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Positive solutions of singular fractional order differential system with Riemann-Stieltjes integral boundary condition

Jingjing Tan*

*Correspondence:
tanjingjing1108@163.com
School of Mathematics and Information Science, Weifang University, Dongfeng Road, Weifang, 261061, P.R. China

Abstract

In this paper, we study the existence of positive solutions to a class of higher-order nonlinear fractional functional differential system with Riemann-Stieltjes integral boundary conditions. Our method relies upon the upper and lower solutions and the Schauder fixed point theorem. Furthermore, we constructed an iterative scheme to approximate the positive solution. We also give an example to illustrate the main results.

Keywords: fractional differential equation; boundary value problem; fixed point theorem; Riemann-Stieltjes integral

1 Introduction

In 1927, Fermi and Thomas studied the problem of how to determine the electric potential in an atom. They found that this problem can be translated into the following second order differential equation, that is, two point singular boundary value problems:

$$\begin{cases} u'' - t^{-\frac{1}{2}} u^{\frac{3}{2}} = 0, \\ u(0) = 1, \qquad u(b) = 0, \end{cases}$$

where

$$\lim_{t\to 0^+} u''(t) = \lim_{t\to 0^+} t^{-\frac{1}{2}} u^{\frac{3}{2}} = \infty.$$

Since then, many scholars began to research this kind of singular boundary value problem. Consequently, the differential equation singular boundary value problem and its applications in various fields of science has received much attention (see [1–25]). It should be noted that most of the papers are devoted to the solvability of the existence of positive solutions for a singular differential equation boundary value problem. However, there are few papers to deal with the existence of a high-order singular differential equation system boundary value problem, especially with Riemann-Stieltjes integral boundary condition.

In [26], Zhang and Han investigated the following singular fractional differential equation boundary value problem:

$$\begin{cases} -D_{0+}^{\alpha}x(t) = f(t,x(t)), & 0 < t < 1, \alpha \in (n-1,n], \alpha \geq 2, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, & x(1) = \int_0^1 x(s)\,dA(s), \end{cases}$$

where $f(t,x)$ satisfies some decreasing conditions. The authors obtained the existence and uniqueness of the positive solutions of the above boundary value problem.

In [27], the authors studied the following nonlinear fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,v(t)) = 0, & 0 < t < 1, 2 < \alpha \leq 3, \\ D_{0+}^{\beta}v(t) + g(t,u(t)) = 0, & 0 < t < 1, 2 < \beta \leq 3, \\ u(0) = u(1) = u'(0) = v(0) = v(1) = v'(0) = 0, \end{cases}$$

where $f,g : (0,1) \times R \to R$ are continuous, $\lim_{t\to 0+} f(t,\cdot) = +\infty$, $\lim_{t\to 0+} g(t,\cdot) = +\infty$. They established the existence of solutions of the above boundary value problem by using the Krasnoselskii fixed point theorem.

Motivated by the results mentioned above, we study the following system of high-order nonlinear fractional differential equations with Riemann-Stieltjes integral boundary conditions:

$$\begin{cases} -D_{0+}^{\alpha}u(t) = \lambda f(t,v(t)), & 0 < t < 1, n-1 < \alpha \leq n, \\ -D_{0+}^{\beta}v(t) = \mu g(t,u(t)), & 0 < t < 1, m-1 < \beta \leq m, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & u(1) = \int_0^1 u(s)\,dH(s), \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, & v(1) = \int_0^1 v(s)\,dK(s), \end{cases} \tag{1}$$

where $n,m \in N$, $n,m \geq 3$, D_{0+}^{α}, D_{0+}^{β} are the standard Riemann-Liouville fractional derivative, λ and μ are two positive parameters. $f : (0,1) \times (0,\infty) \to [0,\infty)$ and $g : [0,1] \times [0,\infty) \to (0,\infty)$ are continuous functions and $f(t,v)$ may be singular at $t = 0,1$ and $v = 0$. $\int_0^1 u(s)\,dH(s)$ and $\int_0^1 v(s)\,dK(s)$ are Riemann-Stieltjes integrals. $H,K : [0,1] \to R$ are the function of bounded variation with $\int_0^1 s^{\alpha-1}\,dH(s) \neq 1$ and $\int_0^1 s^{\beta-1}\,dK(s) \neq 1$, dH and dK can be signed measures. Webb and Infante [28, 29] were first to use the idea of Riemann-Stieltjes integral with a signed measure.

Obviously, system (1) is more general than the problems discussed in some recent literature. Firstly, the system depends on two parameters; secondly, the nonlinear terms f and g are allowed to have different nonlinear character; finally, the boundary conditions involve the Riemann-Stieltjes integral. This case covers the multi-point boundary conditions and integral boundary conditions as special cases.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and facts. In Section 3, the main results are discussed. Finally, in Section 4, an illustrative example is also presented.

2 Preliminaries

For convenience, we use the following notations in this paper:

$$h_\alpha = \int_0^1 t^{\alpha-1}\,dH(t), \qquad k_\beta = \int_0^1 t^{\beta-1}\,dK(t).$$

Now we begin this section with some preliminaries of fractional calculus.

Definition 1 ([30]) Let $\alpha > 0$ with $\alpha \in R$. Suppose that $x : [0, \infty) \to R$. Then the Riemann-Liouville fractional integral is defined to be

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\,ds,$$

whenever the right side is defined. Similarly, $\alpha > 0$ with $\alpha \in R$, we define the αth Riemann-Liouville fractional derivative to be

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t (t-s)^{n-\alpha-1} x(s)\,ds,$$

whenever the right side is defined, where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α and $t > 0$.

Lemma 1 ([23]) *Given $y \in L^1[0,1]$, if $h_\alpha \neq 1$, then the problem*

$$\begin{cases} -D_{0+}^{\alpha} u(t) = y(t), & 0 < t < 1, n-1 < \alpha \leq n, n \geq 3, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & u(1) = \int_0^1 u(s)\,dH(s), \end{cases}$$

has a unique solution,

$$u(t) = \int_0^1 G_\alpha(t,s) y(s)\,ds,$$

where

$$G_\alpha(t,s) = g_\alpha(t,s) + \frac{t^{\alpha-1}}{1-h_\alpha} H_\alpha(s),$$

$$g_\alpha(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$H_\alpha(s) = \int_0^1 g_\alpha(t,s)\,dH(t).$$

By Lemma 1, similar results are valid for the problem

$$\begin{cases} -D_{0+}^{\beta} v(t) = y(t), & 0 < t < 1, m-1 < \beta \leq m, m \geq 3, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, & v(1) = \int_0^1 v(s)\,dK(s). \end{cases}$$

If $h_\beta \neq 1$, we adopt the following corresponding notations:

$$G_\beta(t,s) = g_\beta(t,s) + \frac{t^{\beta-1}}{1-k_\beta} K_\beta(s),$$

$$g_\beta(t,s) = \begin{cases} \frac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}, & 0 \leq t \leq s \leq 1, \\ \frac{[t(1-s)]^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$K_\beta(s) = \int_0^1 g_\beta(t,s)\,dK(t).$$

Lemma 2 ([26]) *Let $h_\alpha, k_\beta \in [0,1)$ and $H_\alpha(s), K_\beta(s) > 0$ for $s \in [0,1]$. Then the Green function $G_\alpha(t,s)$ and $G_\beta(t,s)$ satisfy the following properties:*

(1) $G_\alpha(t,s) > 0$ *and* $G_\beta(t,s) > 0$ *for all* $t,s \in (0,1)$;

(2) *there exist functions* $m_\alpha(s)$, $m_\beta(s)$, $M_\alpha(s)$, *and* $M_\beta(s)$ *such that*

$$m_\alpha(s)t^{\alpha-1} \le G_\alpha(t,s) \le M_\alpha(s)t^{\alpha-1}, \quad \text{for } t,s \in [0,1], \tag{2}$$

$$m_\beta(s)t^{\beta-1} \le G_\beta(t,s) \le M_\beta(s)t^{\beta-1}, \quad \text{for } t,s \in [0,1], \tag{3}$$

where

$$m_\alpha(s) = \frac{H_\alpha(s)}{1-h_\alpha}, \qquad M_\alpha(s) = \frac{\|H_\alpha(s)\|}{1-h_\alpha} + \frac{1}{\Gamma(\alpha-1)},$$
$$m_\beta(s) = \frac{K_\beta(s)}{1-k_\beta}, \qquad M_\beta(s) = \frac{\|K_\beta(s)\|}{1-k_\beta} + \frac{1}{\Gamma(\beta-1)}.$$

Lemma 3 *Assume $n-1 < \alpha \le n$, $u \in C([0,1],R)$ satisfies $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, $u(1) = \int_0^1 u(s)\,dH(s)$, and $-D_{0+}^\alpha u(t) \ge 0$ for any $t \in (0,1)$. Then $u(t) \ge 0$, $t \in (0,1)$.*

Proof From Lemma 1, it is easy to see that Lemma 3 holds. □

It is easy to see that $(u,v) \in C[0,1] \times C[0,1]$ is a pair of solution to the system (1) if and only if (u,v) is a pair of solution of the following nonlinear integral system:

$$\begin{cases} u(t) = \lambda \int_0^1 G_\alpha(t,s)f(s,v(s))\,ds, \\ v(t) = \mu \int_0^1 G_\beta(t,s)g(s,u(s))\,ds. \end{cases} \tag{4}$$

Obviously, we can convert the system (4) to the following equivalent integral equation:

$$u(t) = \lambda \int_0^1 G_\alpha(t,s)f\left(s, \mu \int_0^1 G_\beta(s,\tau)g\bigl(\tau,u(\tau)\bigr)\,d\tau\right)ds, \quad t \in [0,1].$$

We consider operator T defined by

$$(Tu)(t) = \lambda \int_0^1 G_\alpha(t,s)f\left(s, \mu \int_0^1 G_\beta(s,\tau)g\bigl(\tau,u(\tau)\bigr)\,d\tau\right)ds, \quad t \in [0,1]. \tag{5}$$

It is simple to show that if $u^*(t)$ is a fixed point of T in $C[0,1]$, then the system (1) has a pair of solutions $(u(t),v(t))$ expressed as

$$\begin{cases} u(t) = u^*(t), \\ v(t) = \mu \int_0^1 G_\beta(t,s)g(s,u^*(s))\,ds. \end{cases}$$

In the following, we consider the following boundary value problem:

$$\begin{cases} -D_{0+}^\alpha u(t) = \lambda f(t, \mu \int_0^1 G_\beta(t,s)g(s,u(s)))\,ds, \quad 0 < t < 1, n-1 < \alpha \le n, n \ge 3, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \int_0^1 u(s)\,dH(s). \end{cases} \tag{6}$$

Firstly, we recall the definitions of the upper solution and the lower solution for the system (6).

Definition 2 A continuous function $\varphi(t)$ is said to be an upper solution of the problem (6) if it satisfies

$$\begin{cases} -D_{0+}^{\alpha}\varphi(t) \geq \lambda f(t, \mu \int_0^1 G_\beta(t,s) g(s, \varphi(s)))\, ds, \\ \varphi(0) \leq 0, \varphi'(0) \leq 0, \ldots, \varphi^{(n-2)}(0) \leq 0, \qquad \varphi(1) \leq \int_0^1 \varphi(s)\, dH(s). \end{cases}$$

Definition 3 A continuous function $\phi(t)$ is said to be a lower solution of the problem (6) if it satisfies

$$\begin{cases} -D_{0+}^{\alpha}\phi(t) \leq \lambda f(t, \mu \int_0^1 G_\beta(t,s) g(s, \phi(s)))\, ds, \\ \phi(0) \geq 0, \phi'(0) \geq 0, \ldots, \phi^{(n-2)}(0) \geq 0, \qquad \phi(1) \geq \int_0^1 \phi(s)\, dH(s). \end{cases}$$

3 Main results

In this section, we establish the existence of positive solutions results for the system (1).

Let $E = C[0,1]$. It is easy to see that E is a Banach space with the norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$ for any $x \in E$. Let $P = \{x \in E \mid x(t) \geq 0, t \in [0,1]\}$. Clearly P is a normal cone in Banach space E. The normality constant is 1. The space E can be equipped with a partial order as follows: $x, y \in E, x \leq y \Longleftrightarrow x(t) \leq y(t)$ for $t \in [0,1]$. Define

$$\begin{aligned} P^* = \bigl\{x \in P :\ & \text{there exist two positive numbers } L_x > 1 > l_x, \\ & \text{such that } l_x t^{\alpha-1} \leq x(t) \leq L_x t^{\alpha-1}, t \in [0,1]\bigr\}. \end{aligned}$$

Obviously, P^* is nonempty since $t^{\alpha-1} \in P^*$.

Now, we give the existence of positive solutions to the system (1).

Theorem 1 *Assume that*:

- (H_1) *H and K are two functions of bounded variation such that $h_\alpha, k_\beta \in [0,1)$ and $H_\alpha(s), K_\beta(s) > 0$ for $t \in [0,1]$*;
- (H_2) *$f(t,v) : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and decreasing with respect to the second argument, and $g(t,u) : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and increasing with respect to the second argument*;
- (H_3) *for any real numbers l, $\mu > 0$,*

$$\int_0^1 M_\alpha(s) f\left(s, \mu \int_0^1 G_\beta(s,\tau) g\bigl(\tau, l\tau^{\alpha-1}\bigr)\, d\tau\right) ds < +\infty.$$

Then for any $(\lambda, \mu) \in (0,+\infty) \times (0,+\infty)$, the system (1) *has at least one pair of positive solution (u^*, v^*) and there exist positive constants r_α, R_α, r_β, and R_β such that*

$$\begin{aligned} & r_\alpha t^{\alpha-1} \leq u^*(t) \leq R_\alpha t^{\alpha-1}, \quad t \in [0,1], \\ & r_\beta t^{\beta-1} \leq v^*(t) \leq R_\beta t^{\beta-1}, \quad t \in [0,1]. \end{aligned}$$

Proof It is easy to see that the existence of solutions to the system of (1) is equivalent to the existence of fixed point of the nonlinear operator T. Therefore it suffices to prove the existence of fixed point of the operator T. To begin with, we assert that the operator T is well defined and $T(P^*) \subset P^*$. For any $u \in P^*$, there exist two positive numbers $l_u < 1 < L_u$ such that $l_u t^{\alpha-1} \leq u(t) \leq L_u t^{\alpha-1}$. From (2), (5), ($H_2$), and ($H_3$) we have

$$\begin{aligned}(Tu)(t) &= \lambda \int_0^1 G_\alpha(t,s) f\left(s, \mu \int_0^1 G_\beta(s,\tau) g\bigl(\tau, u(\tau)\bigr)\, d\tau\right) ds \\ &\leq \lambda t^{\alpha-1} \int_0^1 M_\alpha(s) f\left(s, \mu \int_0^1 G_\beta(s,\tau) g\bigl(\tau, l_u \tau^{\alpha-1}\bigr)\, d\tau\right) ds \\ &< +\infty, \\ (Tu)(t) &\geq \lambda t^{\alpha-1} \int_0^1 m_\alpha(s) f\left(s, \mu \int_0^1 G_\beta(s,\tau) g\bigl(\tau, L_u \tau^{\alpha-1}\bigr)\, d\tau\right) ds.\end{aligned}$$

Let

$$\begin{aligned}l'_u &= \min\left\{1, \lambda \int_0^1 m_\alpha(s) f\left(s, \mu \int_0^1 G_\beta(s,\tau) g\bigl(\tau, L_u \tau^{\alpha-1}\bigr)\, d\tau\right) ds\right\}, \\ L'_u &= \max\left\{1, \lambda \int_0^1 M_\alpha(s) f\left(s, \mu \int_0^1 G_\beta(s,\tau) g\bigl(\tau, l_u \tau^{\alpha-1}\bigr)\, d\tau\right) ds\right\}.\end{aligned}$$

Thus

$$l'_u t^{\alpha-1} \leq (Tu)(t) \leq L'_u t^{\alpha-1}.$$

Therefore $(Tu)(t)$ is well defined and $T(P^*) \subseteq P^*$. Taking (H_2) into consideration, we see that the operator $(Tu)(t)$ is decreasing on u. Moreover, using Lemma 1, we obtain

$$\begin{cases} -D_{0+}^\alpha (Tu)(t) = \lambda f(t, \mu \int_0^1 G_\beta(t,s) g(s, u(s))\, ds), \\ (Tu)(0) = 0, (Tu)'(0) = 0, \ldots, (Tu)^{(n-2)}(0) = 0, \\ (Tu)(1) = \int_0^1 (Tu)(s)\, dH(s). \end{cases} \tag{7}$$

Let

$$\varphi(t) = \min\bigl\{t^{\alpha-1}, T\bigl(t^{\alpha-1}\bigr)\bigr\}, \qquad \phi(t) = \max\bigl\{t^{\alpha-1}, T\bigl(t^{\alpha-1}\bigr)\bigr\}. \tag{8}$$

If $t^{\alpha-1} = T(t^{\alpha-1})$, then

$$u^*(t) = t^{\alpha-1}, \qquad v^*(t) = \mu \int_0^1 G_\beta(t,s) g\bigl(s, s^{\alpha-1}\bigr)\, ds \tag{9}$$

is a pair of positive solution of (1). If $t^{\alpha-1} \neq T(t^{\alpha-1})$, we find

$$\varphi(t), \phi(t) \in P^*(t), \qquad \varphi(t) \leq t^{\alpha-1} \leq \phi(t). \tag{10}$$

Take

$$\xi(t) = (T\varphi)(t), \qquad \psi(t) = (T\phi)(t), \tag{11}$$

together with (H_2), we know that T is non-increasing on u. Using (8) and (10) we can show that

$$\psi(t) = (T\phi)(t) \le (T\varphi)(t) = \xi(t), \tag{12}$$

$$(T\phi)(t) \le T\bigl(t^{\alpha-1}\bigr) \le \phi(t), \qquad (T\varphi)(t) \ge T\bigl(t^{\alpha-1}\bigr) \ge \varphi(t), \tag{13}$$

and $\xi(t), \psi(t) \in P^*$. According to (7)-(13), it follows that

$$\begin{aligned} &D_{0+}^{\alpha}\xi(t) + \lambda f\left(t, \mu\int_0^1 G_\beta(t,s)g\bigl(s,\xi(s)\bigr)\,ds\right) \\ &\quad = D_{0+}^{\alpha}(T\varphi)(t) + \lambda f\left(t, \mu\int_0^1 G_\beta(t,s)g\bigl(s,(T\varphi)(s)\bigr)\,ds\right) \\ &\quad \le D_{0+}^{\alpha}(T\varphi)(t) + \lambda f\left(t, \mu\int_0^1 G_\beta(t,s)g\bigl(s,\varphi(s)\bigr)\,ds\right) \\ &\quad = 0, \end{aligned} \tag{14}$$

$$\xi(0) = \xi'(0) = \cdots = \xi^{(n-2)}(0) = 0, \qquad \xi(1) = \int_0^1 \xi(s)\,dH(s), \tag{15}$$

$$\begin{aligned} &D_{0+}^{\alpha}\psi(t) + \lambda f\left(t, \mu\int_0^1 G_\beta(t,s)g\bigl(s,\psi(s)\bigr)\,ds\right) \\ &\quad = D_{0+}^{\alpha}(T\phi)(t) + \lambda f\left(t, \mu\int_0^1 G_\beta(t,s)g\bigl(s,(T\phi)(s)\bigr)\,ds\right) \\ &\quad \ge D_{0+}^{\alpha}(T\phi)(t) + \lambda f\left(t, \mu\int_0^1 G_\beta(t,s)g\bigl(s,\phi(s)\bigr)\,ds\right) \\ &\quad = 0, \end{aligned} \tag{16}$$

$$\psi(0) = \psi'(0) = \cdots = \psi^{(n-2)}(0) = 0, \qquad \psi(1) = \int_0^1 \psi(s)\,dH(s). \tag{17}$$

From (14)-(17) we see that $\xi(t), \psi(t) \in P^*$ are an upper solution and a lower solution of the problem (6), respectively. Define a function $F : (0,1) \times E \to E$:

$$F\bigl(t,u(t)\bigr) = \begin{cases} f(t, \mu\int_0^1 G_\beta(t,s)g(s,\psi(s))\,ds), & u(t) < \psi(t), \\ f(t, \mu\int_0^1 G_\beta(t,s)g(s,u(s))\,ds), & \psi(t) \le u(t) \le \xi(t), \\ f(t, \mu\int_0^1 G_\beta(t,s)g(s,\xi(s))\,ds), & u(t) > \xi(t). \end{cases}$$

Clearly, $F \in C((0,1) \times E, E)$. Now, we define an operator $\overline{T}$ in E by

$$(\overline{T}u)(t) = \lambda\int_0^1 G_\alpha(t,s)F\bigl(s,u(s)\bigr)\,ds.$$

Consider the following boundary value problem:

$$\begin{cases} -D_{0+}^{\alpha}u(t) = \lambda F(t,u(t)), \\ u(0) = 0, u'(0) = 0, \ldots, u^{(n-2)}(0) = 0, \qquad u(1) = \int_0^1 u(s)\,dH(s). \end{cases} \tag{18}$$

Applying Lemma 1, the existence of solutions to the boundary value problem (18) is equivalent to the existence of a fixed point of the nonlinear operator $\overline{T}$. Thanks to (2) and (H_2), we have

$$\begin{aligned}
(\overline{T}u)(t) &\leq \lambda t^{\alpha-1} \int_0^1 M_\alpha(s) F\bigl(s, u(s)\bigr)\, ds \\
&\leq \lambda t^{\alpha-1} \int_0^1 M_\alpha(s) f\left(s, \mu \int_0^1 G_\beta(s,\tau) g\bigl(\tau, \psi(\tau)\bigr)\, d\tau\right) ds \\
&\leq \lambda t^{\alpha-1} \int_0^1 M_\alpha(s) f\left(s, \mu \int_0^1 G_\beta(s,\tau) g\bigl(\tau, l_\phi \tau^{\alpha-1}\bigr)\, d\tau\right) ds \\
&< +\infty,
\end{aligned}$$

that is, $\overline{T}$ is bounded. It is easy to see that $\overline{T}: E \to E$ is continuous from the continuity of $G_\alpha(t,s)$. Let $\Omega \subset E$ be bounded, together with the uniform continuity of $G_\alpha(t,s)$ and the Lebesgue dominated convergence theorem, we see that $\overline{T}(\Omega)$ is equicontinuous. From the Arzela-Ascoli theorem, we see that $\overline{T}: E \to E$ is completely continuous. An application of Schauder's fixed point theorem shows that $\overline{T}$ has at least one fixed point $u^*(t)$ such that $u^*(t) = (Tu^*)(t)$. Our task now is to prove

$$\psi(t) \leq u^*(t) \leq \xi(t), \quad \forall t \in [0,1]. \tag{19}$$

Since $u^*(t)$ is a fixed point of $\overline{T}$, we obtain

$$u^*(0) = 0, \qquad \bigl(u^*\bigr)'(0) = 0, \qquad \ldots, \qquad \bigl(u^*\bigr)^{(n-2)}(0) = 0, \qquad u^*(1) = \int_0^1 u^*(s)\, dH(s).$$

Firstly, we verify $u^*(t) \leq \xi(t)$. Otherwise, there exists some t_0 such that $u^*(t_0) > \xi(t_0)$; together with the definition of F, we get

$$-D_{0+}^\alpha u^*(t_0) = \lambda F\bigl(t_0, u^*(t_0)\bigr) = \lambda f\left(t_0, \mu \int_0^1 G_\beta(t_0,s) g\bigl(s, \xi(s)\bigr)\, ds\right). \tag{20}$$

On the other hand, since ξ is an upper solution of (6), we find

$$-D_{0+}^\alpha \xi(t_0) \geq \lambda f\left(t_0, \mu \int_0^1 G_\beta(t_0,s) g\bigl(s, \xi(s)\bigr)\, ds\right). \tag{21}$$

Taking $x(t_0) = \xi(t_0) - u^*(t_0)$, it follows from (20) and (21) that

$$\begin{aligned}
&-D_{0+}^\alpha x(t_0) = D_{0+}^\alpha u^*(t_0) - D_{0+}^\alpha \xi(t_0) \geq 0, \\
&x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \qquad x(1) = \int_0^1 x(s)\, dH(s).
\end{aligned}$$

According to Lemma 3, we have $x(t_0) \geq 0$, this means $u^*(t_0) \leq \xi(t_0)$, which contradicts $u^*(t_0) > \varphi(t_0)$. Thus we conclude that $\xi(t) \geq u^*(t)$ for any $t \in (0,1)$. In the same way, we obtain $u^*(t) \geq \psi(t)$. Consequently, we infer that (19) holds, and then $u^*(t)$ is a positive solution of the problem (6). As $\xi(t)$, $\psi(t) \in P^*$, there exist four numbers $0 < l_\xi, l_\psi < 1$, $L_\xi, L_\psi > 1$, such that

$$l_\xi t^{\alpha-1} \le \xi(t) \le L_\xi t^{\alpha-1}, \qquad l_\psi t^{\alpha-1} \le \psi(t) \le L_\psi t^{\alpha-1}. \tag{22}$$

It follows from (19) and (22) that

$$r_\alpha t^{\alpha-1} = l_\psi t^{\alpha-1} \le u^*(t) \le L_\xi t^{\alpha-1} = R_\alpha t^{\alpha-1}.$$

Taking

$$\begin{aligned} r_\beta &= \min\left\{\frac{1}{3}, \mu \int_0^1 m_\beta(s) g\bigl(s, r_\alpha s^{\alpha-1}\bigr)\,ds\right\}, \\ R_\beta &= \max\left\{3, \mu \int_0^1 M_\beta(s) g\bigl(s, R_\alpha s^{\alpha-1}\bigr)\,ds\right\}. \end{aligned}$$

From (3) we have

$$\begin{aligned} r_\beta t^{\beta-1} &\le \mu t^{\beta-1} \int_0^1 m_\beta(s) g\bigl(s, r_\alpha s^{\alpha-1}\bigr)\,ds \\ &\le y^*(t) \\ &\le \mu t^{\beta-1} \int_0^1 M_\beta(s) g\bigl(s, R_\alpha s^{\alpha-1}\bigr)\,ds \\ &\le R_\beta t^{\beta-1}. \end{aligned}$$

The proof of Theorem 1 is now complete. □

Remark 1 In Theorem 1, we cannot only give the result of the existence of positive solutions, but also can take r_α, R_α, r_β, and R_β such that

$$\begin{aligned} &r_\alpha t^{\alpha-1} \le u^*(t) \le R_\alpha t^{\alpha-1}, \quad t \in [0,1], \\ &r_\beta t^{\beta-1} \le v^*(t) \le R_\beta t^{\beta-1}, \quad t \in [0,1]. \end{aligned}$$

So the properties of the positive solution are clearer.

4 Example

To illustrate how our main results can be used in practice we present an example. Consider the following system:

$$\begin{cases} -D_{0+}^{\frac{7}{3}} u(t) = \lambda[\sin t + v^{-\frac{1}{3}}(t)], \quad 0 < t < 1, n = 3, \\ -D_{0+}^{\frac{7}{2}} v(t) = \mu[t^{-\frac{1}{2}} + u^2(t)], \quad 0 < t < 1, m = 4, \\ u(0) = u'(0) = 0, \qquad u(1) = \int_0^1 u(s)\,dH(s), \\ v(0) = v'(0) = v''(0) = 0, \qquad v(1) = \int_0^1 v(s)\,dK(s), \end{cases} \tag{23}$$

where $\lambda, \mu > 0$ are two parameters and $H(t) = t^2$, $K(t) = t^3$ for all $t \in [0,1]$. In this case, $\alpha = \frac{7}{3}$, $\beta = \frac{7}{2}$. Problem (23) can be regarded as a system of form (1) with

$$f(t,v) = \sin t + v^{-\frac{1}{3}}(t) \tag{24}$$

and

$$g(t,u) = t^{-\frac{1}{2}} + u^2(t). \tag{25}$$

Now we verify that conditions (H_1)-(H_3) are satisfied. By a simple computation, we have

$$\int_0^1 t^{\frac{4}{3}}\,dH(t) = 2\int_0^1 t^{\frac{7}{3}}\,dt = \frac{3}{5}, \tag{26}$$

$$\int_0^1 t^{\frac{5}{2}}\,dK(t) = 3\int_0^1 t^{\frac{9}{2}}\,dt = \frac{6}{11},$$

$$\begin{aligned}\int_0^1 g_{\frac{7}{3}}(t,s)\,dH(t) &= \frac{2}{\Gamma(\frac{7}{3})}\left\{\int_0^1 t\big[t(1-s)\big]^{\frac{4}{3}}\,dt - \int_s^1 t(t-s)^{\frac{4}{3}}\,dt\right\} \\ &= \frac{2}{\Gamma(\frac{7}{3})}\left[\frac{3}{10}(1-s)^{\frac{4}{3}} - \frac{3}{7}(1-s)^{\frac{7}{3}} + \frac{3}{10}(1-s)^{\frac{10}{3}}\right], \quad s \in [0,1],\end{aligned} \tag{27}$$

$$\begin{aligned}\int_0^1 g_{\frac{7}{2}}(t,s)\,dK(t) &= \frac{3}{\Gamma(\frac{7}{2})}\left\{\int_0^1 t^2\big[t(1-s)\big]^{\frac{5}{2}}\,dt - \int_s^1 t^2(t-s)^{\frac{5}{2}}\,dt\right\} \\ &= \frac{3}{\Gamma(\frac{7}{2})}\left[\frac{2}{11}(1-s)^{\frac{5}{2}} - \frac{2}{7}(1-s)^{\frac{7}{2}} + \frac{8}{56}(1-s)^{\frac{9}{2}} - \frac{16}{692}(1-s)^{\frac{11}{2}}\right], \\ &\quad s \in [0,1].\end{aligned}$$

This means (H_1) holds. Thanks to (24) and (25), we conclude that (H_2) is proved. It follows from Lemma 2 that there exists a positive number $m_0 < +\infty$ such that

$$G_\beta(t,s) \geq m_\beta(s)t^{\frac{5}{2}} \geq m_0 t^{\frac{5}{2}}. \tag{28}$$

Moreover, for any constants $\mu, l > 0$, we obtain

$$\begin{aligned} & f\left(t, \mu\int_0^1 G_\beta(t,s)g\big(s, ls^{\alpha-1}\big)\,ds\right) \\ &\quad = \sin t + \left(\mu\int_0^1 G_\beta(t,s)g\big(s, ls^{\alpha-1}\big)\,ds\right)^{-\frac{1}{3}} \\ &\quad = \sin t + \left(\mu\int_0^1 G_{\frac{7}{2}}(t,s)\big(s^{-\frac{1}{2}} + l^2 s^{\frac{8}{3}}\big)\,ds\right)^{-\frac{1}{3}} \\ &\quad \leq \sin t + (\mu m_0)^{-\frac{1}{3}}\left(\int_0^1 \big[s^{-\frac{1}{2}} + l^2 s^{\frac{8}{3}}\big]\,ds\right)^{-\frac{1}{3}} \\ &\quad \leq \sin 1 + (\mu m_0)^{-\frac{1}{3}}\left(2 + \frac{3}{11}l^2\right)^{-\frac{1}{3}} \\ &\quad < \sin 1 + (3\mu m_0)^{-\frac{1}{3}}. \end{aligned} \tag{29}$$

According to (26), (27), and (29), we obtain

$$\begin{aligned}
M_\alpha(s) &= \frac{1}{\Gamma(\frac{4}{3})} + \frac{\int_0^1 g_{\frac{7}{3}}(t,s)\,dH(t)}{1-\int_0^1 t^{\frac{4}{3}}\,dH(t)} \\
&= \frac{1}{\Gamma(\frac{4}{3})} + \frac{5(1-s)^{\frac{4}{3}}[\frac{36}{245}+\frac{3}{10}(s-\frac{2}{7})^2]}{\Gamma(\frac{7}{3})} \\
&< \frac{1}{\Gamma(\frac{4}{3})} + \frac{2}{\Gamma(\frac{7}{3})}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 M_\alpha(s) f\left(s, \mu\int_0^1 G_\beta(s,\tau) g\left(\tau, l\tau^{\alpha-1}\right)d\tau\right)ds \\
&\quad< \int_0^1 \left(\sin 1 + (3\mu m_0)^{-\frac{1}{3}}\right)\left(\frac{1}{\Gamma(\frac{4}{3})} + \frac{2}{\Gamma(\frac{7}{3})}\right)ds \\
&\quad< +\infty,
\end{aligned}$$

which implies that (H_3) is satisfied. Thus from Theorem 1, for any $(\lambda,\mu) \in (0,+\infty) \times (0,+\infty)$, we can show that the system (23) has at least one pair of positive solutions (u^*, v^*) and there exist four positive constants $r_{\frac{7}{3}}$, $r_{\frac{7}{2}}$, $R_{\frac{7}{3}}$, and $R_{\frac{7}{3}}$ such that

$$\begin{aligned}
&r_{\frac{7}{3}} t^{\frac{4}{3}} \le u^*(t) \le R_{\frac{7}{3}} t^{\frac{4}{3}}, \\
&r_{\frac{7}{2}} t^{\frac{5}{2}} \le v^*(t) \le R_{\frac{7}{2}} t^{\frac{5}{2}}.
\end{aligned}$$

Competing interests
The author declares that he has no competing interests.

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