RESEARCH

Open Access

CrossMark

Dynamic analysis of effects of phytoplankton dispersal on zooplankton

Zhong Zhao[*], Lin Shen[*] and Chengguang Luo

[*]Correspondence:
shenlin@huanghuai.edu.cn;
zhaozhong8899@163.com
School of Mathematics and
Statistics, Huanghuai University,
Zhumadian, Henan 463000,
P.R. China

Abstract

In this paper, we formulate a delayed phytoplankton-zooplankton model with impulsive diffusion on phytoplankton. Using the discrete dynamical system determining the stroboscopic map, we obtain the zooplankton-extinction periodic solution which is globally attractive. The conditions of the permanence are given by using the theory on the delay and impulsive differential equation. Finally, some numerical simulations are presented to illustrate the results.

MSC: 34C05; 92D25

Keywords: nonlinear measure of intra-species interference; impulsive dispersal; global attractivity; permanence

1 Introduction

Plankton, including phytoplankton and zooplankton, are an important food source for organisms in an aquatic environment. Phytoplankton perform a great service for the earth by absorbing the carbon dioxide from the surrounding environments and releasing the oxygen into the atmosphere [1, 2]. As a primary producer, phytoplankton are most favorable food sources for fish and other aquatic animals [3]. An obvious feature of the phytoplankton is a rapid appearance and disappearance resulting in the formation of bloom, which causes a great harm to the human health and zooplankton population [4, 5]. Therefore, it is necessary to investigate the effect of zooplankton and phytoplankton on the occurrence of bloom.

Many mathematical models have been formulated to describe the dynamical interaction between zooplankton and phytoplankton [6–10]. In [6], deterministic and stochastic models of nutrient-phytoplankton-zooplankton interaction are proposed to investigate the impact of toxin-producing phytoplankton upon persistence of the populations. The author of [7] formulated a toxin-producing phytoplankton-zooplankton model with stochastic perturbation and investigated the global stability of the positive equilibrium by means of constructing suitable Lyapunov functions. Chowdhury *et al.* [9] proposed a mathematical model of NTP-TPP-zooplankton with constant and variable zooplankton migration. The asymptotic dynamics of the system around the biologically feasible equilibria was explored through local stability analysis. The authors of [11] analyzed a mathematical model for the interactions between phytoplankton and zooplankton in a periodic environment and obtained the permanent condition.

Springer

As we know, population dispersal has a great effect on the dynamics [12–17]. Hong *et al.* [12] investigated a single species model with intermittent unilateral diffusion in two patches. The global attractivity of positive periodic solution and the extinction of species were established by using Lyapunov function approach. Shao [16] formulated a delayed predator-prey system with impulsive diffusion between two patches:

$$
\begin{cases}
\frac{dx_1}{dt} = x_1(r_1 - a_1 x_1 - b_1 y), & \\
\frac{dx_2}{dt} = x_2(r_2 - a_2 x_2), & t \neq nT, \\
\frac{dy}{dt} = y(-r_3 + a_3 x_1(t - \tau_1) - b_2 y(t - \tau_2)), & \\
\Delta x_1 = d_1(x_2 - x_1), & \\
\Delta x_2 = d_2(x_1 - x_2), \Delta y = 0, & t = nT,
\end{cases}
\tag{1.1}
$$

with the initial condition

$$
x_1(s) = \phi_1(s), \qquad x_2(s) = \phi_2(s), \qquad y(s) = \phi_3(s),
$$

$$
\phi = (\phi_1, \phi_2, \phi_3) \in C\big([-\tau, 0], R_+^3\big), \phi_i(0) > 0, i = 1, 2, 3,
$$

where phytoplankton are structured into two patches connected by impulsive diffusion. Wang and Jia [18] proposed a single species model with impulsive diffusion and pulsed harvesting at the different fixed time as follows:

$$
\begin{cases}
\frac{dx_1}{dt} = x_1(a_1 - b_1 x_1^{\theta_1}), & \\
\frac{dx_2}{dt} = x_2(a_2 - b_2 x_2^{\theta_2}), & t \neq (n + l - 1)T, t \neq nT, \\
\Delta x_1 = -p_1 x_1, & \\
\Delta x_2 = -p_2 x_2, & t = (n + l - 1)T, \\
\Delta x_1 = d_1(x_2 - x_1), & \\
\Delta x_2 = d_2(x_1 - x_2), \Delta y = 0, & t = nT,
\end{cases}
\tag{1.2}
$$

where the system is composed of two patches connected by diffusion. x_i $(i = 1, 2)$ is the density of species in the ith patch. Wang *et al.* [19] proposed a single species model with impulsive diffusion between two patches and obtained a globally stable positive periodic solution by using the discrete dynamical system generated by a monotone, concave map for the population. However, little information is available about the application of impulsive diffusion to plankton model. In this paper, we will formulate a nonlinear modeling of the interaction between phytoplankton and zooplankton with impulsive dispersal on the phytoplankton.

The outline of this paper is as follows: a delayed phytoplankton-zooplankton model with impulsive diffusion is presented in Section 2. In addition, some important lemmas are also given in Section 2. In Sections 3 and 4, we obtain the sufficient conditions for global attractivity of zooplankton-extinction periodic solution and the permanence of the system. Finally, we give some numerical simulations and a brief discussion.

2 Development model and preliminaries

Although phytoplankton are single-celled organisms, they play an important role in the marine ecosystem. To describe the complex effect of the phytoplankton on zooplankton,

Roy *et al.* [20] considered a significant number of species of phytoplankton that have the ability to produce toxic or inhibitory compounds and they formulated the following model:

$$
\begin{cases}
\frac{dP_1}{dt} = P_1\{r_1(1 - \frac{P_1 + \alpha_1 P_2}{K}) - \frac{w_1 Z}{D_1 + P_1}\}, \\
\frac{dP_2}{dt} = P_2\{r_2(1 - \frac{P_2 + \alpha_2 P_1}{K}) - \frac{w_2 Z}{D_2 + P_2}\}, \\
\frac{dZ}{dt} = Z\{\frac{\xi_1 P_1}{D_1 + P_1} - \frac{\xi_2 P_2}{D_2 + P_2} - c\}.
\end{cases}
\tag{2.1}
$$

Motivated by [16, 18, 19], we are concerned with the effects of the phytoplankton-impulsive diffusion in two patches on the dynamics of a phytoplankton-zooplankton system,

$$
\begin{cases}
\frac{dP_1}{dt} = P_1(r_1 - a_1 P_1^{\theta_1}) - \alpha_1 P_1 Z, \\
\frac{dP_2}{dt} = P_2(r_2 - a_2 P_2^{\theta_2}) - \alpha_2 P_2 Z, & t \neq nT, \\
\frac{dZ}{dt} = Z(\varrho_1 P_1 + \varrho_2 P_2 - a_3 Z(t - \tau) - \mu - \frac{\beta P_2}{K + P_2}), \\
\Delta P_1 = d(P_2 - P_1), \\
\Delta P_2 = d(P_1 - P_2), \Delta Z = 0, & t = nT,
\end{cases}
\tag{2.2}
$$

with the initial condition

$$
P_1(s) = \phi_1(s), \qquad P_2(s) = \phi_2(s), \qquad Z(s) = \phi_3(s),
$$

$$
\phi = (\phi_1, \phi_2, \phi_3) \in C([-\tau, 0], R_+^3), \phi_i(0) > 0, i = 1, 2, 3,
\tag{2.3}
$$

where $P_1(t)$ denotes the concentration of the nontoxic phytoplankton (NPP) and $P_2(t)$ is the concentration of the toxin-producing phytoplankton (TPP). $Z(t)$ is the concentration of the zooplankton. r_1 and r_2 are the intrinsic growth rates of NTP and TPP population, respectively. θ_i ($i = 1, 2$) present nonlinear measure of intra-species interference. a_i ($i = 1, 2$) are the coefficients of intra-specific competition. α_i ($i = 1, 2$) denote the capturing rates of the zooplankton. $\frac{\varrho_i}{\alpha_i}$ ($i = 1, 2$) denote the conversion rate of nutrient into the production rate of the zooplankton. The term $-a_3 Z(t - \tau)$ denotes the negative feedback of zooplankton crowding. T is the impulsive diffusion because of the external perturbation. d ($0 < d < 1$) is the diffusion rate. μ is the death rate of zooplankton. The term $\frac{\beta P}{K + P_2}$ ($\beta > 0, K > 0$) contributes to the death of zooplankton population, where K is a half-saturation constant, β denotes the rate of toxin liberation by toxin-producing phytoplankton. $\Delta P_i(t^+) = P_i(t^+) - P_i(t)$ ($i = 1, 2$), $\Delta Z(t^+) = Z(t^+) - Z(t)$, $n \in N$.

For convenience, we first give some lemmas.

Lemma 2.1 *All the solutions $(P_1(t), P_2(t), Z(t))$ of system* (2.2) *with the initial conditions are positive for all $t \geq 0$.*

Lemma 2.2 *Let $(P_1(t), P_2(t), Z(t))$ be any solution of system* (2.2), *there exists a constant $M > 0$, such that $P_i(t) \leq M$ ($i = 1, 2$) and $Z(t) \leq M$ for t large enough.*

Proof Define $V(t) = \frac{\theta_1}{\varrho_1} P_1(t) + \frac{\varrho_2}{\alpha_2} P_2(t) + Z(t)$. When $t \neq nT$, we have

$$
D^+ V + \mu V(t) \leq \frac{\varrho_1}{\alpha_1}\left((r_1 + \mu)P_1 - a_1 P_1^{\theta_1}\right) + \frac{\varrho_2}{\alpha_2}\left((r_2 + \mu)P_2 - a_2 P_2^{\theta_2}\right)
$$

$$
\leq \frac{(r_1 + \mu)\theta_1 \varrho_1}{\alpha_1(\theta_1 + 1)}\left(\frac{r_1 + \mu}{a_1(\theta_1 + 1)}\right)^{\frac{1}{\theta_1}} + \frac{(r_2 + \mu)\theta_2 \varrho_2}{\alpha_2(\theta_2 + 1)}\left(\frac{r_2 + \mu}{a_2(\theta_2 + 1)}\right)^{\frac{1}{\theta_2}} \overset{\triangle}{=} \xi.
$$

For $t = nT$, $V(nT^+) \leq V(nT)$, hence we obtain

$$V(t) \leq V(0^+)e^{-\mu t} + \frac{\xi}{\mu}\left(1 - e^{-\mu t}\right) \to \frac{\xi}{\mu}$$

as $t \to \infty$, which shows $V(t)$ is uniformly ultimately bounded. Thus, we have $P_i(t) \leq M$ ($i = 1, 2$), $Z(t) \leq M$.

Consider the following system:

$$\begin{cases} \frac{dP_1}{dt} = P_1(r_1 - a_1 P_1^{\theta_1}), & t \neq nT, \\ \frac{dP_2}{dt} = P_2(r_2 - a_2 P_2^{\theta_2}), & \\ \Delta P_1 = d(P_2 - P_1), & \\ \Delta P_2 = d(P_1 - P_2), & t = nT. \end{cases} \tag{2.4}$$

Integrating system (2.4) on $(nT, (n+1)T]$, we have

$$P_i(t) = \left(P_1^{-\theta_i}\left(nT^+\right)e^{-r_i\theta_i(t-nT)} + \frac{a_i}{r_i}\left(1 - e^{-r_i\theta_i(t-nT)}\right) \right)^{-\frac{1}{\theta_i}},$$

$$t \in \left(nT, (n+1)T\right], i = 1, 2. \tag{2.5}$$

Similar to [18], we derive the difference equation at the impulsive moment according to system (2.5).

$$\begin{cases} P_1(n+1)T^+ = \left(\frac{P_1^{\theta_1}(nT^+)}{\beta_1 + \gamma_1 P_1^{\theta_1}(nT^+)}\right)^{\frac{1}{\theta_1}} + d\left(\left(\frac{P_2^{\theta_2}(nT^+)}{\beta_2 + \gamma_2 P_2^{\theta_2}(nT^+)}\right)^{\frac{1}{\theta_2}} - \left(\frac{P_1^{\theta_1}(nT^+)}{\beta_1 + \gamma_1 P_1^{\theta_1}(nT^+)}\right)^{\frac{1}{\theta_1}}\right), \\ P_2(n+1)T^+ = \left(\frac{P_2^{\theta_2}(nT^+)}{\beta_2 + \gamma_2 P_2^{\theta_2}(nT^+)}\right)^{\frac{1}{\theta_2}} + d\left(\left(\frac{P_1^{\theta_1}(nT^+)}{\beta_1 + \gamma_1 P_1^{\theta_1}(nT^+)}\right)^{\frac{1}{\theta_1}} - \left(\frac{P_2^{\theta_2}(nT^+)}{\beta_2 + \gamma_2 P_2^{\theta_2}(nT^+)}\right)^{\frac{1}{\theta_2}}\right), \end{cases} \tag{2.6}$$

where $\beta_i = e^{-r_i\theta_i T} < 1$, $\gamma_i = \frac{a_i}{r_i}(1 - e^{-r_i\theta_i T})$, $i = 1, 2$. Equation (2.6) presents the phytoplankton concentration between patches after diffusion at the moment $t = nT$.

To investigate the dynamics of system (2.6), we define a continuous map $F : R_+^2 \to R_+^2$,

$$\begin{cases} F_1(x) = \left(\frac{x_1^{\theta_1}}{\beta_1 + \gamma_1 x_1^{\theta_1}}\right)^{\frac{1}{\theta_1}} + d\left(\left(\frac{x_2^{\theta_2}}{\beta_2 + \gamma_2 x_2^{\theta_2}}\right)^{\frac{1}{\theta_2}} - \left(\frac{x_1^{\theta_1}}{\beta_1 + \gamma_1 x_1^{\theta_1}}\right)^{\frac{1}{\theta_1}}\right), \\ F_2(x) = \left(\frac{x_2^{\theta_2}}{\beta_2 + \gamma_2 x_2^{\theta_2}}\right)^{\frac{1}{\theta_2}} + d\left(\left(\frac{x_1^{\theta_1}}{\beta_1 + \gamma_1 x_1^{\theta_1}}\right)^{\frac{1}{\theta_1}} - \left(\frac{x_2^{\theta_2}}{\beta_2 + \gamma_2 x_2^{\theta_2}}\right)^{\frac{1}{\theta_2}}\right). \end{cases} \tag{2.7}$$

$\square$

Lemma 2.3 [18] *There exists a unique positive fixed point $q = (q_1, q_2)$ of map F and for any $x = (x_1, x_2)$, we have $F(x) \to q$ as $n \to \infty$, which implies $q = (q_1, q_2)$ is globally asymptotically stable. Therefore, system (2.4) has a positive periodic solution $(P_1^*(t), P_2^*(t))$, where*

$$P_i^*(t) = \left(\frac{a_i}{r_i} + \left(\frac{1}{q_i^{\theta_i}} - \frac{a_i}{r_i} \right)e^{-r_i\theta_i(t-nT)} \right)^{-\frac{1}{\theta_i}}, \quad t \in \left(nT, (n+1)T\right], i = 1, 2. \tag{2.8}$$

Thus, system (2.2) has a zooplankton-extinction periodic solution $(P_1^*(t), P_2^*(t), 0)$.

Lemma 2.4 [16] *Let us consider the following equality:*

$$\frac{dx}{dt} \leq x(t)\big(a - bx(t - \tau)\big), \tag{2.9}$$

where $a, b, \tau > 0, x(t) > 0$ *for* $t \in [-\tau, 0]$. *Then we obtain* $\frac{a}{b}e^{a-ae^{a\tau}} \leq x(t) \leq \frac{a}{b}e^{a\tau}$ *for* t *large enough.*

3 Global attractivity of the zooplankton-extinction periodic solution

Theorem 3.1 *The zooplankton-extinction periodic solution* $(P_1^*(t), P_2^*(t), 0)$ *is globally attractive if* $\sum_{i=1}^{i=1} \varrho_i \left(\frac{a_i}{r_i} + \left(\frac{1}{q_i^{\theta_i}} - \frac{a_i}{r_i} \right) e^{-r_i \theta_i T} \right)^{-\frac{1}{\theta_i}} < \mu$ *holds.*

Proof From the first two equations of system (2.1), we have

$$\begin{aligned}
\frac{dP_1}{dt} &\leq P_1(r_1 - a_1 P_1^{\theta_1}), \\
\frac{dP_2}{dt} &\leq P_2(r_2 - a_2 P_2^{\theta_2}).
\end{aligned} \tag{3.1}$$

We consider the comparison system as follows:

$$\begin{cases}
\frac{du_1}{dt} = u_1(r_1 - a_1 u_1^{\theta_1}), \\
\frac{du_2}{dt} = u_2(r_2 - a_2 u_2^{\theta_2}), & t \neq nT, \\
\Delta u_1 = d(u_2 - P_1), \\
\Delta u_2 = d(u_1 - u_2), & t = nT.
\end{cases} \tag{3.2}$$

From Lemma 2.3, we see that system (3.2) has a periodic solution $(u_1^*(t), u_2^*(t))$,

$$u_i^*(t) = \left(\frac{a_i}{r_i} + \left(\frac{1}{q_i^{\theta_i}} - \frac{a_i}{r_i} \right) e^{-r_i \theta_i (t - nT)} \right)^{-\frac{1}{\theta_i}}, \quad t \in \big(nT, (n+1)T\big], i = 1, 2, \tag{3.3}$$

which is globally asymptotically stable. There exist an integer $k_1 > 0$ and $\varepsilon_1 > 0$ such that $P_i(t) \leq u_i(t) \leq u_i^*(t) + \varepsilon$ for $kT \leq t \leq (k+1)T$, that is,

$$P_i(t) \leq u_i^*(t) + \varepsilon \leq \left(\frac{a_i}{r_i} + \left(\frac{1}{q_i^{\theta_i}} - \frac{a_i}{r_i} \right) e^{-r_i \theta_i T} \right)^{-\frac{1}{\theta_i}} + \varepsilon \overset{\Delta}{=} \kappa_i$$

$$(i = 1, 2), kT \leq t \leq (k+1)T, k > k_1 > 0. \tag{3.4}$$

Again from system (2.2), we have

$$\frac{dZ}{dt} \leq Z\big(\varrho_1 \kappa_1 + \varrho_2 \kappa_2 - \mu - a_3 Z(t - \tau)\big), \quad t > k_1 T + \tau. \tag{3.5}$$

From the condition of the Theorem 3.1, we have $\theta_1 \kappa_1 + \theta_2 \kappa_2 < \mu$ for ε small enough. From system (3.4) and Lemma 2.4, we easily obtain $Z(t) \leq 0$ using the comparison theorem of impulsive differential equations. Again from the positivity of $Z(t)$, we derive $\lim_{t \to \infty} Z(t) = 0$. Therefore, there exists an integer $k_2 > k_1$ such that $Z(t) < \varepsilon_1$ for $t > k_2 T$.

From the first equations of system (2.2), we have

$$\begin{cases} \frac{dP_1}{dt} \geq P_1(r_1 - \alpha_1\varepsilon_1 - a_1P_1^{\theta_1}), \\ \frac{dP_2}{dt} \geq P_2(r_2 - \alpha_2\varepsilon_1 - a_2P_2^{\theta_2}). \end{cases} \tag{3.6}$$

Integrating the comparison system, we have

$$\begin{cases} \frac{du_3}{dt} = u_3(r_1 - \alpha_1\varepsilon_1 - a_1u_3^{\theta_1}), & t \neq nT, \\ \frac{du_4}{dt} = u_4(r_2 - \alpha_2\varepsilon_1 - a_2u_4^{\theta_2}), \\ \Delta u_3 = d(u_4 - u_3), & t = nT. \\ \Delta u_4 = d(u_3 - u_4), \end{cases} \tag{3.7}$$

From Lemma 2.3, we get system (3.7) has a periodic solution $(u_3^*(t), u_4^*(t))$ as follows:

$$u_i^*(t) = \left(\frac{a_i}{r_i - \alpha_i\varepsilon_1} + \left(\frac{1}{q_{*i}^{\theta_i}} - \frac{a_i}{r_i - \alpha_i\varepsilon_1} \right) e^{-(r_i - \alpha_i\varepsilon_1)\theta_i(t-nT)} \right)^{-\frac{1}{\theta_i}},$$

$$t \in \left(nT, (n+1)T \right] (i = 3, 4),$$

which is globally asymptotically stable and q_{*i} can be computed similar to Lemma 2.3.

By the comparison theorem of impulsive different equations, for any $\varepsilon_2 > 0$, there exists an integer k_3 ($k_3 > k_2$) such that $P_1(t) \geq u_3^*(t) - \varepsilon_2$ and $P_2(t) > u_4^*(t) - \varepsilon_2$, $kT < t < (k+1)T$, $k > k_3$.

Let $\varepsilon_1 \to 0$, we have $P_i(t) \to P_i^*(t), i = 1, 2$. Therefore, the zooplankton-extinction periodic solution $(P_1^*(t), P_2^*(t), 0)$ is globally attractive if $\sum_{i=1}^{i=1} \varrho_i(\frac{a_i}{r_i} + (\frac{1}{q_i^{\theta_i}} - \frac{a_i}{r_i})e^{-r_i\theta_iT})^{-\frac{1}{\theta_i}} < \mu$ holds. The proof is completed. □

4 Permanence

In this section, we investigate system (2.2) is permanent if the zooplankton population is above a certain threshold level for sufficiently large time.

Theorem 4.1 *System* (2.2) *is permanent if* $\theta_1a_1 + \theta_2a_2 > \mu + \frac{\beta M}{K+M}$ *holds, where M is a positive constant.*
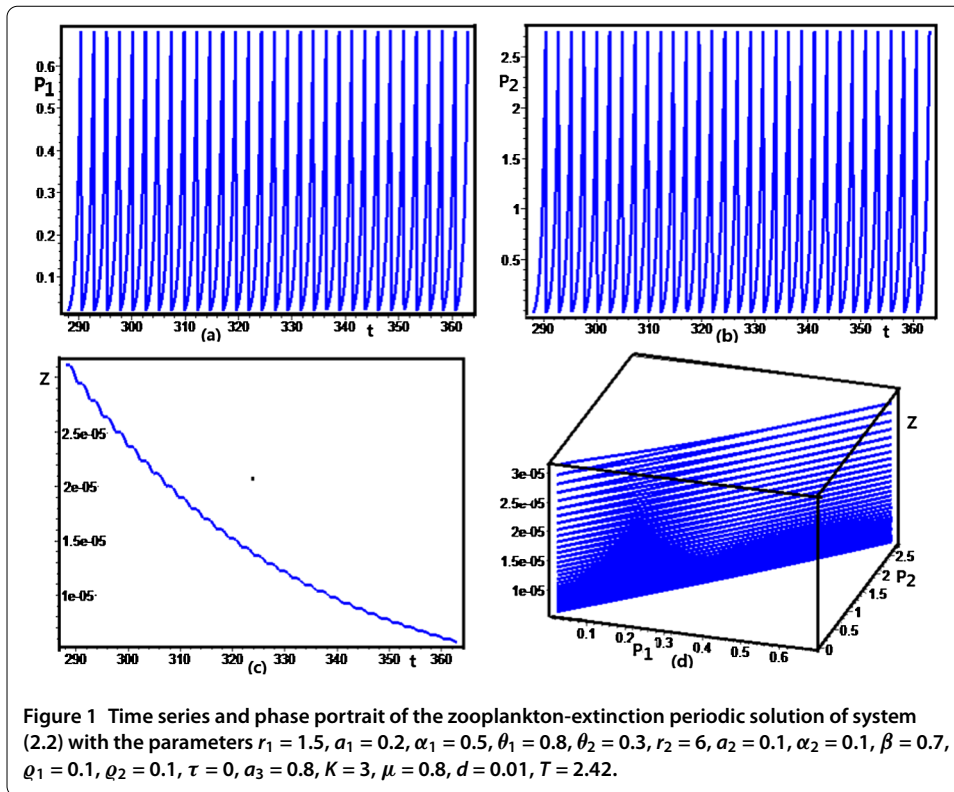
Proof Let $(P_1(t), P_2(t), Z(t))$ be any solution of system (2.2) with initial value (2.3). From Lemma 2.2, we obtain $P_1(t) < M, P_2(t) < M, Z(t) < M$ for $t \to \infty$. Next, we need to prove there exists a positive constant $m > 0$ such that $P_i(t) > m$ ($i = 1, 2$) and $Z(t) > m$ as $t \to \infty$.

From system (3.4), we have $P_i(t) \leq \kappa_i$ ($i = 1, 2$) for t large enough. According to system (2.2), we have $\frac{dZ}{dt} \leq Z(t)(\varrho_1\kappa_1 + \varrho_2\kappa_2 - \mu - a_3Z(t-\tau))$. According to Lemma 2.4, we derive

$$Z(t) \leq \frac{(\varrho_1\kappa_1 + \varrho_2\kappa_2 - \mu)}{a_3} e^{(\varrho_1\kappa_1 + \varrho_2\kappa_2 - \mu)\tau} \stackrel{\Delta}{=} \vartheta. \tag{4.1}$$

From system (2.2), we get

$$\begin{cases} \frac{dP_1}{dt} \geq P_1(r_1 - a_1P_1^{\theta_1} - \alpha_1P_1\vartheta), \\ \frac{dP_2}{dt} \geq P_2(r_2 - a_2P_2^{\theta_2} - \alpha_2\vartheta). \end{cases} \tag{4.2}$$

Figure 1 Time series and phase portrait of the zooplankton-extinction periodic solution of system (2.2) with the parameters $r_1 = 1.5$, $a_1 = 0.2$, $\alpha_1 = 0.5$, $\theta_1 = 0.8$, $\theta_2 = 0.3$, $r_2 = 6$, $a_2 = 0.1$, $\alpha_2 = 0.1$, $\beta = 0.7$, $\varrho_1 = 0.1$, $\varrho_2 = 0.1$, $\tau = 0$, $a_3 = 0.8$, $K = 3$, $\mu = 0.8$, $d = 0.01$, $T = 2.42$.

We consider the comparison system:

$$
\begin{cases}
\frac{dv_1}{dt} = v_1(r_1 - \alpha_1\vartheta - a_1 v_1^{\theta_1}), & \\
\frac{dv_2}{dt} = v_2(r_2 - \alpha_2\vartheta - a_2 v_2^{\theta_2}), & t \neq nT, \\
\Delta v_1 = d(v_2 - v_1), & \\
\Delta v_2 = d(v_1 - v_2), & t = nT.
\end{cases}
\tag{4.3}
$$

From Lemma 2.3, we know that system (4.3) has a periodic solution $(v_1^*(t), v_2^*(t))$ as follows:
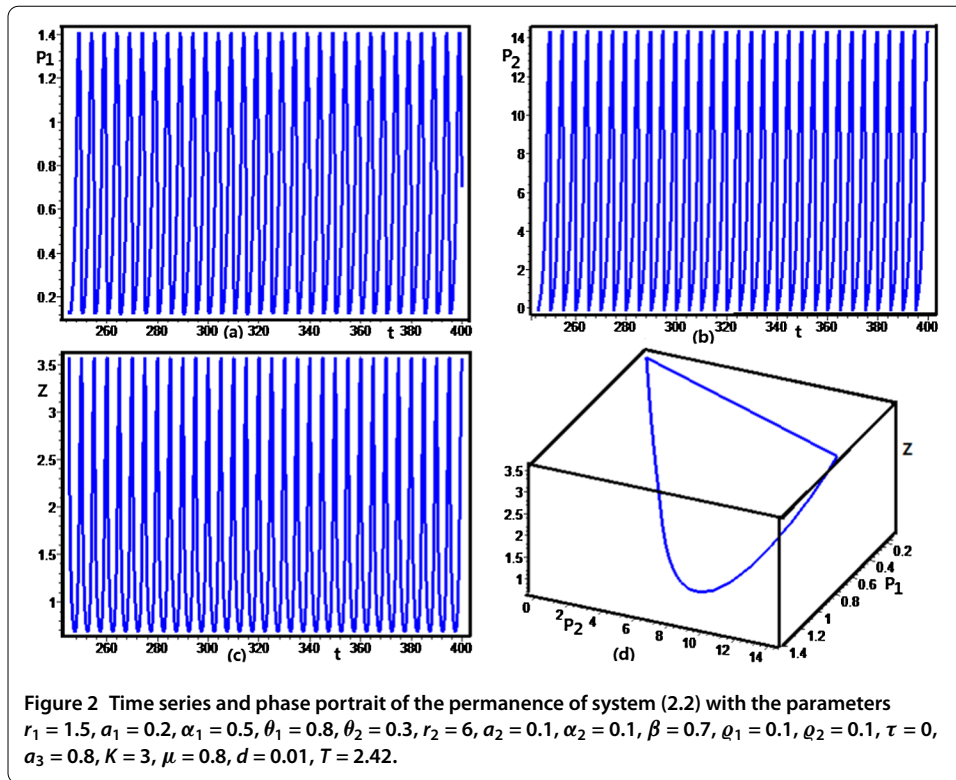
$$
v_i^*(t) = \left(\frac{a_i}{r_i - \alpha_i\vartheta} + \left(\frac{1}{\overline{q}_i^{\theta_i}} - \frac{a_i}{r_i - \alpha_i\vartheta} \right) e^{-(r_i - \alpha_i\vartheta)\theta_i(t - nT)} \right)^{-\frac{1}{\theta_i}}, \quad i = 1, 2,
\tag{4.4}
$$

which is globally asymptotically stable, where $\overline{q}_i$ can be obtained similar to Lemma 2.3. We obtain $P_i(t) \geq v_i(t)$ $(i = 1, 2)$ for $nT < t \leq (n+1)T$ by using the comparison theorem of the impulsive differential equations. Thus, there exists $\varepsilon_3 > 0$ such that

$$
P_i(t) \geq \left(\frac{a_i}{r_i - \alpha_i\vartheta} + \left(\frac{1}{\overline{q}_i^{\theta_i}} - \frac{a_i}{r_i - \alpha_i\vartheta} \right) e^{-(r_i - \alpha_i\vartheta)\theta_i(t - nT)} \right)^{-\frac{1}{\theta_i}} - \varepsilon_3 \overset{\Delta}{=} m_i \quad (i = 1, 2)
\tag{4.5}
$$

for t large enough.

In the following, we will show there exists a positive constant $m_3 > 0$ such that $Z(t) \geq m_3$ as $t \to \infty$.

Figure 2 Time series and phase portrait of the permanence of system (2.2) with the parameters $r_1 = 1.5$, $a_1 = 0.2$, $\alpha_1 = 0.5$, $\theta_1 = 0.8$, $\theta_2 = 0.3$, $r_2 = 6$, $a_2 = 0.1$, $\alpha_2 = 0.1$, $\beta = 0.7$, $\varrho_1 = 0.1$, $\varrho_2 = 0.1$, $\tau = 0$, $a_3 = 0.8$, $K = 3$, $\mu = 0.8$, $d = 0.01$, $T = 2.42$.

Again from system (2.2), we get

$$\frac{dZ}{dt} \geq Z\left(\varrho_1 m_1 + \varrho_2 m_2 - \mu - \frac{\beta M}{K+M} - a_3 Z(t-\tau) \right). \tag{4.6}$$

Considering the following comparison system:

$$\frac{dw}{dt} = w\left(\varrho_1 m_1 + \varrho_2 m_2 - \mu - \frac{\beta M}{K+M} - a_3 w(t-\tau) \right), \tag{4.7}$$

we get

$$Z(t) \geq w(t) \geq \frac{A}{a_3} e^{A - Ae^{A\tau}}, \tag{4.8}$$

where $A = \varrho_1 m_1 + \varrho_2 m_2 - \mu - \frac{\beta M}{K+M}$.

From system (4.5), we derive

$$m_i > \overline{q}_i^{\theta_i} - \varepsilon_3 \quad (i = 1, 2) \tag{4.9}$$

for t large enough. From system (4.9) and the condition

$$\theta_1 a_1 + \theta_2 a_2 > \mu + \frac{\beta M}{K+M},$$

we obtain $\varrho_1 m_1 + \varrho_2 m_2 - \mu - \frac{\beta M}{K+M} > 0$. Therefore, there exists $\varepsilon_4 > 0$ such that $Z(t) \geq \frac{A}{a_3} e^{A-Ae^{A\tau}} - \varepsilon_4 \overset{\Delta}{=} m_3$ as $t \to \infty$. Define $m = \min\{m_1, m_2, m_3\}$, we have $P_1(t) \geq m, P_2(t) \geq m, Z(t) \geq m$ holding for $t \to \infty$. The proof is completed. $\qquad\square$

5 Discussion

To investigate the effect of the phytoplankton diffusion on the dynamics, we formulate a phytoplankton-zooplankton model with impulsive diffusion. By using impulsive differential equations, we prove the zooplankton-extinction is globally attractive if $\sum_{i=1}^{i=1} \theta_i \left(\frac{a_i}{r_i} + \left(\frac{1}{\theta_i} \frac{1}{q_i} - \frac{a_i}{r_i} \right) e^{-r_i \theta_i T} \right)^{-\frac{1}{\theta_i}} < \mu$, which is simulated in Figure 1 with parameters $r_1 = 1.5$, $a_1 = 0.2$, $\alpha_1 = 0.5$, $\theta_1 = 0.8$, $\theta_2 = 0.3$, $r_2 = 6$, $a_2 = 0.1$, $\alpha_2 = 0.1$, $\beta = 0.7$, $\varrho_1 = 0.1$, $\varrho_2 = 0.1$, $\tau = 0$, $a_3 = 0.8$, $K = 3$, $\mu = 0.8$, $d = 0.01$, $T = 2.42$. The phytoplankton and zooplankton coexist when $\theta_1 a_1 + \theta_2 a_2 > \mu + \frac{\beta M}{K+M}$. Let parameters be $r_1 = 1.5$, $a_1 = 0.2$, $\alpha_1 = 0.8$, $\theta_1 = 0.8$, $\theta_2 = 0.3$, $r_2 = 6$, $a_2 = 0.1$, $\alpha_2 = 0.1$, $\beta = 0.5$, $\varrho_1 = 0.1$, $\varrho_2 = 0.1$, $\tau = 0$, $a_3 = 0.2$, $K = 3$, $\mu = 0.2$, $d = 0.01$, we can see the phytoplankton and zooplankton oscillate in an impulsive period, which shows phytoplankton and zooplankton are permanent (see Figure 2).

References
1. Jiang, ZH, Ma, WB, Li, D: Dynamical behavior of a delay differential equation system on toxin producing phytoplankton and zooplankton interaction. Jpn. J. Ind. Appl. Math. **31**, 583-609 (2014)
2. Javidi, M, Ahmad, B: Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system. Ecol. Model. **318**, 8-18 (2015)
3. Rehim, M, Imran, M: Dynamical analysis of a delay model of phytoplankton-zooplankton interaction. Appl. Math. Model. **36**, 638-647 (2012)
4. Zhang, ZZ, Rehim, M: Global qualitative analysis of a phytoplankton-zooplankton model in the presence of toxicity. Int. J. Dyn. Control **8**, 1-12 (2016)
5. Pal, R, Basu, D, Banerjee, M: Modelling of phytoplankton allelopathy with Monod-Haldane-type functional response - a mathematical study. Biosystems **95**, 243-253 (2009)
6. Jang, RJ, Allen, EJ: Deterministic and stochastic nutrient-phytoplankton-zooplankton models with periodic toxin producing phytoplankton. Appl. Math. Comput. **271**, 52-67 (2015)
7. Rao, F: The complex dynamics of a stochastic toxic-phytoplankton-zooplankton model. Adv. Differ. Equ. **2014**, 22 (2014)
8. Lv, YF, Yuan, R, Pei, YZ: Stable coexistence mediated by specialist harvesting in a two zooplankton-phytoplankton system. Appl. Math. Model. **37**, 9012-9030 (2013)
9. Chowdhury, T, Roy, S, Chattopadhyay, J: Modeling migratory grazing of zooplankton on toxic and non-toxic phytoplankton. Appl. Math. Comput. **197**, 659-671 (2008)
10. Ruan, SG: Persistence and coexistence in zooplankton-phytoplankton-nutrient models with instantaneous nutrient recycling. J. Math. Biol. **31**(6), 633-645 (1993)
11. Luo, J: Phytoplankton-zooplankton dynamics in periodic environments taking into account eutrophication. Math. Biosci. **245**, 126-136 (2013)
12. Hong, LL, Zhang, L, Teng, ZD, Jiang, YL: A periodic single species model with intermittent unilateral diffusion in two patches. J. Appl. Math. Comput. **12**, 1-12 (2015)
13. Jiao, JJ, Cai, SH, Li, LM: Impulsive vaccination and dispersal on dynamics of an SIR epidemic model with restricting infected individuals boarding transports. Physica A **449**, 145-159 (2016)
14. Jiao, JJ, Chen, LS, Cai, SH: Dynamical analysis on a single population model with state-dependent impulsively unilateral diffusion between two patches. Adv. Differ. Equ. **2012**, 155 (2012)

15. Sun, MJ, Liu, YL, Liu, SJ, Hu, ZL, Chen, LS: A novel method for analyzing the stability of periodic solution of impulsive state feedback model. Appl. Math. Comput. **273**(15), 425-434 (2016)
16. Shao, YF: Analysis of a delayed predator-prey system with impulsive diffusion between two patches. Math. Comput. Model. **52**, 120-127 (2010)
17. Yang, JL, Tang, SY: Effects of population dispersal and impulsive control tactics on pest management. Nonlinear Anal. Hybrid Syst. **3**, 487-500 (2009)
18. Wang, XH, Jia, JW: A single species model with impulsive diffusion and pulsed harvesting. Appl. Math. Sci. **8**(123), 6141-6149 (2014)
19. Wang, LM, Liu, ZJ, Hui, J, Chen, LS: Impulsive diffusion in single species model. Chaos Solitons Fractals **33**, 1213-1219 (2007)
20. Roy, SL, Alam, S, Chattopadhyay, J: Competing effects of toxin-producing phytoplankton on overall plankton populations in the Bay of Bengal. Bull. Math. Biol. **68**, 2303-2320 (2006)