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Fractional optimal control problem for infinite order system with control constraints

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Abstract

In this article, we study a homogeneous infinite order Dirichlet and Neumann boundary fractional equations in a bounded domain. The fractional time derivative is considered in a Riemann-Liouville sense. Constraints on controls are imposed. The existence results for equations are obtained by applying the classical Lax-Milgram Theorem. The performance functional is in quadratic form. Then we show that the optimal control problem associated to the controlled fractional equation has a unique solution. Interpreting the Euler-Lagrange first order optimality condition with an adjoint problem defined by means of the right fractional Caputo derivative, we obtain an optimality system. The obtained results are well illustrated by examples.

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Keywords: fractional optimal control problems; parabolic systems; Dirichlet and Neumann conditions; existence and uniqueness of solutions; infinite order operators; Riemann-Liouville sense; Caputo derivative

1 Introduction

The fractional calculus is nowadays an excellent mathematical tool which opens the gates for finding hidden aspects of the dynamics of the complex processes which appear naturally in many branches of science and engineering. The methods and techniques of this type of calculus are continuously generalized and improved especially during the last few decades. The recent trend in the mathematical modeling of several phenomena indicates the popularity of fractional calculus modeling tools due to the nonlocal characteristic of fractional-order differential and integral operators, which are capable of tracing the past history of many materials and processes; see, for instance, [1–13] and the references therein.

The study of fractional optimal control involving first and second order operators has recently attracted the attention of many researchers and modelers see, for instance, [2–5, 9, 11, 12] and the references therein.

In this paper we try to extend the previous results. We consider here a different type of evolution equations, namely, fractional partial differential equations involving infinite order operators see Bahaa and Kotarski [14] and papers therein. Such an infinite order system can be treated as a generalization of the mathematical model for a plasma control process. The existence and uniqueness of solutions for such equations are proved.

Fractional optimal control is characterized by the adjoint problem. By using this characterization, particular properties of fractional optimal control are proved.

This paper is organized as follows. In Section 2, we introduce Sobolev spaces with infinite order and we introduce some definitions and preliminary results. In Section 3, we formulate the fractional Dirichlet problem for infinite order equations. In Section 4, we show that our fractional optimal control problem holds and gives the optimality system for the optimal control. In Section 5, we formulate the fractional Neumann problem. In Section 6, we formulate the minimization problem and we state some illustrated examples. In Section 7 we state our conclusion of the paper.

2 Sobolev spaces with infinite order and fractional derivatives

The object of this section is to give the definition of some function spaces of infinite order, and the chains of the constructed spaces which will be used later; see Dubinskii [15, 16].

Let Ω be a bounded open set of $\mathbb{R}^n$ with a smooth boundary Γ. We define the infinite order Sobolev space $H^\infty\{a_\alpha, 2\}(\Omega)$ of functions $\phi(x)$ defined on Ω as follows:

$$H^\infty\{a_\alpha, 2\}(\Omega) = \left\{ \phi(x) \in C^\infty(\Omega) : \sum_{|\alpha|=0}^{\infty} a_\alpha \left\| D^\alpha \phi \right\|_2^2 < \infty \right\},$$

where $C^\infty(\Omega)$ is the space of infinitely differentiable functions, $a_\alpha \geq 0$ is a numerical sequence and $\|\cdot\|_2$ is the canonical norm in the space $L^2(\Omega)$, and

$$D^\alpha \phi(x) := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \cdot \phi(x_1, \dots, x_n),$$
$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}},$$

$\alpha = (\alpha_1, \dots, \alpha_n)$ being a multi-index for differentiation, $|\alpha| = \sum_{i=1}^n \alpha_i$.

The space $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ is defined as the formal conjugate space to the space $H^\infty\{a_\alpha, 2\}(\Omega)$, namely

$$H^{-\infty}\{a_\alpha, 2\}(\Omega) = \left\{ \psi(x) \in L^2(\Omega) : \psi(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha \psi_\alpha(x) \right\},$$

where $\psi_\alpha \in L^2(\Omega)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha \|\psi_\alpha\|_2^2 < \infty$.

The duality pairing of the spaces $H^\infty\{a_\alpha, 2\}(\Omega)$ and $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ is postulated by the formula

$$(\phi, \psi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega \psi_\alpha(x) D^\alpha \phi(x)\, dx,$$

where

$$\phi \in H^\infty\{a_\alpha, 2\}(\Omega), \qquad \psi \in H^{-\infty}\{a_\alpha, 2\}(\Omega).$$

From Bahaa and Kotarski [14], $H^\infty\{a_\alpha, 2\}(\Omega)$ is everywhere dense in $L^2(\Omega)$ with topological inclusions and $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ denotes the topological dual space with respect to

$L^2(\Omega)$, so we have the following chain of inclusions:

$$H^\infty\{a_\alpha, 2\}(\Omega) \subseteq L^2(\Omega) \subseteq H^{-\infty}\{a_\alpha, 2\}(\Omega).$$

The space $H_0^\infty\{a_\alpha, 2\}(\Omega)$ is proper subspace of $H^\infty\{a_\alpha, 2\}(\Omega)$ consisting of all the functions $\phi \in H^\infty\{a_\alpha, 2\}(\Omega)$ such that $\phi|_\Gamma = 0$.

Clearly, $H_0^\infty\{a_\alpha, 2\}(\Omega)$ is dense in $L^2(\Omega)$. Constructing the corresponding negative space $H_0^{-\infty}\{a_\alpha, 2\}(\Omega)$.

Then we have the following chains:

$$\begin{aligned} &H_0^\infty\{a_\alpha, 2\}(\Omega) \subseteq H^\infty\{a_\alpha, 2\}(\Omega) \subseteq L^2(\Omega), \\ &H_0^\infty\{a_\alpha, 2\}(\Omega) \subseteq L^2(\Omega) \subseteq H_0^{-\infty}\{a_\alpha, 2\}(\Omega). \end{aligned} \tag{2.1}$$

We shall use the following notation:

$$\begin{aligned} &Q = Q_T = \Omega \times]0, T[, \quad \Omega \text{ an open subset of } \mathbb{R}^n, \\ &\Sigma = \Sigma_T = \Gamma \times]0, T[, \\ &\Gamma = \text{boundary of } \Omega, \qquad \Sigma = \text{lateral boundary of } Q. \end{aligned}$$

We now introduce $L^2(0, T; L^2(\Omega))$ which we shall denoted by $L^2(Q)$, denotes the space of measurable functions $t \to \phi(t)$ such that

$$\|\phi\|_{L^2(Q)} = \left(\int_0^T \|\phi(t)\|_2^2 \, dt\right)^{\frac{1}{2}} < \infty,$$

endowed with the scalar product $(f, g) = \int_0^T (f(t), g(t))_{L^2(\Omega)} \, dt$, $L^2(Q)$ is a Hilbert space.

In the same manner we define the spaces $L^2(0, T; H^\infty\{a_\alpha, 2\}(\Omega))$, and $L^2(0, T; H^{-\infty}\{a_\alpha, 2\}(\Omega))$, as their formal conjugates.

Also, we have the following chain of inclusions:

$$\begin{aligned} &L^2\big(0, T; H^\infty\{a_\alpha, 2\}(\Omega)\big) \subseteq L^2(Q) \subseteq L^2\big(0, T; H^{-\infty}\{a_\alpha, 2\}(\Omega)\big), \\ &L^2\big(0, T; H_0^\infty\{a_\alpha, 2\}(\Omega)\big) \subseteq L^2(Q) \subseteq L^2\big(0, T; H_0^{-\infty}\{a_\alpha, 2\}(\Omega)\big). \end{aligned}$$

The following definitions and lemmas can be found in Agrawal [1, 2] and Mophou [11, 12].

Definition 2.1 Let $f : R_+ \to R$ be a continuous function on R^+ and $\beta > 0$. Then the expression:

$$I_+^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} f(s) \, ds, \quad t > 0,$$

is called the Riemann-Liouville integral of order β.

Definition 2.2 Let $f : R_+ \to R$. The left Riemann-Liouville fractional derivative of order β of f is defined by

$$D_+^\beta f(t) = \frac{1}{\Gamma(n - \beta)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n - \beta - 1} f(s) \, ds, \quad t > 0,$$

where $\beta \in (n-1,n)$, $n \in N$. When β is an integer the left derivative are replaced with D, the ordinary differential operator.

Definition 2.3 Let $f : R_+ \to R$. The left Caputo fractional derivative of order β of f is defined by

$$D_0^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} f^{(n)}(s)\,ds, \quad t > 0,$$

where $\beta \in (n-1,n)$, $n \in N$.

The Caputo fractional derivative is a sort of regularization in the time origin for the Riemann-Liouville fractional derivative.

Lemma 2.4 *Let $T > 0$, $u \in C^m([0,T])$, $p \in (m-1,m)$, $m \in N$ and $v \in C^1([0,T])$. Then for $t \in [0,T]$ the following properties hold*:

$$\begin{aligned}
&D_+^p v(t) = \frac{d}{dt} I_+^{1-p} v(t), \quad m = 1,\\
&D_+^p I_+^p v(t) = v(t);\\
&I_+^p D_0^p u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0);\\
&\lim_{t\to 0^+} D_0^p u(t) = \lim_{t\to 0^+} I_+^p u(t) = 0.
\end{aligned}$$

From now on we set

$$D^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_t^T (s-t)^{-\beta} f'(s)\,ds.$$

Remark 2.5 $-D^\beta f(t)$ is the so-called right fractional Caputo derivative. It represents the future state of $f(t)$. For more details on the derivative we refer to Agrawal [1, 2] and Mophou [11, 12]. Note also that when $T = +\infty$, $D^\beta f(t)$ is the Weyl fractional integral of order β of f'.

Lemma 2.6 [11] *Let $0 < \beta < 1$. Then for any $\phi \in C^\infty(\overline{Q})$ we have*

$$\begin{aligned}
&\int_0^T \int_\Omega \bigl(D_+^\beta y(x,t) + \mathcal{A} y(x,t)\bigr)\phi(x,t)\,dx\,dt\\
&\quad = \int_\Omega \phi(x,T) I_+^{1-\beta} y(x,T)\,dx - \int_\Omega \phi(x,0) I_+^{1-\beta} y\bigl(x,0^+\bigr)\,dx\\
&\qquad + \int_0^T \int_{\partial\Omega} y \frac{\partial\phi}{\partial\nu}\,d\Gamma\,dt - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial\nu}\phi\,d\Gamma\,dt\\
&\qquad + \int_0^T \int_\Omega y(x,t)\bigl(-D^\beta \phi(x,t) + \mathcal{A}^* \phi(x,t)\bigr)\,dx\,dt,
\end{aligned}$$

where $\mathcal{A}$ is a given operator which is defined by (3.4) *below and*

$$\frac{\partial y}{\partial \nu_{\mathcal{A}}} = \sum_{|\alpha|=0}^{\infty} \bigl(D^{\alpha} y\bigr) \cos(n,x) \quad \text{on } \Gamma,$$

$\cos(n, x_k)$ *is the kth direction cosine of n, n being the normal at* Γ.

We also introduce the space

$$\mathcal{W}(0,T) := \bigl\{y; y \in L^2\bigl(0,T; H_0^{\infty}\{a_\alpha,2\}(\Omega)\bigr), D_+^{\beta} y(t) \in L^2\bigl(0,T; H_0^{-\infty}\{a_\alpha,2\}(\Omega)\bigr)\bigr\}$$

in which a solution of a parabolic equation with an infinite order is contained. The spaces considered in this paper are assumed to be real.

3 Fractional Dirichlet problem for infinite order system

The object of this section is to formulate the following mixed initial-boundary value fractional Dirichlet evolution problem for infinite order system which defines the state of the system model:

$$D_+^{\beta} y(t) + \mathcal{A} y(t) = f(t), \quad t \in [0,T], \tag{3.1}$$

$$I_+^{1-\beta} y\bigl(0^+\bigr) = y_0, \quad x \in \Omega, \tag{3.2}$$

$$y(x,t) = 0, \quad x \in \Gamma, t \in (0,T), \tag{3.3}$$

where $0 < \beta < 1$, $y_0 \in H_0^{\infty}\{a_\alpha,2\}(\Omega)$, the function f belongs to $L^2(Q)$. The fractional integral $I_+^{1-\beta}$ and the derivative D_+^{β} are understood here in the Riemann-Liouville sense, Ω has the same properties as in Section 2 and $I_+^{1-\beta} y(0^+) = \lim_{t \to 0^+} I_+^{1-\beta} y(t)$. The operator $\mathcal{A}$ in the state equation (3.1) is an infinite order parabolic operator; see Dubinskii [15, 16], Bahaa and Kotarski [14], $\mathcal{A}$ is given by

$$\mathcal{A} y = \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1\right) y, \tag{3.4}$$

and $\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha}$ is an infinite order self-adjoint elliptic partial differential operator.

The operator

$$\mathcal{A} \in \mathcal{L}\bigl(H_0^{\infty}\{a_\alpha,2\}(\Omega), H_0^{-\infty}\{a_\alpha,2\}(\Omega)\bigr).$$

For this operator we define the bilinear form as follows.

Definition 3.1 For each $t \in]0,t[$, we define a family of bilinear forms on $H_0^{\infty}\{a_\alpha,2\}(\Omega)$ by

$$\pi(t; y, \phi) = (\mathcal{A} y, \phi)_{L^2(\Omega)}, \quad y, \phi \in H_0^{\infty}\{a_\alpha,2\}(\Omega), \tag{3.5}$$

where $\mathcal{A}$ maps $H_0^\infty\{a_\alpha,2\}(\Omega)$ onto $H_0^{-\infty}\{a_\alpha,2\}(\Omega)$ and takes the form (3.4). Then

$$\begin{aligned}\pi(t;y,\phi) &= (\mathcal{A}y,\phi)_{L^2(\Omega)}\\ &= \left(\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2\alpha}+1\right)y,\phi(x)\right)_{L^2(\Omega)}\\ &= \int_\Omega\sum_{|\alpha|=0}^{\infty}a_\alpha D^\alpha y(x)D^\alpha\phi(x)\,dx+\int_\Omega y(x)\phi(x)\,dx.\end{aligned}$$

Lemma 3.2 *The bilinear form $\pi(t;y,\phi)$ is coercive on $H_0^\infty\{a_\alpha,2\}(\Omega)$, that is,*

$$\pi(t;y,y)\geq\lambda\|y\|^2_{H_0^\infty\{a_\alpha,2\}(\Omega)},\quad \lambda>0. \tag{3.6}$$

Proof It is well known that the ellipticity of $\mathcal{A}$ is sufficient for the coerciveness of $\pi(t;y,\phi)$ on $H_0^\infty\{a_\alpha,2\}(\Omega)$.

Since

$$\pi(t;\phi,\psi)=\int_\Omega\sum_{|\alpha|=0}^{\infty}a_\alpha D^\alpha\phi D^\alpha\psi\,dx+\int_\Omega y(x)\phi(x)\,dx,$$

we get

$$\begin{aligned}\pi(t;y,y) &= \int_\Omega\sum_{|\alpha|=0}^{\infty}a_\alpha D^\alpha yD^\alpha y\,dx+\int_\Omega y(x)y(x)\,dx\\ &= \sum_{|\alpha|=0}^{\infty}a_\alpha\left\|D^{2\alpha}y(x)\right\|^2_{L^2(\Omega)}+\left\|y(x)\right\|^2_{L^2(\Omega)}\\ &\geq\lambda\|y\|^2_{H_0^\infty\{a_\alpha,2\}(\Omega)},\quad\lambda>0.\end{aligned}$$ □

Also we assume that $\forall y,\phi\in H_0^\infty\{a_\alpha,2\}(\Omega)$ the function $t\to\pi(t;y,\phi)$ is continuously differentiable in $]0,T[$ and the bilinear form $\pi(t;y,\phi)$ is symmetric,

$$\pi(t;y,\phi)=\pi(t;\phi,y)\quad\forall y,\phi\in H_0^\infty\{a_\alpha,2\}(\Omega). \tag{3.7}$$

The Equations (3.1)-(3.3) constitute a fractional Dirichlet problem. First by using the Lax-Milgram lemma, we will prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (3.1)-(3.3).

Lemma 3.3 (see Agrawal [1, 2] and Mophou [11, 12]) (Green's formula) *Let y be the solution of system* (3.1)-(3.3). *Then for any $\phi\in C^\infty(\overline{Q})$ such that $\phi(x,T)=0$ in Ω and $\phi=0$ on Σ, we have*

$$\begin{aligned}&\int_0^T\int_\Omega\big(D_+^\beta y(x,t)+\mathcal{A}y(x,t)\big)\phi(x,t)\,dx\,dt\\ &\quad=-\int_\Omega\phi(x,0)I_+^{1-\beta}y\big(x,0^+\big)\,dx+\int_0^T\int_{\partial\Omega}y\frac{\partial\phi}{\partial\nu}\,d\Gamma\,dt\\ &\qquad-\int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial\nu}\phi\,d\Gamma\,dt+\int_0^T\int_\Omega y(x,t)\big(-D^\beta\phi(x,t)+\mathcal{A}^*\phi(x,t)\big)\,dx\,dt.\end{aligned}$$

Lemma 3.4 *If* (3.6), (3.7) *holds, then the problem* (3.1)-(3.3) *admits a unique solution* $y \in \mathcal{W}(0,T)$.

Proof See Lions, [17], Chapter 2, Theorem 1.2, pp.102-103. From the coerciveness condition (3.6), there exists a unique element $y(t) \in H^\infty\{a_\alpha, 2\}(\Omega)$ such that

$$\bigl(D_+^\beta y(t), \phi\bigr)_{L^2(Q)} + \pi(t; y, \phi) = L(\phi) \quad \text{for all } \phi \in H_0^\infty\{a_\alpha, 2\}(\Omega),$$

which is equivalent to the existence of a unique solution $y(t) \in H^\infty\{a_\alpha, 2\}(\Omega)$ for

$$\bigl(D_+^\beta y(t), \phi\bigr)_{L^2(Q)} + \bigl(\mathcal{A}y(t), \phi\bigr)_{L^2(Q)} = L(\phi) \quad \text{for all } \phi \in H_0^\infty\{a_\alpha, 2\}(\Omega),$$

i.e. for

$$\bigl(D_+^\beta y(t) + \mathcal{A}y(t), \phi(x)\bigr)_{L^2(Q)} = L(\phi)$$

which can be written as

$$\int_Q \bigl(D_+^\beta y(t) + \mathcal{A}y(t)\bigr)\phi(x)\,dx\,dt = L(\phi) \quad \text{for all } \phi \in H_0^\infty\{a_\alpha, 2\}(\Omega). \tag{3.8}$$

This is known as the variational fractional Dirichlet problem, where $L(\phi)$ is a continuous linear form on $H_0^\infty\{a_\alpha, 2\}(\Omega)$ and takes the form

$$\begin{aligned} L(\phi) &= \int_Q f\phi\,dx\,dt + \int_\Omega y_0\phi(x,0)\,dx, \\ &f \in L^2(Q), y_0 \in L^2(\Omega). \end{aligned} \tag{3.9}$$

Then equation (3.8) is equivalent to

$$\begin{aligned} &\int_Q \bigl(D_+^\beta y(t) + \mathcal{A}y(t)\bigr)\phi(x)\,dx\,dt \\ &\quad= \int_Q f\phi\,dx\,dt + \int_\Omega y_0\phi(x,0)\,dx \quad \text{for all } \phi \in H_0^\infty\{a_\alpha, 2\}(\Omega) \end{aligned}$$

that is, the PDE

$$D_+^\beta y(t) + \mathcal{A}y(t) = f \tag{3.10}$$

'tested' against $\phi(x)$.

Let us multiply both sides in (3.10) by ϕ and applying Green's formula (Lemma 3.2), we have

$$\begin{aligned} &\int_Q \bigl(D_+^\beta y + \mathcal{A}y\bigr)\phi\,dx\,dt \\ &\quad= \int_Q f\phi\,dx\,dt - \int_\Omega \phi(x,0) I_+^{1-\beta} y\bigl(x, 0^+\bigr)\,dx \end{aligned}$$

$$+\int_0^T\int_{\partial\Omega} y\frac{\partial\phi}{\partial\nu}\,d\Gamma\,dt-\int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial\nu}\phi\,d\Gamma\,dt$$
$$+\int_0^T\int_\Omega y(x,t)\bigl(-D^\beta\phi(x,t)+\mathcal{A}^*\phi(x,t)\bigr)\,dx\,dt=\int_Q f\phi\,dx\,dt$$

whence comparing with (3.8), (3.9)

$$\int_\Omega \phi(x,0)I_+^{1-\beta}y\bigl(x,0^+\bigr)\,dx-\int_0^T\int_{\partial\Omega} y\frac{\partial\phi}{\partial\nu}\,d\Gamma\,dt=\int_\Omega y_0\phi(x,0)\,dx.$$

From this we deduce (3.2), (3.3). □

4 Optimization theorem and the control problem

For a control $u\in L^2(Q)$ the state $y(u)$ of the system is given by

$$D_+^\beta y+\mathcal{A}y(u)=u,\quad (x,t)\in Q, \tag{4.1}$$
$$y(u)|_\Sigma=0, \tag{4.2}$$
$$I_+^{1-\beta}y(x,0;u)=y_0(x),\quad x\in\Omega. \tag{4.3}$$

The observation equation is given by

$$z(u)=y(u).$$

The cost function $J(v)$ is given by

$$J(v)=\int_Q\bigl(y(v)-z_d\bigr)^2\,dx\,dt+(Nv,v)_{L^2(Q)},$$

where z_d is a given element in $L^2(\Sigma)$ and $N\in\mathcal{L}(L^2(Q),L^2(Q))$ is a hermitian positive definite operator:

$$(Nu,u)\geq c\|u\|^2_{L^2(Q)},\quad c>0.$$

Control constraints: We define U_{ad} (set of admissible controls) is closed, convex subset of $U=L^2(Q)$.

Control problem: We want to minimize J over U_{ad} *i.e.* find u such that

$$J(u)=\inf_{v\in U_{\mathrm{ad}}}J(v). \tag{4.4}$$

Under the given considerations we have the following theorem.

Theorem 4.1 *The problem* (4.4) *admits a unique solution given by* (4.1)-(4.3) *and by*

$$\int_Q\bigl(p(u)+Nu\bigr)(v-u)\,dx\,dt\geq 0,$$

where $p(u)$ *is the adjoint state.*

Proof In a similar manner to Mophou [11], Proposition 4.1 and Theorem 4.2, and Lions [17], the control $u \in U_{\mathrm{ad}}$ is optimal if and only if

$$J'(u)(v-u) \geq 0 \quad \text{for all } v \in U_{\mathrm{ad}}.$$

The above condition, when explicitly calculated for this case, gives

$$\big(y(u)-z_d, y(v)-y(u)\big)_{L^2(Q)} + (Nu, v-u)_{L^2(Q)} \geq 0$$

i.e.

$$\int_Q \big(y(u)-z_d\big)\big(y(v)-y(u)\big)\,dx\,dt + (Nu, v-u)_{L^2(Q)} \geq 0. \tag{4.5}$$

For the control $u \in L^2(Q)$ the adjoint state $p(u) \in L^2(Q)$ is defined by

$$\begin{aligned} &-D^\beta p(u) + \mathcal{A}^* p(u) = y(u) - z_d \quad \text{in } Q, \\ &p(u) = 0 \quad \text{on } \Sigma, \\ &p(x,T;u) = 0 \quad \text{in } \Omega. \end{aligned} \tag{4.6}$$

Now, multiplying equation (4.6) by $(y(v)-y(u)) \in C^\infty(\Omega)$ and applying Green's formula, we obtain

$$\begin{aligned} &\int_Q \big(y(u)-z_d\big)\big(y(v)-y(u)\big)\,dx\,dt \\ &\quad = \int_Q \big(-D^\beta p(u) + \mathcal{A}^* p(u)\big)\big(y(v)-y(u)\big)\,dx\,dt \\ &\quad = \int_\Omega p(x,0) I_+^{1-\beta}\big(y\big(v;x,0^+\big) - y\big(u;x,0^+\big)\big)\,dx \\ &\qquad + \int_\Sigma p(u)\left(\frac{\partial y(v)}{\partial \nu_{\mathcal{A}}} - \frac{\partial y(u)}{\partial \nu_{\mathcal{A}}}\right) d\Sigma - \int_\Sigma \frac{\partial p(u)}{\partial \nu_{\mathcal{A}}}\big(y(v)-y(u)\big)\,d\Sigma \\ &\qquad + \int_Q p(u)\big(D_+^\beta + \mathcal{A}\big)\big(y(v)-y(u)\big)\,dx\,dt. \end{aligned}$$

From (4.1), (4.2) we have

$$\big(D_+^\beta + \mathcal{A}\big)\big(y(v)-y(u)\big) = v - u, \qquad y(u)|_\Sigma = 0, \qquad p(u)|_\Sigma = 0.$$

Then we obtain

$$\int_Q \big(y(u)-z_d\big)\big(y(v)-y(u)\big)\,dx\,dt = \int_Q p(u)(v-u)\,dx\,dt$$

and hence (4.5) is equivalent to

$$\int_Q p(u)(v-u)\,dx\,dt + (Nu, v-u)_{L^2(Q)} \geq 0$$

i.e.

$$\int_Q \big(p(u) + Nu\big)(v - u)\,dx\,dt \geq 0$$

which completes the proof. □

Example 4.2 [11] Let $n \in N^*$. and Ω be a bounded open subset of R^n with boundary $\partial\Omega$ of class C^2. For a time $T > 0$. We consider the fractional diffusion equation

$$D_+^\beta y(t) - \Delta y(t) = v, \quad t \in [0, T], \tag{4.7}$$

$$I_+^{1-\beta} y\big(0^+\big) = y_0, \quad x \in \Omega, \tag{4.8}$$

$$y(x,t) = 0, \quad x \in \Gamma, t \in (0, T), \tag{4.9}$$

where $0 < \beta < 1$, $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the control v belongs to $L^2(Q)$. We can minimize

$$J(v) = \big\|y(v) - z_d\big\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}, \quad z_d \in L^2(Q), N > 0$$

subject to system (4.7)-(4.9) and the optimal control v will be characterized by system (4.7)-(4.9) with the adjoint system

$$-D_+^\beta p(t) - \Delta p(t) = y - z_d, \quad t \in [0, T],$$

$$p(x,t) = 0, \quad x \in \Omega, t \in (0, T),$$

$$p(x,T) = 0, \quad x \in \Gamma,$$

and with the optimality condition

$$v = -\frac{p}{N} \quad \text{in } Q.$$

5 Fractional Neumann problem for infinite order system

From (2.1) we can show that the bilinear form (3.5) is coercive in $H^\infty\{a_\alpha, 2\}(\Omega)$, that is,

$$\pi(y,y) \geq c\|y\|^2_{H^\infty\{a_\alpha,2\}(\Omega)}, \quad c > 0 \text{ for all } y \in H^\infty\{a_\alpha, 2\}(\Omega). \tag{5.1}$$

From the above coercitivness condition (5.1) and using the Lax-Milgram lemma we have the following lemma, which defines the Neumann problem for the operator $\mathcal{A}$ with $\mathcal{A} \in \mathcal{L}(H^\infty\{a_\alpha, 2\}(\Omega), H^{-\infty}\{a_\alpha, 2\}(\Omega))$ and enables us to obtain the state of our control problem.

Lemma 5.1 *If* (5.1) *is satisfied then there exists a unique element $y \in H^\infty\{a_\alpha, 2\}(\Omega)$ satisfying the Neumann problem*

$$D_+^\beta y + \mathcal{A}y = f \quad \textit{in } Q, \tag{5.2}$$

$$\frac{\partial y}{\partial \nu_{\mathcal{A}}} = h \quad \textit{on } \Sigma, \tag{5.3}$$

$$I_+^{1-\beta} y\big(0^+\big) = y_0(x), \quad x \in \Omega. \tag{5.4}$$

Proof From the coerciveness condition (5.1) and using the Lax-Milgram lemma, there exists a unique element $y \in H^\infty\{a_\alpha, 2\}(\Omega)$ such that

$$\int_Q y\big(-D^\beta \psi + \mathcal{A}^* \psi\big)\,dx\,dt = M(\psi) \quad \text{for all } \psi \in H^\infty\{a_\alpha, 2\}(\Omega). \tag{5.5}$$

This is known as the fractional Neumann problem, where $M(\psi)$ is a continuous linear form on $H^\infty\{a_\alpha, 2\}(\Omega)$ and takes the form

$$\begin{aligned} M(\psi) &= \int_Q f\psi\,dx\,dt + \int_\Omega y_0 \psi(x,0)\,dx - \int_\Sigma h \frac{\partial \psi}{\partial \nu_{A^*}}\,d\Sigma, \\ &f \in L^2(Q), y_0 \in L^2(\Omega), h \in H^\infty\{a_\alpha, 2\}(\Sigma). \end{aligned} \tag{5.6}$$

The Equation (5.5) is equivalent to (5.2). Let us multiply both sides in (5.2) by ψ and applying Green's formula, we have

$$\begin{aligned} &\int_Q \big(D_+^\beta y + \mathcal{A}y\big)\psi\,dx\,dt \\ &\quad= \int_Q f\psi\,dx\,dt - \int_\Omega \psi(x,0) I_+^{1-\beta} y\big(x,0^+\big)\,dx \\ &\qquad+ \int_0^T \int_{\partial\Omega} y \frac{\partial \psi}{\partial \nu}\,d\Gamma\,dt - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu} \psi\,d\Gamma\,dt \\ &\qquad+ \int_0^T \int_\Omega y(x,t)\big(-D^\beta \psi(x,t) + \mathcal{A}^* \psi(x,t)\big)\,dx\,dt = \int_Q f\psi\,dx\,dt \end{aligned}$$

whence comparing with (5.5), (5.6)

$$\begin{aligned} &\int_\Omega \psi(x,0) I_+^{1-\beta} y\big(x,0^+\big)\,dx + \int_0^T \int_{\partial\Omega} \psi \frac{\partial y}{\partial \nu}\,d\Gamma\,dt \\ &\quad= \int_\Omega y_0 \psi(x,0)\,dx + \int_0^T \int_{\partial\Omega} h\psi\,d\Gamma\,dt. \end{aligned}$$

From this we deduce (5.3), (5.4). □

6 Minimization theorem and boundary control problem

We consider the space $U = L^2(\Sigma)$ (the space of controls), for every control $u \in U$, the state of the system $y(u) \in H^\infty\{a_\alpha, 2\}(\Omega)$ is given by the solution of

$$D_+^\beta y(u) + \mathcal{A}y(u) = f \quad \text{in } Q, \tag{6.1}$$

$$\frac{\partial y(u)}{\partial \nu_{\mathcal{A}}} = u \quad \text{on } \Sigma, \tag{6.2}$$

$$I_+^{1-\beta} y(x,0;u) = y_0(x), \quad x \in \Omega. \tag{6.3}$$

For the observation, we consider the following two cases:

(i)

$$z(u) = y(u), \tag{6.4}$$

(ii) observation of final state

$$z(u) = y(x, T; u). \tag{6.5}$$

Case (i) For the observation (6.4), the cost function is given by

$$J(v) = \int_Q \big(y(v) - z_d\big)^2 \, dx \, dt + (Nv, v)_{L^2(\Sigma)}, \quad z_d \in L^2(Q), \tag{6.6}$$

where $N \in \mathcal{L}(L^2(\Sigma), L^2(\Sigma))$, N is a hermitian positive definite:

$$(Nu, u)_{L^2(\Sigma)} \geq c\|u\|^2_{L^2(\Sigma)}, \quad c > 0. \tag{6.7}$$

Control constraints: We define U_{ad} (set of admissible controls) is closed, convex dense subset of $U = L^2(\Sigma)$.

Control problem: We wish to find

$$\inf_{v \in U_{\mathrm{ad}}} J(v). \tag{6.8}$$

Under the given considerations we have the following theorem.

Theorem 6.1 *Assume that* (6.7) *holds and the cost function being given by* (6.6). *The optimal control u of problem* (6.8) *is characterized by* (6.1), (6.2), (6.3) *together with*

$$-D^\beta p(u) + \mathcal{A}^* p(u) = y(u) - z_d \quad \text{in } Q, \tag{6.9}$$

$$\frac{\partial p(u)}{\partial \nu_{\mathcal{A}^*}} = 0 \quad \text{on } \Sigma, \tag{6.10}$$

$$p(x, T; u) = 0, \quad x \in \Omega, \tag{6.11}$$

and the optimality condition is

$$\int_\Sigma \big(p(u) + Nu\big)(v - u) \, d\Sigma \geq 0 \quad \forall v \in U_{\mathrm{ad}}, \tag{6.12}$$

where $p(u)$ *is the adjoint state.*

Proof By similar manner as in Mophou [11], Proposition 4.1 and Theorem 4.2 and Lions [17], the control $u \in U_{\mathrm{ad}}$ is optimal if and only if

$$J'(u)(v - u) \geq 0 \quad \forall v \in U_{\mathrm{ad}}$$

which is equivalent to

$$\big(y(u) - z_d, y(v) - y(u)\big)_{L^2(Q)} + (Nu, v - u)_U \geq 0. \tag{6.13}$$

The adjoint state is given by the solution of the adjoint Neumann problem (6.9), (6.10), (6.11). Now, multiplying the equation in (6.9) by $y(v) - y(u)$ and applying Green's formula, with taking into account the conditions in (6.1), (6.2), we obtain

$$\begin{aligned}
&\int_Q \big(y(u) - z_d\big)\big(y(v) - y(u)\big)\,dx\,dt \\
&\quad = \int_Q \big(-D^\beta p(u) + \mathcal{A}^* p(u)\big)\big(y(v) - y(u)\big)\,dx\,dt \\
&\quad = -\int_\Omega p(x,0) I_+^{1-\beta}\big(y\big(v;x,0^+\big) - y\big(u;x,0^+\big)\big)\,dx + \int_\Sigma p(u)\left(\frac{\partial}{\partial \nu_{\mathcal{A}}} y(v) - \frac{\partial}{\partial \nu_{\mathcal{A}}} y(u)\right) d\Sigma \\
&\qquad - \int_\Sigma \frac{\partial}{\partial \nu_{\mathcal{A}^*}} p(u)\big(y(v) - y(u)\big)\,d\Sigma + \int_Q p(u)\big(D_+^\beta + \mathcal{A}\big)\big(y(v) - y(u)\big)\,dx\,dt \\
&\quad = \int_\Sigma p(u)(v-u)\,d\Sigma. \qquad (6.14)
\end{aligned}$$

Hence we substituted from (6.14) in (6.13), we get

$$\int_\Sigma p(u)(v-u)\,d\Sigma + (Nu, v-u)_{L^2(\Sigma)} \ge 0$$

i.e.

$$\int_\Sigma \big(p(u) + Nu\big)(v-u)\,d\Sigma \ge 0 \quad \forall v \in \mathcal{U}_{\mathrm{ad}}$$

which completes the proof. □

Example 6.2 In the case of no constraints on the control ($\mathcal{U}_{\mathrm{ad}} = \mathcal{U}$). Then (6.12) reduce to

$$p + Nu = 0 \quad \text{on } \Sigma.$$

The optimal control is obtained by the simultaneous solution of the following system of partial differential equations:

$$\begin{aligned}
&D_+^\beta + \mathcal{A}y = f, \qquad -D^\beta p + \mathcal{A}^* p = y - z_d \quad \text{in } Q, \\
&\left.\frac{\partial y}{\partial \nu_{\mathcal{A}}}\right|_\Sigma + N^{-1} p|_\Sigma = 0, \qquad \frac{\partial p}{\partial \nu_{\mathcal{A}^*}} = 0 \quad \text{on } \Sigma, \\
&I_+^{1-\beta} y(x,0) = y_0(x), \qquad p(x,T) = 0, \quad x \in \Omega,
\end{aligned}$$

further

$$u = -N^{-1}(P|_\Sigma).$$

Example 6.3 Take

$$\mathcal{U}_{\mathrm{ad}} = \big\{u | u \in L^2(\Sigma), u \ge 0 \text{ almost everywhere on } \Sigma\big\}.$$

The optimal control is obtained by the solution of the problem

$$\begin{aligned}
&D_+^\beta y + \mathcal{A}y = f, \qquad -D^\beta p + \mathcal{A}^* p = y - z_d \quad \text{in } Q,\\
&\frac{\partial y}{\partial \nu_{\mathcal{A}}} \geq 0, \qquad \frac{\partial p}{\partial \nu_{\mathcal{A}}^*} = 0 \quad \text{on } \Sigma,\\
&p + N\frac{\partial y}{\partial \nu_{\mathcal{A}}} \geq 0, \qquad \frac{\partial y}{\partial \nu_{\mathcal{A}}}\left[p + N\frac{\partial y}{\partial \nu_{\mathcal{A}}}\right] = 0 \quad \text{on } \Sigma,\\
&I_+^{1-\beta} y(x,0) = y_0(x), \qquad p(x,T) = 0, \quad x \in \Omega,
\end{aligned}$$

hence

$$u = \left.\frac{\partial y}{\partial \nu_{\mathcal{A}}}\right|_{\Sigma}.$$

Case (ii) for observation of final state (6.5), the cost function is given by

$$J(v) = \int_\Omega \bigl(y(x,T;v) - z_d\bigr)^2\,dx + (Nv,v)_{L^2(\Sigma)}, \quad z_d \in L^2(\Omega).$$

The adjoint state is defined by

$$\begin{aligned}
&-D^\beta p(u) + \mathcal{A}^* p(u) = 0 \quad \text{in } Q,\\
&\frac{\partial p(u)}{\partial \nu_{\mathcal{A}^*}} = 0 \quad \text{on } \Sigma,\\
&p(x,T;u) = y(x,T;u) - z_d(x), \quad x \in \Omega,
\end{aligned}$$

and the optimality condition is

$$\int_\Sigma (p + Nu)(v - u)\,d\Sigma \geq 0 \quad \forall v \in \mathcal{U}_{\text{ad}}, \tag{6.15}$$

where $p(u)$ is the adjoint state.

Example 6.4 Take the case of no constraints on the control ($\mathcal{U}_{\text{ad}} = \mathcal{U}$). Then (6.15) reduces to

$$p + Nu = 0 \quad \text{on } \Sigma.$$

The optimal control is obtained by the simultaneous solution of the following system of partial differential equations:

$$\begin{aligned}
&D_+^\beta y + \mathcal{A}y = f, \qquad -D^\beta p + \mathcal{A}^* p = 0 \quad \text{in } Q,\\
&\left.\frac{\partial y}{\partial \nu_{\mathcal{A}}}\right|_{\Sigma} + N^{-1} p|_\Sigma = 0, \qquad \frac{\partial p}{\partial \nu_{\mathcal{A}}^*} = 0 \quad \text{on } \Sigma,\\
&I_+^{1-\beta} y(x,0) = y_0(x), \qquad p(x,T) = y(x,T;u) - z_d(x), \quad x \in \Omega,
\end{aligned}$$

further

$$u = -N^{-1}(P|_{\Sigma}).$$

Example 6.5 Take

$$\mathcal{U}_{\text{ad}} = \{u | u \in L^2(\Sigma), u \geq 0 \text{ almost everywhere on } \Sigma\}.$$

Then (6.15) is equivalent to

$$u \geq 0, \qquad p(u) + Nu \geq 0, \qquad u(p(u) + Nu) = 0 \quad \text{on } \Sigma.$$

7 Conclusions

An analytical scheme for fractional optimal control of infinite order systems is considered. The fractional derivatives were defined in the Riemann-Liouville sense. The analytical results were given in terms of the Euler-Lagrange equations for the fractional optimal control problems. The formulation presented and the resulting equations are very similar to those for classical optimal control problems. The optimization problem presented in this paper constitutes a generalization of the optimal control problem of parabolic systems with the Dirichlet and Neumann boundary conditions considered in Lions [17], Lions and Magenes [18] to fractional optimal control problem for infinite order systems. Also the main result of the paper contains necessary and sufficient conditions of optimality for infinite order systems that give a characterization of optimal control (Theorems 4.1 and 6.1).

Competing interests
The author declares that he has no competing interests.

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