j+1)$th system, a proper Lyapunov function for $e_j(t)$ is constructed as

$$V_j = \frac{1}{2}\left(e_j^T e_j + \bar{\phi}_j^T \bar{\phi}_j + \bar{\phi}_{j+1}^T \bar{\phi}_{j+1}\right),$$

where $\bar{\phi}_j = \phi_j - \hat{\phi}_j$ and $\bar{\phi}_{j+1} = \phi_{j+1} - \hat{\phi}_{j+1}$ are the parameters errors.

Introducing the controller

$$v_j - v_{j-1} = \left(A_{j+1}e_j - (A_{j+1} - A_j)x_j - g_{j+1}(y_{j+1}) + g_j(y_j) - G_{j+1}(y_{j+1})\phi_{j+1} + G_j(y_j)\phi_j + H_je_j\right), \quad j = 2, \dots, N-1,$$

into the derivative of V_j, one can easily obtain

$$\begin{aligned} \dot{V}_j &= \frac{1}{2}\left(\dot{e}_j^T e_j + e_j^T \dot{e}_j + \dot{\bar{\phi}}_j^T \bar{\phi}_j + \bar{\phi}_j^T \dot{\bar{\phi}}_j + \dot{\bar{\phi}}_{j+1}^T \bar{\phi}_{j+1} + \bar{\phi}_{j+1}^T \dot{\bar{\phi}}_{j+1}\right) \\ &= \frac{1}{2}\left[A_{j+1}e_j + (A_{j+1} - A_j)y_j + g_{j+1}(y_{j+1}) - g_j(y_j) + G_{j+1}(y_{j+1})\hat{\phi}_{j+1} - G_j(y_j)\hat{\phi}_j + v_i - v_{j-1}\right]^T e_j \\ &\quad + \frac{1}{2}e_j{}^T\left[A_{j+1}e_j + (A_{j+1} - A_j)y_j + g_{j+1}(y_{j+1}) - g_j(y_j)\right. \\ &\quad \left. + G_{j+1}(y_{j+1})\hat{\phi}_{j+1} - G_j(y_j)\hat{\phi}_j + v_i - v_{j-1}\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\dot{\phi}_j^T\bar{\phi}_j+\bar{\phi}_j^T\dot{\phi}_j+\dot{\phi}_{j+1}^T\bar{\phi}_{j+1}+\bar{\phi}_{j+1}^T\dot{\phi}_{j+1}\right)\\
&=\frac{1}{2}\left[A_{j+1}e_j+G_j(y_j)(\phi_j-\hat{\phi}_j)-G_{j+1}(y_{j+1})(\phi_{j+1}-\hat{\phi}_{j+1})+H_je_j\right]^T e_j\\
&+\frac{1}{2}e_j{}^T\left[A_{j+1}e_j+G_j(y_j)(\phi_j-\hat{\phi}_j)-G_{j+1}(y_{j+1})(\phi_{j+1}-\hat{\phi}_{j+1})+H_je_j\right]\\
&+\frac{1}{2}\left(\dot{\phi}_j^T\bar{\phi}_j+\bar{\phi}_j^T\dot{\phi}_j+\dot{\phi}_{j+1}^T\bar{\phi}_{j+1}+\bar{\phi}_{j+1}^T\dot{\phi}_{j+1}\right)\\
&=\frac{1}{2}e_j^T\left(A_{j+1}^T+A_{j+1}+H_j+H_j^T\right)e_j+\frac{1}{2}\left[\bar{\phi}_j^TG_j^T(y_j)e_j-\bar{\phi}_{j+1}^TG_{j+1}^T(y_{j+1})e_j\right]\\
&+\frac{1}{2}\left[e_j{}^TG_j(y_j)\bar{\phi}_j-e_j{}^TG_{j+1}(y_{j+1})\bar{\phi}_{j+1}\right]\\
&+\frac{1}{2}\left(\dot{\phi}_j^T\bar{\phi}_j+\bar{\phi}_j^T\dot{\phi}_j+\dot{\phi}_{j+1}^T\bar{\phi}_{j+1}+\bar{\phi}_{j+1}^T\dot{\phi}_{j+1}\right).
\end{aligned}
\tag{2.18}
$$

Substituting the adaptive laws $\dot{\phi}_j = -G_j^T(j_1)e_j$ and $\dot{\phi}_{j+1} = G_{j+1}^T(y_{j+1})e_j$ into (2.18) and simplifying it, one has

$$\dot{V}_j = \frac{1}{2}e_j^T\left(A_{j+1}^T + A_{j+1} + H_j + H_j^T\right)e_j. \tag{2.19}$$

Similarly, by constructing the appropriate coefficient matrix H_j to guarantee $\dot{V}_j < 0$, the error dynamic systems $e_j(t)$ will be asymptotically stable, that is, the adaptive synchronization between the j system and the $j + 1$ system is realized. With the above discussions, it is easy to see that the adaptive transmission synchronization is reached among N systems with unknown parameters. Hence, the proof is completed. □

Remark 3 According to Theorem 2, when the unknown parameters ϕ_j and ϕ_{j+1} are estimated and the adaptive synchronization between the jth system and the $(j + 1)$th system is investigated, the adaptive laws should satisfy $\dot{\phi}_j = -G_j^T(y_j)e_j$ and $\dot{\phi}_{j+1} = G_{j+1}^T(y_{j+1})e_{j+1}$. For the $(j - 1)$th system and the jth system, when the unknown parameters ϕ_{j-1} and ϕ_j are estimated, the conditions $\dot{\phi}_{j-1} = -G_{j-1}^T(y_{j-1})e_{j-1}$ and $\dot{\phi}_j = G_j^T(y_j)e_j$ should hold, then it is easy to conclude that

$$\dot{\phi}_j = G_j^T(y_j)e_{j-1} = -G_j^T(y_j)e_j, \quad j = 2,\ldots,N-1. \tag{2.20}$$

Remark 4 According to Definition 2 and Theorem 2, the transmission synchronization with ring connection can be effectively applied in the many engineering fields, such as circuits, mobile ad hoc networks, etc. [33–35]. The nodes in such networks can exhibit phase synchronization or can be synchronized in the same state to keep such networks stable.

3 Numerical examples and simulation

In the section, two numerical examples are given to validate the effectiveness of the proposed synchronization schemes by choosing three chaotic systems and three hyperchaotic systems [36–38], respectively, which satisfy the above proposed synchronization connection modes.

Example 1 Consider one Lorenz system [36] and two Lü systems [37] as an example, which can be described as follows:

$$\begin{cases} \dot{x}_{11} = \hat{\theta}_{11}(x_{12} - x_{11}), \\ \dot{x}_{12} = \hat{\theta}_{12}x_{11} - x_{11}x_{13} - x_{12}, \\ \dot{x}_{13} = x_{11}x_{12} - \hat{\theta}_{13}x_{13}, \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{x}_{21} = \hat{\theta}_{21}(x_{22} - x_{21}) + u_{11}, \\ \dot{x}_{22} = -x_{21}x_{23} + \hat{\theta}_{22}x_{22} + u_{12}, \\ \dot{x}_{23} = x_{21}x_{22} - \hat{\theta}_{23}x_{23} + u_{13}, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \dot{x}_{31} = \hat{\theta}_{31}(x_{32} - x_{31}) + u_{21}, \\ \dot{x}_{32} = -x_{31}x_{33} + \hat{\theta}_{32}x_{32} + u_{22}, \\ \dot{x}_{33} = x_{31}x_{32} - \hat{\theta}_{33}x_{33} + u_{33}, \end{cases} \tag{3.3}$$

where $\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\theta}_{23}, \hat{\theta}_{31}, \hat{\theta}_{32}, \hat{\theta}_{33}$ are the unknown parameters. When $\hat{\theta}_{11} = 10, \hat{\theta}_{12} = 28, \hat{\theta}_{13} = -\frac{8}{3}, \hat{\theta}_{21} = \hat{\theta}_{31} = 36, \hat{\theta}_{22} = \hat{\theta}_{32} = 20, \hat{\theta}_{23} = \hat{\theta}_{33} = 3$, then (3.1), (3.2), and (3.3) are chaotic systems. $u_1 = [u_{11}, u_{12}, u_{13}]^T$ and $u_2 = [u_{21}, u_{22}, u_{23}]^T$ are the control inputs, and

$$f_1(x_1) = \begin{bmatrix} 0 \\ -x_{11}x_{13} - x_{12} \\ x_{11}x_{12} \end{bmatrix}, \qquad F_1(x_1) = \begin{bmatrix} x_{12} - x_{11} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & -x_{13} \end{bmatrix},$$

$$f_2(x_2) = \begin{bmatrix} 0 \\ -x_{21}x_{23} \\ x_{21}x_{22} \end{bmatrix}, \qquad F_2(x_2) = \begin{bmatrix} x_{22} - x_{21} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & -x_{23} \end{bmatrix},$$

$$f_3(x_3) = \begin{bmatrix} 0 \\ -x_{31}x_{33} \\ x_{31}x_{32} \end{bmatrix}, \qquad F_3(x_3) = \begin{bmatrix} x_{32} - x_{31} & 0 & 0 \\ 0 & x_{32} & 0 \\ 0 & 0 & -x_{33} \end{bmatrix}.$$

The error dynamic systems can be obtained:

$$\begin{aligned} \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} &= \begin{bmatrix} \dot{x}_2 - \dot{x}_1 \\ \dot{x}_3 - \dot{x}_1 \end{bmatrix} = \begin{bmatrix} f_2(x_2) - f_1(x_1) + F_2(x_2)\hat{\theta}_2 - F_1(x_1)\hat{\theta}_1 + u_1 \\ f_3(x_3) - f_1(x_1) + F_3(x_3)\hat{\theta}_3 - F_1(x_1)\hat{\theta}_2 + u_2 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\theta}_{21}(x_{22} - x_{21}) - \hat{\theta}_{11}(x_{12} - x_{11}) + u_{11} \\ -x_{21}x_{23} + \hat{\theta}_{22}x_{22} - \hat{\theta}_{12}x_{11} + x_{11}x_{13} + x_{12} + u_{12} \\ x_{21}x_{22} - \hat{\theta}_{23}x_{23} - x_{11}x_{12} + \hat{\theta}_{13}x_{13} + u_{13} \\ \hat{\theta}_{31}(x_{32} - x_{31}) - \hat{\theta}_{11}(x_{12} - x_{11}) + u_{21} \\ -x_{31}x_{33} + \hat{\theta}_{32}x_{32} - \hat{\theta}_{12}x_{11} + x_{11}x_{13} + x_{12} + u_{22} \\ x_{31}x_{32} - \hat{\theta}_{33}x_{33} - x_{11}x_{12} + \hat{\theta}_{13}x_{13} + u_{23} \end{bmatrix}. \end{aligned}$$

According to (2.6) and (2.7), the proper coefficient matrices are constructed as

$$K_1 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -2 & -2 & -1 \\ 2 & -2 & -2 \\ 1 & 2 & -2 \end{bmatrix};$$

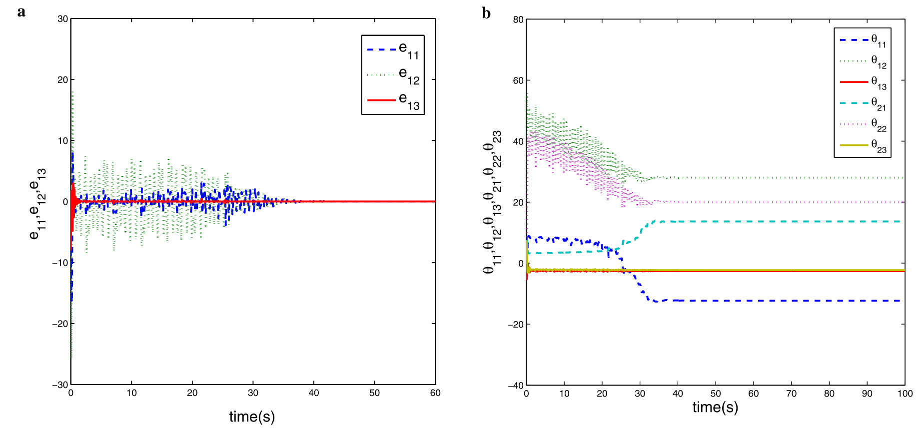

Figure 2 Synchronization errors e_{11}, e_{12}, e_{13} between drive system (3.1) and response system (3.2), and the time response of the adaptive vector parameters $\theta_{11}, \theta_{12}, \theta_{13}, \theta_{21}, \theta_{22}, \theta_{23}$.

then the controllers and the adaptive laws can be obtained as follows:

$$\begin{cases} u_{11} = \theta_{11}(x_{12} - x_{11}) - \theta_{21}(x_{22} - x_{21}) - e_{11} - e_{13}, \\ u_{12} = x_{21}x_{23} - \theta_{22}x_{22} + \theta_{12}x_{11} - x_{11}x_{13} - x_{12} - e_{12} - e_{13}, \\ u_{13} = -x_{21}x_{22} + \theta_{23}x_{23} + x_{11}x_{12} - \theta_{13}x_{13} + e_{11} + e_{12} - e_{13}, \\ u_{21} = \theta_{11}(x_{12} - x_{11}) - \theta_{31}(x_{32} - x_{31}) - 2e_{21} - 2e_{22} - e_{23}, \\ u_{22} = x_{31}x_{33} - \theta_{32}x_{32} + \theta_{12}x_{11} - x_{11}x_{13} - x_{12} + 2e_{21} - 2e_{22} - 2e_{23}, \\ u_{23} = -x_{31}x_{32} + \theta_{33}x_{33} + x_{11}x_{12} - \theta_{13}x_{13} + e_{11} + 2e_{22} - e_{23}, \end{cases} \tag{3.4}$$

and

$$\begin{bmatrix} \dot{\theta}_{11} \\ \dot{\theta}_{12} \\ \dot{\theta}_{13} \end{bmatrix} = -\begin{bmatrix} x_{12} - x_{11} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & -x_{13} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = -\begin{bmatrix} x_{12} - x_{11} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & -x_{13} \end{bmatrix} \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix},$$

$$\begin{bmatrix} \dot{\theta}_{21} \\ \dot{\theta}_{22} \\ \dot{\theta}_{23} \end{bmatrix} = \begin{bmatrix} x_{22} - x_{21} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & -x_{23} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix},$$

$$\begin{bmatrix} \dot{\theta}_{31} \\ \dot{\theta}_{32} \\ \dot{\theta}_{33} \end{bmatrix} = \begin{bmatrix} x_{32} - x_{31} & 0 & 0 \\ 0 & x_{32} & 0 \\ 0 & 0 & -x_{33} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix},$$

where $\theta_{11}, \theta_{12}, \theta_{13}, \theta_{21}, \theta_{22}, \theta_{23}, \theta_{31}, \theta_{32}, \theta_{33}$ are the estimations of the above chaotic system parameters $\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\theta}_{23}, \hat{\theta}_{31}, \hat{\theta}_{32}, \hat{\theta}_{33}$.

For simulations, the initial conditions of one drive system and two response chaotic systems are chosen as $(x_{11}(0), x_{12}(0), x_{13}(0)) = (10, 10, 10)$, $(x_{21}(0), x_{22}(0), x_{23}(0)) = (2, 2, 2)$, and $(x_{31}(0), x_{32}(0), x_{33}(0)) = (3, 3, 3)$. It is assumed that the initial value of the adaptive parameters are $(\hat{\theta}_{11}(0), \hat{\theta}_{12}(0), \hat{\theta}_{13}(0)) = (3, 3, 3)$, $(\hat{\theta}_{21}(0), \hat{\theta}_{22}(0), \hat{\theta}_{23}(0)) = (3, 3, 3)$, $(\hat{\theta}_{31}(0), \hat{\theta}_{32}(0), \hat{\theta}_{33}(0)) = (4, 4, 4)$. Then the state trajectories of the error dynamic systems e_1 and e_2 are shown in Figure 2(a) and Figure 3(a). The adaptive laws $\theta_{11}, \theta_{12}, \theta_{13}, \theta_{21}, \theta_{22}, \theta_{23}, \theta_{31}, \theta_{32}, \theta_{33}$ can be depicted in Figure 2(b) and Figure 3(b). It is clear that the trajectories of the errors

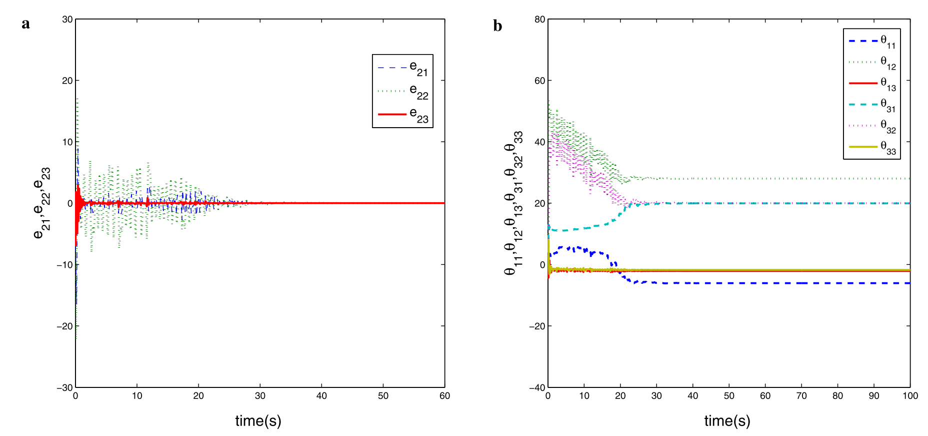

Figure 3 Synchronization errors e_{21}, e_{22}, e_{23} between drive system (3.1) and response system (3.3), and the time response of the adaptive vector parameters $\theta_{11}, \theta_{12}, \theta_{13}, \theta_{31}, \theta_{32}, \theta_{33}$.

systems converge to 0 quickly, and synchronization among the three chaotic systems is achieved. From Figure 2(b) and Figure 3(b), it is easy to see that all adaptive laws converge to some fixed values, which realize the estimation of the unknown parameters of chaotic systems.

Remark 5 In the above simulations, the reasonable value K_1 and K_2 can be chosen according to the corresponding chaotic complex system to achieve the desired result in the simple way.

Example 2 Consider two hyperchaotic Chen systems [37] and one hyperchaotic Rössler system [38] as an example, which satisfy the ring networks connection mode and can be described by

$$\begin{bmatrix} \dot{y}_{11} \\ \dot{y}_{12} \\ \dot{y}_{13} \\ \dot{y}_{14} \end{bmatrix} = A_1 \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \end{bmatrix} + g_1(y_1) + G_1(y_1) \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ r_1 \end{bmatrix}, \tag{3.5}$$

$$\begin{bmatrix} \dot{y}_{21} \\ \dot{y}_{22} \\ \dot{y}_{23} \\ \dot{y}_{24} \end{bmatrix} = A_2 \begin{bmatrix} y_{21} \\ y_{j2} \\ y_{j3} \\ y_{j4} \end{bmatrix} + g_2(y_2) + G_2(y_2) \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} + \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \end{bmatrix}, \tag{3.6}$$

and

$$\begin{bmatrix} \dot{y}_{31} \\ \dot{y}_{32} \\ \dot{y}_{33} \\ \dot{y}_{34} \end{bmatrix} = A_3 \begin{bmatrix} y_{31} \\ y_{32} \\ y_{33} \\ y_{34} \end{bmatrix} + g_3(y_3) + G_3(y_3) \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \\ r_3 \end{bmatrix} + \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \end{bmatrix}, \tag{3.7}$$

where $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{r}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2$, and $\hat{a}_3, \hat{b}_3, \hat{c}_3, \hat{d}_3, \hat{r}_3$ are the systems parameters. When $\hat{a}_l = 35, \hat{b}_l = 3, \hat{c}_l = 12, \hat{d}_l = 7, \hat{r}_l = 0.5, l = 1, 3,\ a_2 = 0.25, b_2 = 3, c_2 = 0.5, d_2 = 0.05$, (3.5), (3.6), and (3.7) are chaotic systems. $v_1 = [v_{11}, v_{12}, v_{13}]^T$ and $v_2 = [v_{21}, v_{22}, v_{23}]^T$ are the control inputs. And

$$A_l = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad g_l(y_l) = \begin{bmatrix} 0 \\ -y_{l1}y_{l3} \\ y_{l1}y_{l2} \\ y_{l2}y_{l3} \end{bmatrix},$$

$$G_1(y_1) = \begin{bmatrix} y_{l2} - y_{l1} & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{l2} & y_{l1} & 0 \\ 0 & -y_{l3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_{l4} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$g_2(y_2) = \begin{bmatrix} 0 \\ 0 \\ y_{21}y_{23} \\ 0 \end{bmatrix}, \qquad G_2(y_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ y_{22} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -y_{23} & y_{24} \end{bmatrix}.$$

The error systems $\dot{e}_1 = \dot{y}_2 - \dot{y}_1$ and $\dot{e}_2 = \dot{y}_3 - \dot{y}_2$ can be obtained as follows:

$$\begin{cases} \dot{e}_{11} = -y_{22} - y_{23} - y_{14} - \hat{a}_1(y_{12} - y_{11}) + v_{11}, \\ \dot{e}_{12} = y_{21} + y_{24} + \hat{a}_2 y_{22} + y_{11}y_{13} - \hat{c}_1 y_{12} - \hat{d}_1 y_{11} + v_{12}, \\ \dot{e}_{13} = y_{21}y_{23} + \hat{b}_2 - y_{11}y_{12} + \hat{b}_1 y_{13} + v_{13}, \\ \dot{e}_{14} = -\hat{c}_2 y_{23} + \hat{d}_2 y_{24} - y_{12}y_{13} - \hat{r}_1 y_{14} + v_{14}, \\ \dot{e}_{21} = y_{22} + y_{23} + y_{34} + \hat{a}_2(y_{12} - y_{11}) + v_{21} - v_{11}, \\ \dot{e}_{22} = -y_{21} - y_{24} - \hat{a}_3 y_{22} - y_{31}y_{33} + \hat{c}_3 y_{32} + \hat{d}_3 y_{31} + v_{22} - v_{12}, \\ \dot{e}_{23} = -y_{21}y_{23} - \hat{b}_2 + y_{31}y_{32} - \hat{b}_3 y_{33} + v_{23} - v_{13}, \\ \dot{e}_{24} = \hat{c}_2 y_{23} - \hat{d}_2 y_{24} + y_{32}y_{33} + \hat{r}_3 y_{34} + v_{24} - v_{14}. \end{cases} \tag{3.8}$$

According to (2.16) and (2.17), the controllers and the adaptive laws can be designed as

$$\begin{cases} v_{11} = y_{22} + y_{23} + y_{11} + a_1(y_{12} - y_{11}) + h_{11}e_1, \\ v_{12} = -y_{21} - y_{24} - a_2 y_{22} - y_{11}y_{13} + c_1 y_{12} + d_1 y_{11} + h_{12}e_1, \\ v_{13} = -y_{21}y_{23} - b_2 + y_{11}y_{12} - b_1 y_{13} + h_{13}e_1, \\ v_{14} = c_2 y_{23} - d_2 y_{24} + y_{12}y_{13} + r_1 y_{14} + h_{14}e_1, \\ v_{21} = -y_{22} - y_{23} - y_{34} - a_2(y_{12} - y_{11}) + v_{11} + h_{21}e_2, \\ v_{22} = y_{21} + y_{24} + a_3 y_{22} + y_{31}y_{33} - c_3 y_{32} - d_3 y_{31} + v_{12} + h_{22}e_2, \\ v_{23} = y_{21}y_{23} + b_2 - y_{31}y_{32} + b_3 y_{33} + v_{13} + h_{23}e_2, \\ v_{24} = -c_2 y_{23} + d_2 y_{24} - y_{32}y_{33} - r_3 y_{34} + v_{14} + h_{24}e_2, \end{cases} \tag{3.9}$$

and

$$\begin{bmatrix} \dot{a}_1 \\ \dot{b}_1 \\ \dot{c}_1 \\ \dot{d}_1 \\ \dot{r}_1 \end{bmatrix} = -\begin{bmatrix} (y_{12} - y_{11})e_{11} \\ -y_{13}e_{13} \\ y_{12}e_{12} \\ y_{11}e_{12} \\ y_{14}e_{14} \end{bmatrix}, \qquad \begin{bmatrix} \dot{a}_3 \\ \dot{b}_3 \\ \dot{c}_3 \\ \dot{d}_3 \\ \dot{r}_3 \end{bmatrix} = \begin{bmatrix} (y_{32} - y_{31})e_{21} \\ -y_{33}e_{23} \\ y_{32}e_{22} \\ y_{31}e_{22} \\ y_{34}e_{24} \end{bmatrix}, \tag{3.10}$$

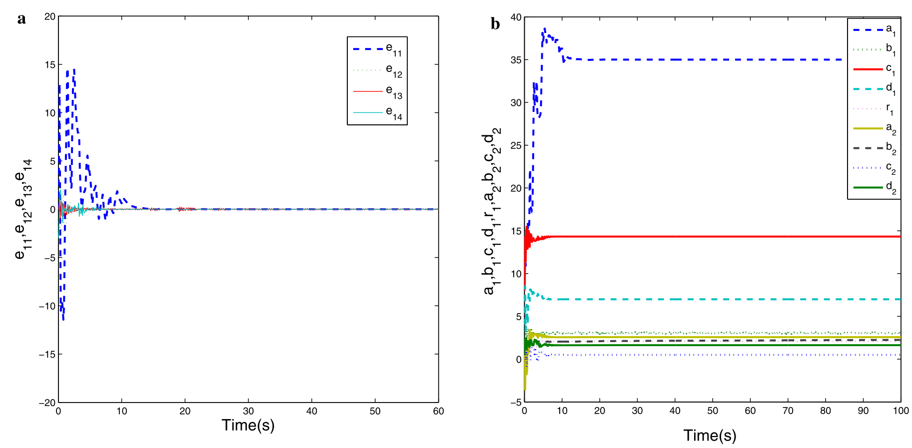

Figure 4 Synchronization errors $e_{11}, e_{12}, e_{13}, e_{14}$ between drive system (3.5) and response system (3.6) in (a), and the time response of the adaptive vector parameters $a_1, b_1, c_1, d_1, r_1, a_2, b_2, c_2, d_2$ in (b).

$$\begin{bmatrix} \dot{a}_2 \\ \dot{b}_2 \\ \dot{c}_2 \\ \dot{d}_2 \end{bmatrix} = \begin{bmatrix} y_{22}e_{12} \\ e_{13} \\ -y_{23}e_{14} \\ y_{24}e_{14} \end{bmatrix} = -\begin{bmatrix} y_{22}e_{22} \\ e_{23} \\ -y_{23}e_{24} \\ y_{24}e_{24} \end{bmatrix},$$

where H_1 and H_2 can be constructed as

$$H_1 = \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{14} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} h_{21} \\ h_{22} \\ h_{23} \\ h_{24} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

and $a_1, b_1, c_1, d_1, r_1, a_2, b_2, c_2, d_2, a_3, b_3, c_3, d_3, r_3$ are the estimations of the unknown parameters $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{r}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2, \hat{a}_3, \hat{b}_3, \hat{c}_3, \hat{d}_3, \hat{r}_3$.

The initial conditions of three chaotic systems are chosen as $(y_{11}(0), y_{12}(0), y_{13}(0), y_{14}(0)) = (1, -2, -1, -2)$, $(y_{21}(0), y_{22}(0), y_{23}(0), y_{24}(0)) = (1, -1, 1, -5)$, $(y_{31}(0), y_{32}(0), y_{33}(0), y_{34}(0)) = (1, -2, -1, -2)$, respectively. The initial values of the adaptive parameters can be assumed as $(a_{10}, b_{10}, c_{10}, d_{10}, r_{10}) = (a_{30}, b_{30}, c_{30}, d_{30}, r_{30}) = (10, 2, 8, 4, 1)$, $(a_{20}, b_{20}, c_{20}, d_{20}) = (1, 2, 0.5, 0.5)$. The state trajectories of the error dynamic systems e_1 and e_2 are shown in Figure 4(a) and Figure 5(a). Meanwhile, the adaptive laws $a_1, b_1, c_1, d_1, r_1, a_2, b_2, c_2, d_2, a_3, b_3, c_3, d_3, r_3$ can be found in Figure 4(b) and Figure 5(b). It implies that the error dynamic systems e_1 and e_2 converge to 0 quickly, that is, (3.5) synchronizes with (3.6), and (3.6) synchronizes with (3.7). It is concluded that adaptive transmission synchronization among three chaotic systems is achieved. According to Figure 4(b) and Figure 5(b), it is clear that the designed adaptive laws (2.17) are appropriate, and they converge to some fixed values, which realize the estimation of the unknown parameters of chaotic systems.

Remark 6 From a simulation analysis of Examples 1 and 2, it is easy to see that two different synchronization modes can be effectively applied in cluster synchronization and transmission synchronization for complex networks. And cluster synchronization and trans-

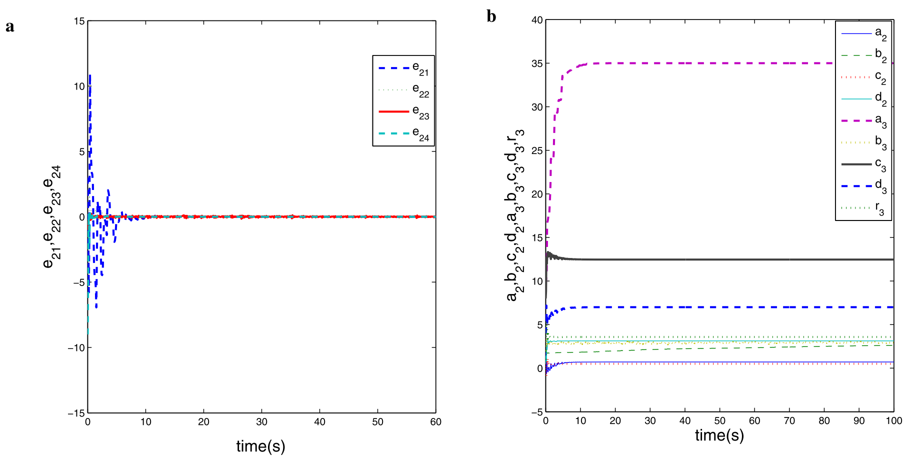

Figure 5 Synchronization errors $e_{21}, e_{22}, e_{23}, e_{24}$ between drive system (3.6) and response system (3.7) in (a), and the time response of the adaptive vector parameters $a_2, b_2, c_2, d_2, a_3, b_3, c_3, d_3, r_3$ in (b).

mission synchronization for special networks connection composed of chaotic systems lead to improved applications in secure communications [32] and other fields.

4 Conclusions

In this paper, we have introduced two classes of different chaos synchronization modes among multiple chaotic systems, and we discussed the adaptive synchronization problems of multiple chaotic systems with unknown parameters. By using the adaptive control method, adaptive controllers and adaptive laws have been designed, and new synchronization criteria are given, which can effectively stabilize the error systems and estimate the unknown parameters, Simulation results have shown the effectiveness of the proposed controllers for synchronizing multiple uncertain chaotic systems by using adaptive control techniques. The main limitation of this work is that the synchronization problems of multiple chaotic systems with unknown parameters are only discussed. To investigate the synchronization among multiple chaotic systems with uncertainty and external disturbances, and to have do rigorous research on their application on multilateral communications [32] and signal transmission of ring networks [33–35] are our future works.

Competing interests
All authors declare that they have no competing interests to this work.

Authors' contributions
Each of the authors contributed to each part of this work equally and all authors read and approved the final manuscript.

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