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Analysis of a nonautonomous stochastic predator-prey model with Crowley-Martin functional response

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Abstract
The objective of this paper is to study some qualitative dynamic properties of a nonautonomous predator-prey model with stochastic perturbation and Crowley-Martin functional response. The existence of a global positive solution and stochastically ultimate boundedness are obtained. Sufficient conditions for extinction, persistence in the mean, and stochastic permanence of the system are established. We also derive conditions to guarantee the global attractiveness and stochastic persistence in probability of the model. Our theoretical results are confirmed by numerical simulations.

Keywords: stochastic nonautonomous model; persistence; extinction; functional response

1 Introduction

Predator-prey systems play an important role in studying the dynamics of interacting species. During the last decades, lots of predator-prey models have been proposed and analyzed from various prospectives. When investigating biological phenomena, functional response is one of the most important factors that affect dynamical properties of biological and mathematical models [1–4]. Many researchers have paid their attention to predator-prey systems with prey-dependent functional response. However, the predator functional response occurs quite frequently in nature and laboratory, such as searching for food and sharing, or competing for, food [5, 6]. Therefore, we must not ignore the predator functional response to prey because of the effect of such a response on dynamical system properties, and many types of predator-dependent functions have been proposed and analyzed. The deterministic predator-prey model with Crowley-Martin functional response can be expressed as follows:

$$\begin{aligned} \dot{x} &= x\left(r(t) - k(t)x - \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right), \\ \dot{y} &= y\left(-g(t) - h(t)y + \frac{f(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right), \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ represent the population densities of the prey and predator at time t, respectively. $r(t)$ and $g(t)$ are the growth rate of the prey and predator, respectively, $k(t)$ and

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$h(t)$ stand for the density-dependent coefficients of species x and y, $\omega(t)$ is the capturing rate of predator, and $f(t)$ denotes the rate of conversion of nutrients into the production of predator at time t. The ratio $\frac{\omega(t)x(t)y(t)}{1+a(t)x(t)+b(t)y(t)+a(t)b(t)x(t)y(t)}$ is the functional response, where $a(t)$ and $b(t)$ describe the effects of handling time and the magnitude of interference among predators.

Meanwhile, population models in the real world are always affected by a lot of unpredictably environmental noises. To predict richer and more complex dynamics of the model, stochastic perturbations are introduced into the population models (see, *e.g.*, [7–10]). In common, there are four approaches including stochastic effects in the model [11], that is, through time Markov chain model [12], parameter perturbation [13], being proportional to the variables [11], and robusting the positive equilibria of deterministic models. Stochastic perturbation will bring effect on almost all parameters of the model in various different ways, and it is valuable to consider more than one approach to describe the random effects on the system. In this paper, we adopt a combination of the second and third approaches to include stochastic perturbations, that is, we assume that the stochastic perturbations are of white noise type and proportional to $x(t)$, $y(t)$, influenced respectively on $\dot{x}(t)$ and $\dot{y}(t)$ in system (1); Moreover, the capturing and conversion rate coefficients $\omega(t)$ and $f(t)$ are changed as $\omega(t) + \sigma_2(t)\dot{B}_1(t)$, and $f(t) + \delta_2(t)\dot{B}_2(t)$, respectively. Then, in accordance with system (1), we propose the following stochastic predator-prey model:

$$\begin{aligned} \mathrm{d}x &= x\left(r(t) - k(t)x - \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}t \\ &\quad + x\left(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_1(t), \\ \mathrm{d}y &= y\left(-g(t) - h(t)y + \frac{f(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}t \\ &\quad + y\left(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_2(t), \end{aligned} \tag{2}$$

where all the coefficients are positive, continuous, and differentiable bounded functions on $\mathbb{R}_+ = [0, +\infty)$, $\sigma_i^2(t)$ and $\delta_i^2(t)$ $(i = 1, 2)$ denote the intensities of the white noises, $B_1(t)$, $B_2(t)$ are independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ satisfying the usual conditions (*i.e.*, it is right continuous and increasing with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets) [14]. We denote $\mathbb{R}_+^2 = \{X = (x, y) | x > 0, y > 0\}$ and $|X(t)| = (x^2(t) + y^2(t))^{\frac{1}{2}}$.

To proceed, we present some useful definitions and notations:

$f^u = \sup_{t\geq 0} f(t), f^l = \inf_{t\geq 0} f(t)$, $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s, f^* = \limsup_{t\to+\infty} f(t)$, $f_* = \liminf_{t\to+\infty} f(t)$.

Extinction: $\lim_{t\to+\infty} x(t) = 0$ *a.s.*

Non-persistence in the mean: $\langle x\rangle^* = 0$.

Weak persistence in the mean: $\langle x\rangle^* > 0$.

Strong persistence in the mean: $\langle x\rangle_* > 0$.

Stochastic permanence: *there are constants* $\delta > 0$, $\chi > 0$ *such that* $P_*\{|x(t)| \geq \delta\} \geq 1 - \varepsilon$ *and* $P_*\{|x(t)| \leq \chi\} \geq 1 - \varepsilon$.

This paper is arranged as follows. In Section 2, we show that there exists a unique positive solution of system (2) and prove its boundedness. In Section 3, we obtain sufficient conditions for extinction, persistence in the mean, and stochastic permanence. The global attractiveness and stochastic persistence in probability of system (2) are analyzed in Section 4. Finally, some numerical simulations to support our analytical findings are given in Section 5.

2 Existence, uniqueness, and stochastically ultimate boundedness

Theorem 1 *For any given value* $(x(0), y(0)) = X_0 \in \mathbb{R}_2^+$, *there is a unique solution* $(x(t), y(t))$ *on* $t \geq 0$, *and the solution remains in* $\mathbb{R}_+$ *with probability one.*

The proof of Theorem 1 is standard, and we present it in the Appendix.

Theorem 2 *The solutions of model* (2) *are stochastically ultimately bounded for any initial value* $X_0 = (x_0, y_0) \in \mathbb{R}_2^+$.

Proof We need to show that for any $\varepsilon \in (0,1)$, there exists a positive constant $\delta = \delta(\varepsilon)$ such that for any given initial value $X_0 \in \mathbb{R}_2^+$, the solution $X(t)$ to (2) has the property

$$\limsup_{t\to\infty} P\{|X(t)| > \delta\} < \varepsilon.$$

Let $V_1(x) = x^p$ and $V_2(y) = y^p$ for $(x,y) \in \mathbb{R}_2^+$ and $p > 1$. Then, we obtain

$$\begin{aligned}
\mathrm{d}(x^p) &= \left(px^{p-1}\,\mathrm{d}x + 0.5p(p-1)x^{p-2}(\mathrm{d}x)^2\right) \\
&= px^p\left(r(t) - k(t)x - \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}t \\
&\quad + 0.5p(p-1)x^p\left(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)^2\mathrm{d}t \\
&\quad + px^p\left(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_1(t) \\
&= LV_1(x,y)\,\mathrm{d}t + px^p\left(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_1(t)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}(y^p) &= py^p\left(-g(t) - h(t)y + \frac{f(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}t \\
&\quad + 0.5p(p-1)y^p\left(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)^2\mathrm{d}t \\
&\quad + py^p\left(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_2(t) \\
&= LV_2(x,y) + py^p\left(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_2(t).
\end{aligned}$$

Here, $LV_1(x,y) \leq px^p(r^u - k^l x + 0.5(p-1)(\sigma_1^u + \frac{\sigma_2^u}{b^l})^2)$, and $LV_2(x,y) \leq py^p(\frac{f^u}{a^l} + 0.5p(\delta_1^u + \frac{\delta_2^u}{a^l})^2 - h^l y)$. Thus,

$$\begin{aligned}\frac{\mathrm{d}E[x^p(t)]}{\mathrm{d}t} &\leq p\left\{\left[r^u + 0.5(p-1)\left(\sigma_1^u + \frac{\sigma_2^u}{b^l}\right)^2\right]E\left[x^p(t)\right] - k^l E\left[x^{p+1}(t)\right]\right\} \\ &\leq p\left\{\left[r^u + 0.5(p-1)\left(\sigma_1^u + \frac{\sigma_2^u}{b^l}\right)^2\right]E\left[x^p(t)\right] - k^l\left[E\left(x^p(t)\right)\right]^{1+\frac{1}{p}}\right\} \\ &\leq pE\left[x^p(t)\right]\left\{\left[r^u + 0.5p\left(\sigma_1^u + \frac{\sigma_2^u}{b^l}\right)^2\right] - k^l\left[E\left(x^p(t)\right)\right]^{\frac{1}{p}}\right\} \end{aligned} \tag{3}$$

and

$$\frac{\mathrm{d}E[y^p(t)]}{\mathrm{d}t} \leq pE\left[y^p(t)\right]\left\{\left[\frac{f^u}{a^l} + 0.5p\left(\delta_1^u + \frac{\delta_2^u}{b^l}\right)^2\right] - h^l\left[E\left(y^p(t)\right)\right]^{\frac{1}{p}}\right\}.$$

For (3), we consider the equation

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = pz(t)\left\{\left[r^u + 0.5p\left(\sigma_1^u + \frac{\sigma_2^u}{b^l}\right)^2\right] - k^l z(t)^{\frac{1}{p}}\right\}$$

with initial value $z(0) = z_0$. Obviously, we can obtain that

$$z(t) = \left(\frac{z_0^{1/p}(r^u + 0.5p(\sigma_1^u + \frac{\sigma_2^u}{b^l})^2)}{(r^u + 0.5p(\sigma_1^u + \frac{\sigma_2^u}{b^l})^2)e^{-(r^u + 0.5p(\sigma_1^u + \frac{\sigma_2^u}{b^l})^2)t} + k^l z_0^{1/p}(1 - e^{-(r^u + 0.5p(\sigma_1^u + \frac{\sigma_2^u}{b^l})^2)t})}\right)^p.$$

Letting $t \to \infty$, we have

$$\lim_{t\to\infty} z(t) = \left(\frac{r^u + 0.5p(\sigma_1^u + \frac{\sigma_2^u}{b^l})^2}{k^l}\right)^p.$$

Thus, by the comparison theorem we get

$$\limsup_{t\to\infty} Ex^p \leq \left(\frac{r^u + 0.5p(\sigma_1^u + \frac{\sigma_2^u}{b^l})^2}{k^l}\right)^p \triangleq G_1 < +\infty.$$

Similarly, we obtain

$$\limsup_{t\to\infty} Ey^p \leq \left(\frac{\frac{f^u}{a^l} + 0.5p(\delta_1^u + \frac{\delta_2^u}{b^l})^2}{h^l}\right)^p \triangleq G_2 < +\infty.$$

Thus, for a given constant $\varepsilon > 0$, there exists $T > 0$ such that

$$E\left[x^p(t)\right] \leq G_1 + \varepsilon \quad \text{and} \quad E\left[y^p(t)\right] \leq G_2 + \varepsilon$$

for all $t > T$. In accordance with the continuity of $E[x^p(t)]$ and $E[y^p(t)]$, there are $\overline{M}_1(p)$, $\overline{M}_2(p) > 0$ satisfying $E[x^p(t)] \leq \overline{M}_1(p)$ and $E[y^p(t)] \leq \overline{M}_2(p)$ for $t \leq T$. Denote

$$M_1(p) = \max\left\{\overline{M}_1(p), G_1 + \varepsilon\right\}, \qquad M_2(p) = \max\left\{\overline{M}_2(p), G_2 + \varepsilon\right\}.$$

Then, for all $t \in R_+$, we have

$$E\big[x^p(t)\big] \le M_1(p), \qquad E\big[y^p(t)\big] \le M_2(p).$$

Consequently,

$$E\big|X(t)\big|^p \le M_p < +\infty,$$

where $M_p = 2^{\frac{p}{2}}(M_1(p) + M_2(p))$. By virtue of the Chebyshev inequality, the proof is completed. □

3 Persistence and extinction

In this part, we show the long-time dynamical properties of system (2), including extinction, persistence in the mean, and stochastic permanence in Theorems 3-5. Before giving the theorems, we introduce some assumptions and lemmas.

In [15], the author considered the stochastic differential equation

$$\mathrm{d}x(t) = \operatorname{diag}\big(x_1(t),\dots,x_n(t)\big)\big(\big[b(t) + A(t)x(t)\big]\big)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}w(t), \tag{4}$$

where $x = (x_1,\dots,x_n)^T$, $b = (b_1,\dots,b_n)^T$, $A = (a_{ij})_{n\times n}$, $w(t) = (w_1(t),\dots,w_n(t))^T$, $\sigma(t) = (\sigma_{ij}(t))_{n\times n}$ and obtained the following theorem (Theorem 4.1 in [15]):

Suppose that all the parameters $b_i(t)$, $a_{ij}(t)$, and $\sigma_{ij}(t)$ $(1 \le i,j \le n)$ are bounded on $t \in \mathbb{R}_+$ and there exist positive numbers $c_1,\dots,c_n$ satisfying

$$-\lambda := \sup_{t\ge 0} \lambda^+_{\max}\big(\overline{C}A + A^T\overline{C}\big) < 0, \tag{5}$$

where $\overline{C} = \operatorname{diag}(c_1,\dots,c_n)$ and $\lambda^+_{\max}(A) = \sup_{x\in\mathbb{R}^n_+,|x|=1} x^TAx$. Then, for any initial value $x(0) \in \mathbb{R}^n_+$, the solution $x(t)$ of the SDE (4) *has the property*

$$\limsup_{t\to\infty} \frac{\ln(|x(t)|)}{\ln t} \le 1 \quad a.s.$$

Introducing an auxiliary matrix $\overline{A} = (\bar{a}_{ij})_{n\times n}$, where $\bar{a}_{ij} = \sup_{t\ge 0} \bar{a}_{ij}(t)$, $1 \le i,j \le n$, the author also achieved a more useful conclusion to verify condition (5):

If $-\overline{A}$ is a nonsingular M-matrix, then condition (5) *holds.*

Thus, we obtain the following lemma.

Assumption (H1) $k^l h^l > f^u \omega^u$.

Lemma 1 *If Assumption* (H1) *holds, then the solution $X(t) = (x(t), y(t))$ of system* (2) *with initial value $(x_0, y_0) \in \mathbb{R}^+_2$ has the following properties*:

$$\limsup_{t\to\infty} \frac{\ln x(t)}{\ln t} \le 1, \qquad \limsup_{t\to\infty} \frac{\ln y(t)}{\ln t} \le 1 \quad a.s., \tag{6}$$

and there is a positive constant K such that

$$\limsup_{t\to\infty} E\big(x(t)\big) \le K, \qquad \limsup_{t\to\infty} E\big(y(t)\big) \le K. \tag{7}$$

Proof By virtue of the useful conclusion obtained by Cheng [15], we achieve that under Assumption (H1), the conditions of Theorem 4.1 in [15] are satisfied. Then inequality (6) is proved.

Next, we turn to (7). Denote $V(x,y) = e^t(x+y)$. Then by Itô's formula we have

$$\begin{aligned} \mathrm{d}V(x,y) &= e^t\biggl\{x+y+x\biggl(r(t)-k(t)x-\frac{\omega(t)y}{1+a(t)x+b(t)y+a(t)b(t)xy}\biggr) \\ &\quad + y\biggl(-g(t)-h(t)y+\frac{f(t)x}{1+a(t)x+b(t)y+a(t)b(t)xy}\biggr)\biggr\}\,\mathrm{d}t \\ &\quad + e^t\biggl(\sigma_1(t)x+\frac{\sigma_2(t)xy}{1+a(t)x+b(t)y+a(t)b(t)xy}\biggr)\,\mathrm{d}B_1(t) \\ &\quad + e^t\biggl(\delta_1(t)y+\frac{\delta_2(t)xy}{1+a(t)x+b(t)y+a(t)b(t)xy}\biggr)\,\mathrm{d}B_2(t) \\ &= LV(x,y)\,\mathrm{d}t + e^t\biggl(\sigma_1(t)x+\frac{\sigma_2(t)xy}{1+a(t)x+b(t)y+a(t)b(t)xy}\biggr)\,\mathrm{d}B_1(t) \\ &\quad + e^t\biggl(\delta_1(t)y+\frac{\delta_2(t)xy}{1+a(t)x+b(t)y+a(t)b(t)xy}\biggr)\,\mathrm{d}B_2(t). \end{aligned}$$

Here,

$$LV(x,y) \le e^t\biggl(x+y+r^u x-k^l x^2-g^l y-h^l y^2+\frac{f^u}{a^l b^l}\biggr) \le C,$$

where $C > 0$ is a constant. Therefore, $\limsup_{t\to\infty} E(V(x(t),y(t))) \le C$, and (7) is proved. □

Consider the ordinary differential equations

$$\begin{aligned} \mathrm{d}\bar{x} &= \bar{x}\bigl(r(t)-k(t)\bar{x}\bigr)\,\mathrm{d}t + \bar{x}\biggl(\sigma_1(t)+\frac{\sigma_2(t)\bar{y}}{1+a(t)\bar{x}+b(t)\bar{y}+a(t)b(t)\bar{x}\bar{y}}\biggr)\,\mathrm{d}B_1(t), \\ \mathrm{d}\bar{y} &= \bar{y}\biggl(-h(t)\bar{y}+\frac{f(t)}{a(t)}\biggr)\,\mathrm{d}t + \bar{y}\biggl(\delta_1(t)+\frac{\delta_2(t)\bar{x}}{1+a(t)\bar{x}+b(t)\bar{y}+a(t)b(t)\bar{x}\bar{y}}\biggr)\,\mathrm{d}B_2(t). \end{aligned} \tag{8}$$

Then we have the following results on the persistence and extinction of the populations.

Theorem 3 *In system* (2), *for the prey population* x, *if Assumption* (H1) *holds, then the following conclusions hold*:

(1) *If* $\langle r_1\rangle^* < 0$, *then the prey species* x *ends in extinction with probability* 1, *where* $r_1(t) = r(t) - 0.5\sigma_1^2(t)$.
(2) *If* $\langle r_1\rangle^* = 0$, *then the prey species* x *is nonpersistent in the mean with probability* 1.
(3) *If* $\langle r - 0.5(\sigma_1 + \frac{\sigma_2}{b})^2\rangle^* > 0$, *then the prey species* x *is weakly persistent in the mean with probability* 1.
(4) *If* $\langle r - 0.5(\sigma_1 + \frac{\sigma_2}{b})^2\rangle_* - \langle\frac{\omega}{b}\rangle^* > 0$, *then the species population* x *is strongly persistent in the mean with probability* 1.
(5) *If* $\langle r_1\rangle^* > 0$, *then* $\langle x(t)\rangle^* \le \frac{r^u}{k^l} \triangleq M_x$.

The proof of Theorem 3 is presented in the Appendix.

Theorem 4 *In system* (2), *for the predator population* y, *if Assumption* (H1) *holds, then the following conclusions hold*:

(1) *If* $k_*\langle -g-0.5\delta_1^2\rangle^* + f^*\langle r_1\rangle^* < 0$, *then the predator species y is extinct with probability* 1.
(2) *If* $k_*\langle -g-0.5\delta_1^2\rangle^* + f^*\langle r_1\rangle^* = 0$, *then the predator species y is nonpersistent in the mean with probability* 1.
(3) *If* $\langle -g-0.5(\delta_1+\frac{\delta_2}{a})^2\rangle^* + \langle\frac{f\bar{x}}{1+a\bar{x}+b\bar{y}+ab\bar{x}\bar{y}}\rangle^* - f^u\frac{4b^l\langle\sigma_1\sigma_2\rangle^*+\langle\sigma_2^2\rangle^*}{(b^l)^2k^l} > 0$, *then the predator species y is weakly persistent in the mean with probability* 1, *where* $(\bar{x}(t),\bar{y}(t))$ *is the solution of* (8) *with initial value* $(x_0,y_0)\in\mathbb{R}_+^2$.
(4) *If* $f^l\langle r-0.5(\sigma_1+\frac{\sigma_2}{b})^2\rangle_* + k^u\langle -g-0.5(\delta_1+\frac{\delta_2}{a})^2\rangle^* > 0$, *then the species population y is strongly persistent in the mean with probability* 1.
(5) *If* $\langle -g-0.5\delta_1^2\rangle^* + \langle\frac{f}{a}\rangle^* > 0$, *then* $\langle y(t)\rangle^* \le \frac{\langle -g-0.5\delta_1^2\rangle^*+\langle\frac{f}{a}\rangle^*}{h^l} \triangleq M_y$.

The proof of Theorem 4 is given in the Appendix.

Remark 1 From the proof of Theorem 4 we can observe that if $\langle r_1\rangle^* > 0$ and $k_*\langle -g-0.5\delta_1^2\rangle^* + f^*\langle r_1\rangle^* < 0$, then although the prey population survives, the predators die out because of the too large diffusion coefficient δ_1^2.

Remark 2 From Theorems 3 and 4 we derive that if $\langle r_1\rangle^* < 0$, then both the prey and predator populations eventually end in extinction. Meanwhile, in this case, the functional rate has no influence on the extinction of the system.

Theorem 5 *Suppose that* $2(\max\{\sigma_1^u,\frac{\sigma_2^u}{b^l},\delta_1^u,\frac{\delta_2^u}{a^l}\})^2 < \min\{r^l-\frac{\omega^u}{b^l},\frac{f^l}{a^u}-g^u\}$, *then system* (2) *is stochastically permanent.*

Proof The proof is motivated by Li and Mao [16] and Liu and Wang [17]. The whole proof is divided into two parts. First, we prove that for arbitrary $\varepsilon > 0$, there exists a constant $\delta > 0$ such that $P_*\{|x(t)| \ge \delta\} \ge 1-\varepsilon$.

Above all, we claim that for any initial value $X(0) = (x(0),y(0)) \in \mathbb{R}_2^+$, the solution $X(t) = (x(t),y(t))$ satisfies

$$\limsup_{t\to\infty} E\left(\frac{1}{|X(t)|^\theta}\right) \le M.$$

Here, θ is an arbitrary positive constant satisfying

$$2(\theta+1)\left(\max\left\{\sigma_1^u,\frac{\sigma_2^u}{b^l},\delta_1^u,\frac{\delta_2^u}{a^l}\right\}\right)^2 < \min\left\{r^l-\frac{\omega^u}{b^l},\frac{f^l}{a^u}-g^u\right\}. \tag{9}$$

By (9) there exists a constant $p > 0$ satisfying

$$\min\left\{r^l-\frac{\omega^u}{b^l},\frac{f^l}{a^u}-g^u-2(\theta+1)\left(\max\left\{\sigma_1^u,\frac{\sigma_2^u}{b^l},\delta_1^u,\frac{\delta_2^u}{a^l}\right\}\right)^2-p\right\} > 0. \tag{10}$$

Define $V(x,y) = x+y$. Then

$$\begin{aligned} dV(x,y) = {}& \left\{x\left(r(t)-k(t)x-\frac{\omega(t)y}{1+a(t)x+b(t)y+a(t)b(t)xy}\right)\right. \\ & \left.+y\frac{f(t)x}{1+a(t)x+b(t)y+a(t)b(t)xy}\right\}dt \end{aligned}$$

$$+ y\big(-g(t) - h(t)y\big)\,\mathrm{d}t + \left(\sigma_1(t)x + \frac{\sigma_2(t)xy}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_1(t)$$
$$+ \left(\delta_1(t)y + \frac{\delta_2(t)xy}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_2(t).$$

Letting $U(x,y) = \frac{1}{V(x,y)}$, by Itô's formula we obtain

$$\begin{aligned}
\mathrm{d}U(X) = {} & \Bigg\{-U^2(X)\Bigg(x\Bigg(r(t) - k(t)x - \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\Bigg) \\
& + y\frac{f(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\Bigg)\Bigg\}\,\mathrm{d}t - U^2(X)y\big(-g(t) - h(t)y\big)\,\mathrm{d}t \\
& + U^3(X)x^2\left(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)^2\mathrm{d}t \\
& + U^3(X)y^2\left(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)^2\mathrm{d}t \\
& - U^2(X)\Bigg(\left(\sigma_1(t)x + \frac{\sigma_2(t)xy}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_1(t) \\
& + \left(\delta_1(t)y + \frac{\delta_2(t)xy}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_2(t)\Bigg) \\
= {} & LU(X)\,\mathrm{d}t - U^2(X)\left(\sigma_1(t)x + \frac{\sigma_2(t)xy}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_1(t) \\
& - U^2(X)\left(\delta_1(t)y + \frac{\delta_2(t)xy}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_2(t).
\end{aligned}$$

Choose a positive constant θ such that it obeys (9). Then

$$\begin{aligned}
L\big(1 + U(X)\big)^\theta = {} & \theta\big(1 + U(X)\big)^{\theta-1}LU(X) + \frac{1}{2}\theta(\theta - 1)\big(1 + U(X)\big)^{\theta-2}U^4(X) \\
& \times \Bigg(x^2\left(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)^2 \\
& + y^2\left(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)^2\Bigg).
\end{aligned}$$

Thus, we can choose $p > 0$ sufficiently small such that it satisfies (10). Denote $W(X) = e^{pt}(1 + U(X))^\theta$,

$$\begin{aligned}
LW(X) = {} & pe^{pt}\big(1 + U(X)\big)^\theta + e^{pt}L\big(1 + U(X)\big)^\theta \\
= {} & e^{pt}\big(1 + U(X)\big)^{\theta-2}\Bigg(P\big(1 + U(X)\big)^2 \\
& - \theta U^2(X)x\left(r(t) - k(t)x - \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right) \\
& - \theta U^2(X)y\left(-g(t) - h(t)y + \frac{f(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right) \\
& - \theta U^3(X)y\big(-g(t) - h(t)y\big) \\
& - \theta U^3(X)\Bigg(x\left(r(t) - k(t)x - \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{f(t)xy}{1 + a(t)x + b(t)y + a(t)b(t)xy}\biggr) \\
&+ \theta U^3(X)\biggl(x^2\biggl(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\biggr)^2 \\
&+ y^2\biggl(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\biggr)^2\biggr) \\
&+ \frac{\theta(\theta+1)}{2}U^4(X)\biggl(x^2\biggl(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\biggr)^2 \\
&+ y^2\biggl(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\biggr)^2\biggr)\biggr).
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
&\theta U^3(X)\biggl(x^2\biggl(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\biggr)^2 \\
&\quad + y^2\biggl(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\biggr)^2\biggr) \\
&\quad \leq \theta U(X)\biggl(2\max\biggl\{\sigma_1^u, \frac{\sigma_2^u}{b^l}, \delta_1^u, \frac{\delta_2^u}{a^l}\biggr\}\biggr)^2; \\
&\frac{\theta(\theta+1)}{2}U^4(X)\biggl(x^2\biggl(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\biggr)^2 \\
&\quad + y^2\biggl(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\biggr)^2\biggr) \\
&\quad \leq \frac{\theta(\theta+1)}{2}U^2(X)\biggl(2\max\biggl\{\sigma_1^u, \frac{\sigma_2^u}{b^l}, \delta_1^u, \frac{\delta_2^u}{a^l}\biggr\}\biggr)^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
LW(X) \leq{}& e^{pt}\bigl(1 + U(X)\bigr)^{\theta-2}\biggl\{\bigl(p + \theta\max\{k^u, h^u\}\bigr) + \biggl(2p - \theta\min\biggl\{r^l - \frac{\omega^u}{b^l}, \frac{f^l}{a^u} - g^u\biggr\} \\
&+ \theta\max\{k^u, h^u\} + 4\theta\biggl(\max\biggl\{\sigma_1^u, \frac{\sigma_2^u}{b^l}, \delta_1^u, \frac{\delta_2^u}{a^l}\biggr\}\biggr)^2\biggr)U(X) \\
&+ \biggl(p - \theta\min\biggl\{r^l - \frac{\omega^u}{b^l}, \frac{f^l}{a^u} - g^u\biggr\} \\
&+ \frac{\theta(\theta+1)}{2}\biggl(2\max\biggl\{\sigma_1^u, \frac{\sigma_2^u}{b^l}, \delta_1^u, \frac{\delta_2^u}{a^l}\biggr\}\biggr)^2\biggr)U^2(X)\biggr\}.
\end{aligned}
$$

By (10) there exists a positive constant S such that $LW(X) \leq Se^{pt}$. Thus,

$$
E\bigl[e^{pt}\bigl(1 + U(X)\bigr)^\theta\bigr] \leq \bigl(1 + U(0)\bigr)^\theta + \frac{S(e^{pt} - 1)}{p}.
$$

Then,

$$
\limsup_{t\to\infty} E\bigl[U^\theta\bigl(X(t)\bigr)\bigr] \leq \limsup_{t\to\infty} E\bigl[\bigl(1 + U\bigl(X(t)\bigr)\bigr)^\theta\bigr] \leq \frac{S}{p}.
$$

In other words,

$$\limsup_{t\to\infty} E\left[\frac{1}{|X(t)|^{\theta}}\right] \le 2^{\theta} \limsup_{t\to\infty} EU^{\theta}(X) \le 2^{\theta}\frac{S}{p} := M.$$

Thus, for any $\varepsilon > 0$, letting $\delta = (\frac{\varepsilon}{M})^{\frac{1}{\theta}}$, by the Chebyshev inequality we obtain

$$P\{|X(t)| < \delta\} = P\{|X(t)|^{-\theta} > \delta^{-\theta}\} \le E[|X(t)|^{-\theta}]/\delta^{-\theta} = \delta^{\theta} E[|X(t)|^{-\theta}].$$

Therefore,

$$P_*\{|X(t)| \ge \delta\} \ge 1 - \varepsilon.$$

In the following, we prove that for any $\varepsilon > 0$, there exists a constant $\chi > 0$ such that $P_*\{|X(t)| \le \chi\} \ge 1 - \varepsilon$. Define $V_4(X) = x^q + y^q$, where, $0 < q < 1$ and $X = (x, y) \in \mathbb{R}_2^+$. By Itô's formula we have

$$\begin{aligned} \mathrm{d}V(X(t)) = {}& qx^q\left(r(t) - k(t)x - \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right. \\ & \left. + \frac{q-1}{2}\left(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)^2\right)\mathrm{d}t \\ & + qy^q\left(-g(t) - h(t)y + \frac{f(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right. \\ & \left. + \frac{q-1}{2}\left(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)^2\right)\mathrm{d}t \\ & + qx^q\left(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_1(t) \\ & + qy^q\left(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_2(t). \end{aligned}$$

Let k_0 be so large that X_0 lies within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \ge k_0$, we define the stopping time $\tau_k = \inf\{t \ge 0 : X(t) \notin (1/k, k)\}$. Obviously, τ_k increases as $k \to \infty$. Therefore,

$$\begin{aligned} & E[\exp\{t \wedge \tau_k\} X^q(t \wedge \tau_k)] - X^q(0) \\ & \quad \le qE\int_0^{t\wedge\tau_k} \exp\{s\}x^q(s)\left(1 + q\left(r(s) - k(s)x(s) - \frac{1-q}{2}\sigma_1^2(s)\right)\right)\mathrm{d}s \\ & \qquad + qE\int_0^{t\wedge\tau_k} \exp\{s\}y^q(s)\left(1 + q\left(-g(s) - h(s)y(s) + \frac{f(s)}{a(s)} - \frac{1-q}{2}\delta_1^2(s)\right)\right)\mathrm{d}s \\ & \quad \le E\int_0^{t\wedge\tau_k} (K_1 + K_2)\exp\{s\}\,\mathrm{d}s \\ & \quad \le (K_1 + K_2)(\exp\{t\} - 1), \end{aligned}$$

where K_1, K_2 are positive constants. Letting $k \to +\infty$, we have

$$\exp\{t\}E[X^q(t)] \le X^q(0) + (K_1 + K_2)(\exp\{t\} - 1).$$

In other words, we have shown that $\limsup_{t\to+\infty} E[X^q(t)] \le K_1 + K_2$. Thus, for any given $\varepsilon > 0$, choosing $\chi = \frac{(K_1+K_2)^{1/q}}{\varepsilon^{1/q}}$, by the Chebyshev inequality we get

$$P\{|X(t)| > \chi\} = P\{|X(t)|^q > \chi^q\} \le E[|X(t)|^q]/\chi^q,$$

that is,

$$P_*\{|X(t)| > \chi\} \le E[|X(t)|^q]/\chi^q \le \varepsilon.$$

Consequently, $P_*\{|X(t)| \le \chi\} \ge 1 - \varepsilon$.

Theorem 5 is proved. □

4 Global attractiveness of the system and stochastically persistent in probability

Definition 1 System (2) is globally attractive if

$$\lim_{t\to+\infty} |x_1(t) - x_2(t)| = 0, \qquad \lim_{t\to+\infty} |y_1(t) - y_2(t)| = 0$$

for any two positive solutions $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$ of system (2).

Theorem 6 *Suppose that* $(x(t), y(t))$ *is a solution of system* (2) *on* $t \ge 0$ *with initial value* $(x_0, y_0) \in \mathbb{R}^2_+$. *Then almost every sample path of* $(x(t), y(t))$ *is uniformly continuous.*

Proof From system (2) we have

$$\begin{aligned} x(t) = x_0 &+ \int_0^t x(s)\left(r(s) - k(s)x(s) - \frac{\omega(s)y(s)}{1 + a(s)x(s) + b(s)y(s) + a(s)b(s)x(s)y(s)}\right) \mathrm{d}s \\ &+ \int_0^t x(s)\left(\sigma_1(s) + \frac{\sigma_2(s)y(s)}{1 + a(s)x(s) + b(s)y(s) + a(s)b(s)x(s)y(s)}\right) \mathrm{d}B_1(s). \end{aligned}$$

Set

$$\begin{aligned} f_1(s) &= x(s)\left(r(s) - k(s)x(s) - \frac{\omega(s)y(s)}{1 + a(s)x(s) + b(s)y(s) + a(s)b(s)x(s)y(s)}\right), \\ f_2(s) &= x(s)\left(\sigma_1(s) + \frac{\sigma_2(s)y(s)}{1 + a(s)x(s) + b(s)y(s) + a(s)b(s)x(s)y(s)}\right), \end{aligned}$$

we obtain

$$\begin{aligned} E|f_1(t)|^p &= E\left|x\left(r(t) - k(t)x - \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\right|^p \\ &= E\left[|x|^p\left|r(t) - k(t)x - \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right|^p\right] \\ &\le \frac{1}{2}E|x|^{2p} + \frac{1}{2}E\left|r^u + k^u x + \omega^u y\right|^{2p} \\ &\le \frac{1}{2}E|x|^{2p} + \frac{1}{2}3^{2p-1}\left[\left(r^u\right)^{2p} + \left(k^u\right)^{2p}E|x|^{2p} + \left(\omega^u\right)^{2p}E|y|^{2p}\right] \end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{2}M_1(2p) + \frac{3^{2p-1}}{2}\big[\big(r^u\big)^{2p} + \big(k^u\big)^{2p}M_1(2p) + \big(\omega^u\big)^{2p}M_2(2p)\big]\\&\triangleq F_1(p)\end{aligned}$$

and

$$\begin{aligned}E\big|f_2(t)\big|^p &= E\left|x\left(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\right|^p\\&\leq \left(\sigma_1^u + \frac{\sigma_2^u}{b^l}\right)^P E|x|^p \leq \left(\sigma_1^u + \frac{\sigma_2^u}{b^l}\right)^P M_1(p)\\&\triangleq F_2(p).\end{aligned}$$

In addition, in view of the moment inequality for stochastic integrals, we show that, for $0 \leq t_1 \leq t_2$ and $p > 2$,

$$\begin{aligned}E\left|\int_{t_1}^{t_2} f_2(s)\,\mathrm{d}B_1(s)\right|^p &\leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(t_2 - t_1)^{\frac{p-2}{2}}\int_{t_1}^{t_2} E\big|f_2(s)\big|^p\,\mathrm{d}s\\&\leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(t_2 - t_1)^{\frac{p}{2}}F_2(p).\end{aligned}$$

Thus, for $0 < t_1 < t_2 < \infty$, $t_2 - t_1 \leq 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\begin{aligned}E\big|x(t_2) - x(t_1)\big|^p &= E\left|\int_{t_1}^{t_2} f_1(s)\,\mathrm{d}s + \int_{t_1}^{t_2} f_2(s)\,\mathrm{d}B_1(s)\right|^p\\&\leq 2^{p-1}E\left|\int_{t_1}^{t_2} f_1(s)\,\mathrm{d}s\right|^p + 2^{p-1}E\left|\int_{t_1}^{t_2} f_2(s)\,\mathrm{d}B_1(s)\right|^p\\&\leq 2^{p-1}(t_2 - t_1)^{\frac{p}{q}}E\left[\int_{t_1}^{t_2}\big|f_1(s)\big|^p\,\mathrm{d}s\right] + 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(t_2 - t_1)^{\frac{p}{2}}F_2(p)\\&\leq 2^{p-1}(t_2 - t_1)^{\frac{p}{q}}F_1(p)(t_2 - t_1) + 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(t_2 - t_1)^{\frac{p}{2}}F_2(p)\\&= 2^{p-1}(t_2 - t_1)^p F_1(p) + 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(t_2 - t_1)^{\frac{p}{2}}F_2(p)\\&= 2^{p-1}(t_2 - t_1)^{\frac{p}{2}}\left\{(t_2 - t_1)^{\frac{p}{2}}F_1(p) + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}F_2(p)\right\}\\&\leq 2^{p-1}(t_2 - t_1)^{\frac{p}{2}}\left\{1 + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right\}F(p).\end{aligned}$$

Here, $F(p) = \max\{F_1(p), F_2(p)\}$. By Lemma 3 in [18, 19] we have that almost every sample path of $x(t)$ is locally but uniformly Hölder-continuous with exponent υ for every $\upsilon \in (0, \frac{p-2}{2p})$, and therefore, almost every sample path of $x(t)$ is uniformly continuous on $t \in \mathbb{R}_+$. Similarly, we can prove that almost every sample path of $y(t)$ is also uniformly continuous on $t \in \mathbb{R}_+$.

Theorem 6 is proved. □

Lemma 2 ([18, 19]) *Let f be a nonnegative function defined on $\mathbb{R}_+$ such that f is integrable and uniformly continuous. Then* $\lim_{t\to+\infty} f(t) = 0$.

Theorem 7 *Suppose that $\sigma_2 = 0$, $\delta_2 = 0$, and there exist constants $\mu_i > 0$ ($i = 1, 2$) such that $\liminf_{t\to\infty} A_i(t) > 0$, where*

$$\begin{aligned} A_1(t) &= \mu_1\left[k(t) - 2\frac{\omega(t)a(t)}{b(t)}\right] - 2\mu_2 f(t), \\ A_2(t) &= \mu_2\left[h(t) - 2\frac{f(t)b(t)}{a(t)}\right] - 2\mu_1\omega(t). \end{aligned} \tag{11}$$

Then

$$\lim_{t\to\infty} E\bigl|x_1(t) - x_2(t)\bigr| = 0, \qquad \lim_{t\to\infty} E\bigl|y_1(t) - y_2(t)\bigr| = 0, \tag{12}$$

where $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$ are any two solutions of model (2) *with initial values $X_{10} = (x_1(0), y_1(0)) \in R_+^2$ and $X_{20} = (x_2(0), y_2(0)) \in R_+^2$. Moreover, model* (2) *is globally attractive.*

Proof We construct a Lyapunov function as follows:

$$V(t) = \mu_1\bigl|\ln x_1(t) - \ln x_2(t)\bigr| + \mu_2\bigl|\ln y_1(t) - \ln y_2(t)\bigr|.$$

Then, we achieve

$$\begin{aligned} D^+\bigl(V(t)\bigr) ={}& \mu_1 \operatorname{sgn}(x_1 - x_2) \\ &\times \left(\left(r(t) - k(t)x_1 - \frac{\omega(t)y_1}{1 + a(t)x_1 + b(t)y_1 + a(t)b(t)x_1y_1} - 0.5\sigma_1^2(t)\right)\mathrm{d}t\right. \\ &\left. - \left(r(t) - k(t)x_2 - \frac{\omega(t)y_2}{1 + a(t)x_2 + b(t)y_2 + a(t)b(t)x_2y_2} - 0.5\sigma_1^2(t)\right)\mathrm{d}t\right) \\ &+ \mu_2 \operatorname{sgn}(y_1 - y_2) \\ &\times \left(\left(-g(t) - h(t)y_1 + \frac{f(t)x_1}{1 + a(t)x_1 + b(t)y_1 + a(t)b(t)x_1y_1} - 0.5\delta_1^2(t)\right)\mathrm{d}t\right. \\ &\left. - \left(-g(t) - h(t)y_2 + \frac{f(t)x_2}{1 + a(t)x_2 + b(t)y_2 + a(t)b(t)x_2y_2} - 0.5\delta_1^2(t)\right)\mathrm{d}t\right) \\ \le{}& \mu_1 \operatorname{sgn}(x_1 - x_2)\left(-k(t)(x_1 - x_2) + \omega(t)\left(\frac{y_2}{1 + a(t)x_2 + b(t)y_2 + a(t)b(t)x_2y_2}\right.\right. \\ &\left.\left. - \frac{y_1}{1 + a(t)x_1 + b(t)y_1 + a(t)b(t)x_1y_1}\right)\right)\mathrm{d}t \\ &+ \mu_2 \operatorname{sgn}(y_1 - y_2)\left(-h(t)(y_1 - y_2) + f(t)\left(\frac{x_1}{1 + a(t)x_1 + b(t)y_1 + a(t)b(t)x_1y_1}\right.\right. \\ &\left.\left. - \frac{x_2}{1 + a(t)x_2 + b(t)y_2 + a(t)b(t)x_2y_2}\right)\right)\mathrm{d}t \\ \le{}& -\left(\mu_1\left(k(t) - 2\frac{\omega(t)a(t)}{b(t)}\right) - 2\mu_2 f(t)\right)|x_1 - x_2| \\ &- \left(\mu_2\left(h(t) - 2\frac{f(t)b(t)}{a(t)}\right) - 2\mu_1\omega(t)\right)|y_1 - y_2|. \end{aligned}$$

Since $\liminf_{t\to\infty} A_i(t) > 0$ $(i = 1,2)$, there exist constants $\alpha > 0$ and $T_0 > 0$ such that $A_i(t) \geq \alpha$ $(i = 1,2)$ for all $t \geq T_0$. Thus,

$$D^+\bigl(V(t)\bigr) \leq -\alpha\bigl(|x_1 - x_2| + |y_1 - y_2|\bigr) \tag{13}$$

for all $t \geq T_0$. Integrating (13) from T_0 to t, we get

$$V(t) - V(T_0) \leq -\alpha \int_{T_0}^{t} \bigl(\bigl|x_1(s) - x_2(s)\bigr| + \bigl|y_1(s) - y_2(s)\bigr|\bigr)\,\mathrm{d}s,$$

that is,

$$V(t) + \alpha \int_{T_0}^{t} \bigl(\bigl|x_1(s) - x_2(s)\bigr| + \bigl|y_1(s) - y_2(s)\bigr|\bigr)\,\mathrm{d}s \leq V(T_0) < +\infty. \tag{14}$$

Then, by $V(t) \geq 0$ and (14) we have

$$\bigl|x_1(t) - x_2(t)\bigr| \in L^1[0,+\infty), \qquad \bigl|y_1(t) - y_2(t)\bigr| \in L^1[0,+\infty). \tag{15}$$

According to Theorem 6 and Lemma 2, the model is globally attractive.

On the other hand, by system (2) and inequality (7) we have

$$\begin{aligned}\frac{\mathrm{d}E(x(t))}{\mathrm{d}t} &\leq r^u E\bigl(x(t)\bigr) - k^l E\bigl(x(s)\bigr)^2 - \omega^u E\bigl(x(t)y(t)\bigr)\\ &\leq r^u E\bigl(x(t)\bigr) \leq r^u K.\end{aligned}$$

Therefore, $E(x(t))$ is a uniformly continuous function. Similarly, we can obtain that $E(y(t))$ is uniformly continuous. According to (15) and Barbalat's conclusion [20], assertion (12) is achieved. □

In the following, we discuss the stochastic persistence in probability of our model, which was proposed and discussed by Schreiber *et al.* [21] and Liu *et al.* [22] for

$$\mathrm{d}X_i(t) = X_i(t)F_i\bigl(X(t)\bigr)\,\mathrm{d}t + \sum_{j=1}^{m} \Theta_i^j\bigl(X(t)\bigr)\,\mathrm{d}B^j(t), \quad i = 1,2,\ldots,n, \tag{16}$$

where $X(t) = (X_1(t),\ldots,X_n(t))$. If there exists a unique invariant probability measure V satisfying $V(\Delta_0) = 0$ and the distribution of $X(t)$ converges to V as $t \to +\infty$ whenever $X(0) \in \mathbb{R}_+^n$, where $\Delta_0 = \{a \in \overline{\mathbb{R}}_+^n | a_i = 0 \text{ for some } i, 1 \leq i \leq n\}$, then (16) is stochastically persistent in probability.

Theorem 8 *Suppose that* $\sigma_2 = 0$ *and* $\delta_2 = 0$ *and let the conditions of Theorem* 7 *hold. If* $\langle r - 0.5(\sigma_1 + \frac{\sigma_2}{b})^2\rangle_* > \min\{\langle\frac{\omega}{b}\rangle^*, -\frac{k^u}{f^l}\langle -g - 0.5(\delta_1 + \frac{\delta_2}{a})^2\rangle^*\}$, *then system* (2) *is stochastically persistent in probability.*

Proof The proof is motivated by [22]. First, we prove that system (2) is asymptotically stable in distribution, that is, there exists a unique probability measure μ such that for every $X(0) \in \mathbb{R}_+^2$, the transition probability $p(t, X(0), \cdot)$ of $X(t)$ converges weakly to μ as $t \to +\infty$.

Let $X(t;X_{10})$ be a solution of (2) with initial value $X_1(0) = X_{10} \in \mathbb{R}^2_+$, $p(t,X_{10},\mathrm{d}y)$ be the transition probability of $X(t;X_{10})$, and $P(t,X_{10},B)$ denote the probability of event $X(t;X_{10}) \in B$. Applying inequality (7) and the Chebyshev inequality, $\{p(t,X_{10},\mathrm{d}y) : t \geq 0\}$ is tight.

Denote all the probability measures on $\mathbb{R}^2_+$ by $\mathcal{P}(\mathbb{R}^2_+)$. Then, for any $P_1, P_2 \in \mathcal{P}$, we can define the metric

$$\mathrm{d}_L(P_1,P_2) = \sup_{f\in L}\left|\int_{\mathbb{R}^2_+} f(x)P_1(\mathrm{d}x) - \int_{\mathbb{R}^2_+} f(x)P_2(\mathrm{d}x)\right|,$$

where $L = \{f : \mathbb{R}^2_+ \to \mathbb{R} \mid |f(x) - f(y)| \leq \|x - y\|, |f(\cdot)| \leq 1\}$. For $f \in L$ and $t, s > 0$, we have

$$\begin{aligned}
&\bigl|Ef\bigl(x(t+s;X_{10})\bigr) - Ef\bigl(x(t;X_{10})\bigr)\bigr| \\
&\quad = \bigl|E\bigl[E\bigl(f\bigl(x(t+s;X_{10})\bigr)\bigr)|\mathcal{F}_s\bigr] - Ef\bigl(x(t;X_{10})\bigr)\bigr| \\
&\quad = \left|\int_{\mathbb{R}^2_+} Ef\bigl(x(t;X_{10})\bigr)p(s,X_{10},\mathrm{d}X_{10}) - Ef\bigl(x(t;X_{10})\bigr)\right| \\
&\quad \leq \int_{\mathbb{R}^2_+} \bigl|Ef\bigl(x(t;X_{20})\bigr) - Ef\bigl(x(t;X_{10})\bigr)\bigr| p(s,X_{10},\mathrm{d}X_{20}).
\end{aligned}$$

By (12) there exists a constant time $T > 0$ such that, for all $t \geq T$,

$$\sup_{f\in L}\bigl|Ef\bigl(x(t;X_{20})\bigr) - Ef\bigl(x(t;X_{10})\bigr)\bigr| \leq \varepsilon.$$

Hence,

$$\bigl|Ef\bigl(x(t+s;X_{10})\bigr) - Ef\bigl(x(t;X_{10})\bigr)\bigr| \leq \varepsilon.$$

Since f is arbitrary, we have

$$\mathrm{d}_L\bigl(p(t+s,X_{10},\cdot), p(t,X_{10},\cdot)\bigr) \leq \varepsilon$$

for all $t \geq T$, $s > 0$. Thus, $\{p(t,X_{10},\cdot) : t \geq 0\}$ is Cauchy in the space $\mathcal{P}(\mathbb{R}^2_+)$. Then there exists a unique $\mu \in \mathcal{P}(\mathbb{R}^2_+)$ satisfying

$$\lim_{t\to+\infty} \mathrm{d}_L\bigl(p(t,\varrho,\cdot),\mu\bigr) = 0,$$

where $\varrho = (0.1, 0.1)^T$. In addition, by (12) we obtain

$$\lim_{t\to+\infty} \mathrm{d}_L\bigl(p(t,X_{10},\cdot), p(t,\varrho,\cdot)\bigr) = 0.$$

Therefore,

$$\begin{aligned}
\lim_{t\to+\infty} \mathrm{d}_L\bigl(p(t,X_{10},\cdot),\mu\bigr) &\leq \lim_{t\to+\infty} \mathrm{d}_L\bigl(p(t,X_{10},\cdot), p(t,\varrho,\cdot)\bigr) \\
&\quad + \lim_{t\to+\infty} \mathrm{d}_L\bigl(p(t,\varrho,\cdot), p(t,\varrho,\mu)\bigr) = 0.
\end{aligned}$$

Then system (2) is asymptotically stable in distribution.

On the other hand, by Theorems 3 and 4 we get that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t x(s)\,\mathrm{d}s > 0, \qquad \lim_{t\to+\infty}\frac{1}{t}\int_0^t y(s)\,\mathrm{d}s > 0 \quad \text{a.s.}$$

Therefore, model (2) is stochastically persistent in probability. □

5 Numerical simulation

This section presents a numerical simulation to verify our theoretical analysis of system (2). By means of the Milstein method mentioned in Higham [23], we consider the following discretized equations:

$$\begin{aligned}
x_{i+1} &= x_i + x_i\left(r(i\Delta t) - k(i\Delta t)x_i - \frac{\omega(i\Delta t)y_i}{1+a(i\Delta t)x_i+b(i\Delta t)y_i+a(i\Delta t)b(i\Delta t)x_iy_i}\right)\Delta t \\
&\quad + x_i\left(\left(\sigma_1(i\Delta t) + \frac{\sigma_2(i\Delta t)y(i)}{1+a(i\Delta t)x_i+b(i\Delta t)y_i+a(i\Delta t)b(i\Delta t)x_iy_i}\right)\sqrt{\Delta t}\xi_i \right.\\
&\quad + \frac{\sigma_1^2(i\Delta t)}{2}x_i\left(\xi_i^2-1\right)\Delta t \\
&\quad \left. + \frac{\sigma_2^2(i\Delta t)}{2}\left(\frac{y_i}{1+a(i\Delta t)x_i+b(i\Delta t)y_i+a(i\Delta t)b(i\Delta t)x_iy_i}\right)^2 x_i\left(\xi_i^2-1\right)\Delta t\right), \\
y_{i+1} &= y_i + y_i\left(-g(i\Delta t) - h(i\Delta t)y_i + \frac{f(i\Delta t)x_i}{1+a(i\Delta t)x_i+b(i\Delta t)y_i+a(i\Delta t)b(i\Delta t)x_iy_i}\right)\Delta t \\
&\quad + y_i\left(\left(\delta_1(i\Delta t) + \frac{\delta_2(i\Delta t)x(i)}{1+a(i\Delta t)x_i+b(i\Delta t)y_i+a(i\Delta t)b(i\Delta t)x_iy_i}\right)\sqrt{\Delta t}\eta_i \right.\\
&\quad + \frac{\delta_1^2(i\Delta t)}{2}y_i\left(\eta_i^2-1\right)\Delta t \\
&\quad \left. + \frac{\delta_2^2(i\Delta t)}{2}\left(\frac{x_i}{1+a(i\Delta t)x_i+b(i\Delta t)y_i+a(i\Delta t)b(i\Delta t)x_iy_i}\right)^2 y_i\left(\eta_i^2-1\right)\Delta t\right).
\end{aligned} \tag{17}$$

In Figure 1, we let $r(t) = 0.4 + 0.1\sin t$, $k(t) = 0.7 + 0.01\sin t$, $\omega(t) = 0.1 + 0.02\sin t$, $f(t) = 0.2 + 0.02\sin t$, $g(t) = 0.2 + 0.05\sin t$, $h(t) = 0.2 + 0.01\sin t$, $\frac{\sigma_1^2(t)}{2} = 0.5 + 0.02\sin t$, $\frac{\sigma_2^2(t)}{2} = 0.3 + 0.02\sin t$, $\frac{\delta_1^2(t)}{2} = 0.4 + 0.02\sin t$, $\frac{\delta_2^2(t)}{2} = 0.3 + 0.02\sin t$, and different values of $a(t)$ and $b(t)$ are chosen for Figures 1(a)-1(d). Then, we have $\langle r_1(t)\rangle^* = -0.1 < 0$. According to Theorems 3 and 4, both of the prey and predator populations (x and y, respectively) end in extinction.

In Figure 2, we choose $\frac{\sigma_1^2(t)}{2} = 0.1 + 0.05\sin t$, $\frac{\sigma_2^2(t)}{2} = 0.1 + 0.05\sin t$, $\frac{\delta_1^2(t)}{2} = 0.8 + 0.02\sin t$, $\frac{\delta_2^2(t)}{2} = 0.3 + 0.02\sin t$, $r(t) = 1.8 + 0.01\sin t$, $b(t) = 0.4 + 0.2\sin t$, and the other parameters are the same as those in Figure 1. Then, $\langle r(t) - 0.5(\sigma_1(t) + \frac{\sigma_2(t)}{b})^2\rangle^* \geq 1.2 > 0$ and $(k(t))_*\langle -g(t) - 0.5\delta_1^2(t)\rangle^* + (f(t))^*\langle r_1(t)\rangle^* = -0.316 < 0$. By virtue of Theorems 3 and 4 we get that the prey population x is weakly persistent in the mean, whereas the predator population y is extinct, which is confirmed by Figure 2. Next, set $\frac{\sigma_1^2(t)}{2} = 0.1 + 0.05\sin t$, $\frac{\sigma_2^2(t)}{2} = 0.1 + 0.05\sin t$, $\frac{\delta_1^2(t)}{2} = 0.09 + 0.02\sin t$, $\frac{\delta_2^2(t)}{2} = 0.08 + 0.02\sin t$, $b(t) = 0.4 + 0.2\sin t$, and $f(t) = 0.45 + 0.02\sin t$. The other parameters are the same as those in Figure 1. Then, $\langle r(t) - 0.5(\sigma_1(t) + \frac{\sigma_2(t)}{b})^2\rangle^* \geq 0$, and from Figure 3 we observe that both of prey and predator populations are weakly persistent in the mean.

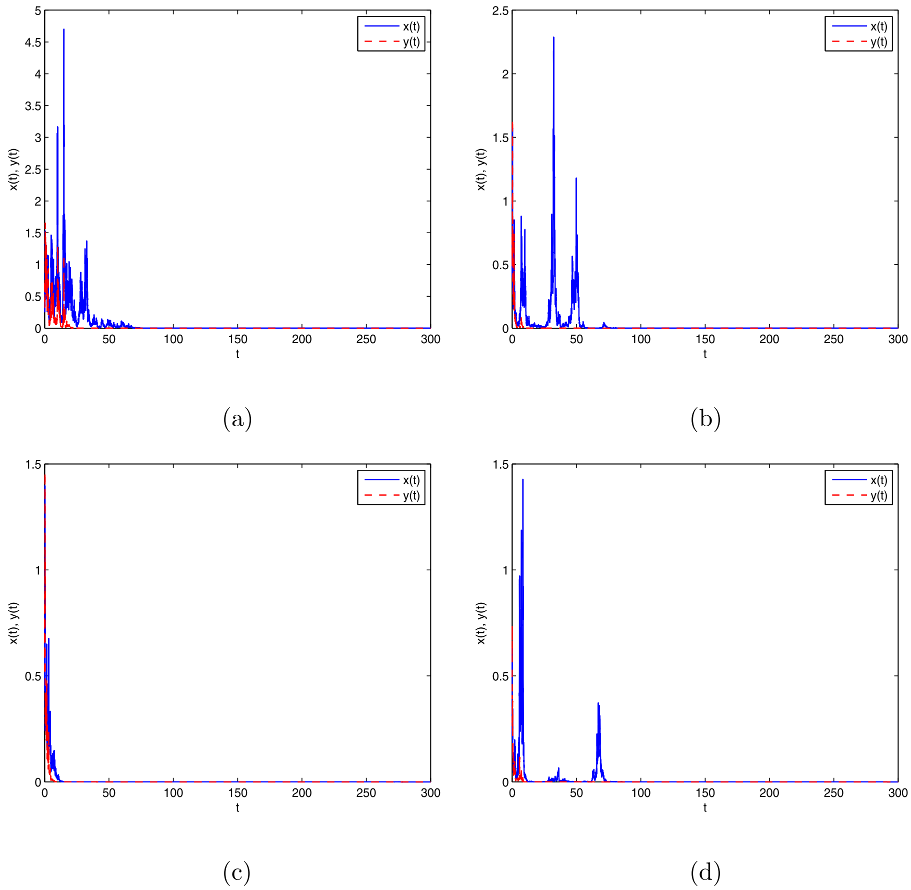
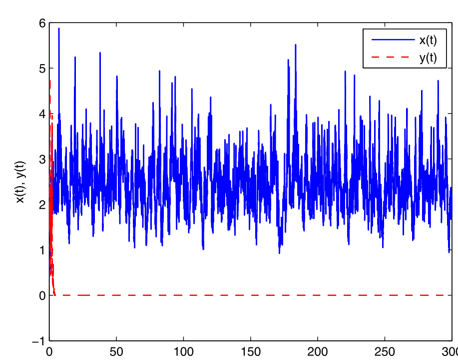


Figure 1 The figure depicts the extinction of the prey and predator species. (a) $a(t) = 0.1 + 0.04\sin t$, $b(t) = 0.5 + 0.05\sin t$. **(b)** $a(t) = b(t) = 0$. **(c)** $a(t) = 0$, $b(t) = 0.5 + 0.05\sin t$. **(d)** $a(t) = 0.1 + 0.04\sin t$, $b(t) = 0$. Both of the prey and predator populations go to extinction.

Figure 2 The prey population is weakly persistent in the mean, whereas the predator species *y* is extinct.



In Figure 4, we let $\sigma_1(t) = 0.05 + 0.02\sin t$, $\sigma_2(t) = 0.02 + 0.01\sin t$, $\delta_1(t) = 0.06 + 0.01\sin t$, $\delta_2(t) = 0.04 + 0.02\sin t$, $r(t) = 1.8 + 0.01\sin t$, $a(t) = 0.46 + 0.04\sin t$, $f(t) = 1.12 + 0.02\sin t$, and $(x_1(0), y_1(0)) = (1.5, 1.2)$. Thus, the conditions of Theorem 5 hold, and model (2) is stochastically permanent.

Moreover, we choose $\frac{\sigma_1^2(t)}{2} = 0.1 + 0.05\sin t$, $\frac{\delta_1^2(t)}{2} = 0.04 + 0.02\sin t$, $r(t) = 1.8 + 0.01\sin t$, $h(t) = 0.05 + 0.01\sin t$, and $f(t) = 1.12 + 0.02\sin t$. The initial conditions are $x_1(0) = 2$,

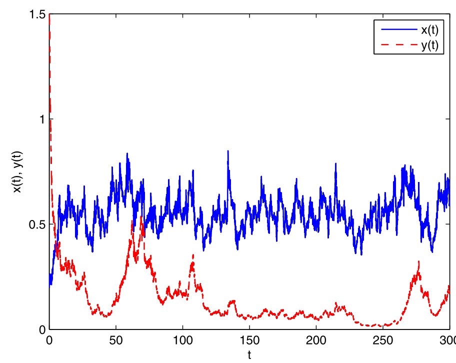


Figure 3 The populations are weakly persistent in the mean for system (2) with $\frac{\sigma_1^2(t)}{2} = \frac{\sigma_2^2(t)}{2} = 0.1 + 0.05\sin t$, $\frac{\delta_1^2(t)}{2} = 0.09 + 0.02\sin t$, and $\frac{\delta_2^2(t)}{2} = 0.08 + 0.02\sin t$.

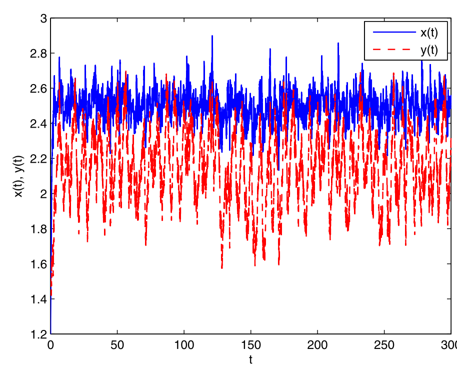


Figure 4 The system is stochastically permanent.

$y_1(0) = 0.6$, and $x_2(0) = 0.5$, $y_2(0) = 6$. The only difference among Figures 5(a)-5(d) is the values of σ_2^2 and δ_2^2, which are chosen as $\sigma_2^2 = \delta_2^2 = 0$ in Figures 5(a), 5(b), whereas $\frac{\sigma_2^2(t)}{2} = 0.06 + 0.01\sin t$, $\frac{\delta_2^2(t)}{2} = 0.03 + 0.02\sin t$ in Figures 5(c), 5(d). From the figures we can observe that system (2) is globally attractive.

In the following example, we investigate the effects of functional response on the species. First, by comparing Figures 1(a)-1(d) we observe that the effects of handling time $a(t)$ and the magnitude of interference among predators $b(t)$ do not influence the extinction of the system. Second, we fix $f(t) = 0.05 + 0.02\sin t$, and the other parameters are the same as in Figure 3. Then we obtain that $(k(t))_*\langle -g(t) - 0.5\delta_1^2(t)\rangle^* + (f(t))^*\langle r_1(t)\rangle^* < 0$. On the basis of Theorem 4, the predator species goes to extinction, and it is confirmed by Figure 6(a). We increase the intensity of conversion rate and choose $f(t) = 0.6 + 0.02\sin t$ and $f(t) = 2.2 + 0.02\sin t$, respectively, for Figures 6(b) and 6(c). From Figures 6(a)-6(c), we observe that the predator changes from extinction to persistence, which shows that increasing the amplitude of periodical conversion rate is benefit for the coexistence of ecosystems.

Appendix

Proof of Theorem 1 Let $k_0 > 0$ be so large that X_0 lies within the interval $[1/k_0, k_0]$. For each integer $k > k_0$, define the stopping times $\tau_k = \inf\{t \in [0, \tau_e] : x(t) \notin (1/k, k) \text{ or } y(t) \notin (1/k, k)\}$. Then, τ_k is increasing as $k \to \infty$. Denote $\tau_\infty = \lim_{k\to+\infty}\tau_k$; thus, $\tau_\infty \le \tau_e$. Next, we show that $\tau_\infty = \infty$. Otherwise, there are constants $T > 0$ and $\varepsilon \in (0,1)$ satisfying $P\{\tau_\infty < \infty\} > \varepsilon$. Then, there exists an integer $k_1 \ge k_0$ such that

$$P\{\tau_k \le T\} \ge \varepsilon$$

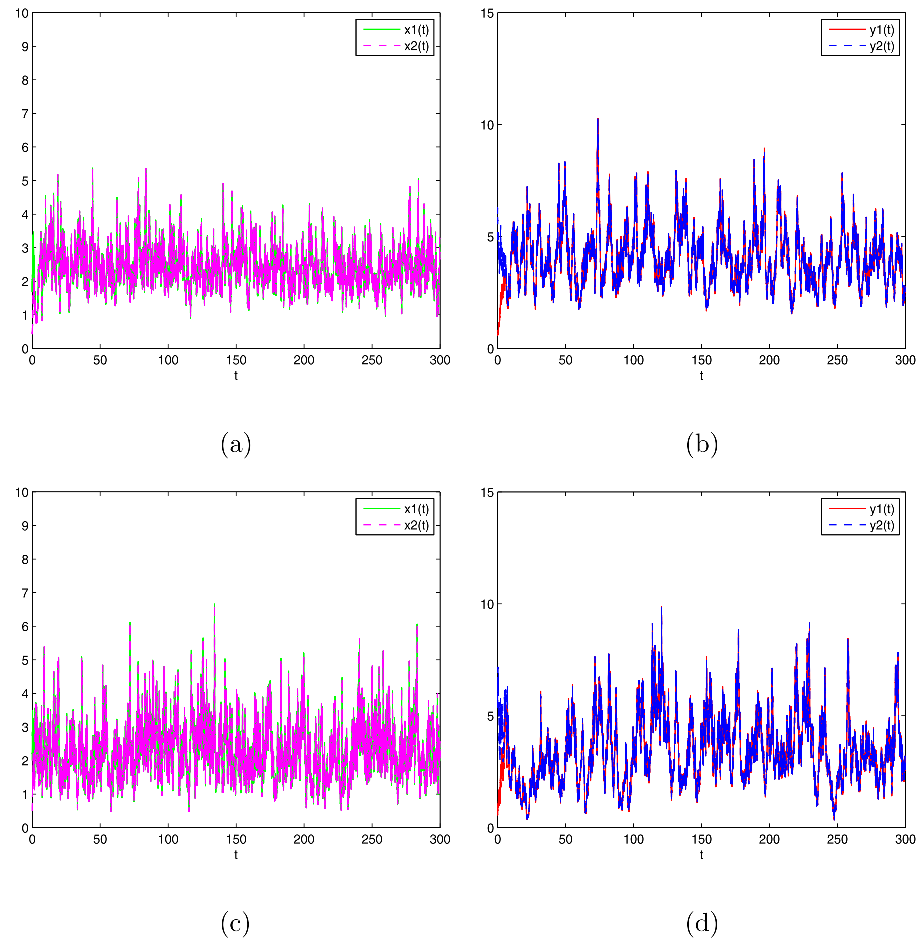


Figure 5 The figure shows the attractiveness of system (2). The only difference between these graphs is the values of σ_2^2 and δ_2^2. **(a)**, **(b)**: $\sigma_2^2 = \delta_2^2 = 0$. **(c)**, **(d)**: $\frac{\sigma_2^2(t)}{2} = 0.06 + 0.01 \sin t$, $\frac{\delta_2^2(t)}{2} = 0.03 + 0.02 \sin t$.

for all $k > k_1$. Define the C^2-function $V : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $V(x,y) = (x - 1 - \ln x) + (y - 1 - \ln y)$, which is nonnegative. If $(x(t), y(t)) \in \mathbb{R}_2^+$, then by Itô's formula we have

$$
\begin{aligned}
\mathrm{d}V(x,y) = {} & V_x\,\mathrm{d}x + 0.5V_{xx}(\mathrm{d}x)^2 + V_y\,\mathrm{d}y + 0.5V_{yy}(\mathrm{d}y)^2 \\
= {} & \biggl((1 - 1/x)x\biggl(r(t) - k(t)x - \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy} \biggr) \\
& + (1 - 1/y)y\biggl(-g(t) - h(t)y + \frac{f(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy} \biggr) \biggr)\,\mathrm{d}t \\
& + 0.5\biggl(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy} \biggr)^2\,\mathrm{d}t \\
& + 0.5\biggl(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy} \biggr)^2\,\mathrm{d}t \\
& + (1 - 1/x)x\biggl(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy} \biggr)\,\mathrm{d}B_1(t) \\
& + (1 - 1/y)y\biggl(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy} \biggr)\,\mathrm{d}B_2(t)
\end{aligned}
$$

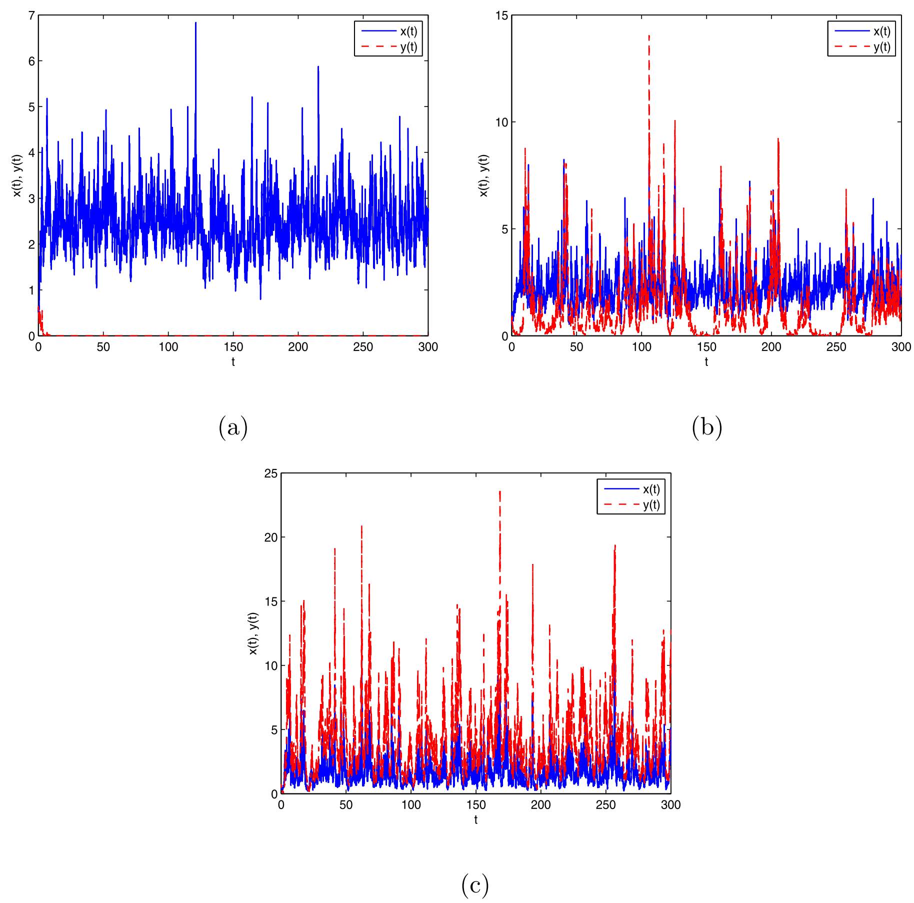


Figure 6 The figure depicts the effects of functional response on the dynamical properties of the model. The difference between the graphs is the values of $f(t)$: **(a)** $f(t) = 0.05 + 0.02\sin t$, **(b)** $f(t) = 0.6 + 0.02\sin t$, and **(c)** $f(t) = 2.2 + 0.02\sin t$.

$$\begin{aligned} &= LV(x,y)\,\mathrm{d}t + (x-1)\left(\sigma_1(t) + \frac{\sigma_2(t)y}{1+a(t)x+b(t)y+a(t)b(t)xy}\right)\mathrm{d}B_1(t) \\ &\quad + (y-1)\left(\delta_1(t) + \frac{\delta_2(t)x}{1+a(t)x+b(t)y+a(t)b(t)xy}\right)\mathrm{d}B_2(t). \end{aligned} \tag{A.1}$$

Here

$$\begin{aligned} LV(x,y) &\le \left(r^u + k^u\right)x - k^l x^2 - r^l + \frac{\omega^u}{b^l} + g^u + 0.5\left(\sigma_1^u + \frac{\sigma_2^u}{b^l}\right)^2 \\ &\quad + \left(h^u + \frac{f^u}{a^l} - g^l\right)y - h^l y^2 + 0.5\left(\delta_1^u + \frac{\delta_2^u}{a^l}\right)^2 \\ &\le G, \end{aligned}$$

where G is a positive number. Integrating both sides of inequality (A.1) from 0 to $\tau_k \wedge T$ ($\tau_k \wedge T = \min\{\tau_k, T\}$) and taking the expectations, we obtain that

$$EV\big(x(\tau_k \wedge T), y(\tau_k \wedge T)\big) \le V\big(x(0), y(0)\big) + GE(\tau_k \wedge T) \le V\big(x(0), y(0)\big) + GT.$$

Let $\Omega_k = \{\tau_k \le T\}$. Then we have $P(\Omega_k) \ge \varepsilon$. For each $\omega \in \Omega_k$, $x(\tau_k,\omega)$ or $y(\tau_k,\omega)$ equals either k or $1/k$, and

$$V\bigl(x(\tau_k,\omega),y(\tau_k,\omega)\bigr) \ge \min\{k-1-\ln k, 1/k-1+\ln k\}.$$

Therefore,

$$\begin{aligned} V\bigl(x(0),y(0)\bigr) + GT &\ge E\bigl[1_{\Omega_k}(\omega)V\bigl(x(\omega),y(\omega)\bigr)\bigr] \\ &\ge \varepsilon \min\{k-1-\ln k, 1/k-1+\ln k\}, \end{aligned}$$

where 1_{Ω_k} is the indicator function of Ω_k. Letting $k \to \infty$, we obtain the contradiction.

The proof is completed. □

Proof of Theorem 3 (1) From system (2) we have that

$$\begin{aligned} \mathrm{d}\ln x &= \biggl(r(t) - k(t)x - \frac{\omega(t)y}{1+a(t)x+b(t)y+a(t)b(t)xy} \\ &\quad - 0.5\biggl(\sigma_1(t) + \frac{\sigma_2(t)y}{1+a(t)x+b(t)y+a(t)b(t)xy}\biggr)^2\biggr)\,\mathrm{d}t \\ &\quad + \biggl(\sigma_1(t) + \frac{\sigma_2(t)y}{1+a(t)x+b(t)y+a(t)b(t)xy}\biggr)\,\mathrm{d}B_1(t), \\ \mathrm{d}\ln y &= \biggl(-g(t) - h(t)y + \frac{f(t)x}{1+a(t)x+b(t)y+a(t)b(t)xy} \\ &\quad - 0.5\biggl(\delta_1(t) + \frac{\delta_2(t)x}{1+a(t)x+b(t)y+a(t)b(t)xy}\biggr)^2\biggr)\,\mathrm{d}t \\ &\quad + \biggl(\delta_1(t) + \frac{\delta_2(t)x}{1+a(t)x+b(t)y+a(t)b(t)xy}\biggr)\,\mathrm{d}B_2(t). \end{aligned} \tag{A.2}$$

Integrating the first equation of (A.2), we have

$$\frac{\ln x(t) - \ln x_0}{t} \le \bigl\langle r_1(t)\bigr\rangle + \frac{\int_0^t (\sigma_1(s) + \frac{\sigma_2(s)y}{1+a(s)x+b(s)y+a(s)b(s)xy})\,\mathrm{d}B_1(s)}{t}. \tag{A.3}$$

Let

$$M_1(t) = \int_0^t \biggl(\sigma_1(s) + \frac{\sigma_2(s)y}{1+a(s)x+b(s)y+a(s)b(s)xy}\biggr)\,\mathrm{d}B_1(s)$$

and

$$M_2(t) = \int_0^t \biggl(\delta_1(s) + \frac{\delta_2(s)x}{1+a(s)x+b(s)y+a(s)b(s)xy}\biggr)\,\mathrm{d}B_2(s).$$

Then, $M_i(t)$ $(i = 1,2)$ is a local martingale, and the quadratic variation satisfies

$$\langle M_1, M_1\rangle_t = \int_0^t \biggl(\sigma_1(s) + \frac{\sigma_2(s)y}{1+a(s)x+b(s)y+a(s)b(s)xy}\biggr)^2\,\mathrm{d}s \le \biggl(\biggl(\sigma_1 + \frac{\sigma_2}{b}\biggr)^u\biggr)^2 t$$

and

$$\langle M_2, M_2\rangle_t = \int_0^t \left(\delta_1(s) + \frac{\delta_2(s)x}{1 + a(s)x + b(s)y + a(s)b(s)xy}\right)^2 \mathrm{d}s \le \left(\left(\delta_1 + \frac{\delta_2}{a}\right)^u\right)^2 t.$$

According to the strong law of large numbers for martingales, we get

$$\limsup_{t\to\infty} \frac{M_i(t)}{t} = 0 \quad \text{a.s.} \tag{A.4}$$

Thus,

$$\left(\frac{\ln x(t) - \ln x_0}{t}\right)^* \le \langle r_1(t)\rangle^* < 0.$$

Then, $\lim_{t\to\infty} x(t) = 0$.

(2) By virtue of the superior limit and (A.4) we can show that, for an arbitrary $\varepsilon > 0$, there exists $T > 0$ such that $\langle r_1(t)\rangle \le \langle r_1(t)\rangle^* + \frac{\varepsilon}{2}$ and $\frac{M_1(t)}{t} \le \frac{\varepsilon}{2}$ for all $t > T$. From (A.2) we get

$$\frac{\ln x(t) - \ln x_0}{t} \le \langle r_1\rangle^* - k^l\langle x\rangle + \varepsilon \le \varepsilon - k^l\langle x\rangle.$$

By Lemma 4 in [8] we have $\langle x(t)\rangle^* \le \frac{\varepsilon}{k^l}$. By the arbitrariness of ε the desired conclusion is obtained.

(3) According to (A.4) and Lemma 1, we have

$$\begin{aligned} k^u\langle x\rangle^* + \omega^u\langle y\rangle^* &\ge \left(\frac{\ln x(t) - \ln x_0}{t}\right)^* + \langle k(t)x\rangle^* + \left\langle \frac{\omega(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right\rangle^* \\ &\ge \left\langle r - 0.5\left(\sigma_1 + \frac{\sigma_2}{b}\right)^2\right\rangle^* > 0. \end{aligned} \tag{A.5}$$

Then, $\langle x\rangle^* > 0$ a.s. If not, for arbitrary $\upsilon \in \{\langle x(t,\upsilon)\rangle^* = 0\}$, by (A.5) we have $\langle y(t,\upsilon)\rangle^* > 0$. Meanwhile, from equation (A.2) we get

$$\left(\frac{\ln y(t,\upsilon) - \ln y_0}{t}\right)^* \le \langle -g - 0.5\delta_1^2\rangle^* + f^u\langle x(t,\upsilon)\rangle^* + h^l\langle -y(t,\upsilon)\rangle^* < 0.$$

Therefore, $\lim_{t\to\infty} y(t,\upsilon) = 0$, which contradicts with $\langle y(t,\upsilon)\rangle^* > 0$. The proof is completed.

(4) By the condition $\langle r - 0.5(\sigma_1 + \frac{\sigma_2}{b})^2\rangle_* - \langle\frac{\omega}{b}\rangle^* > 0$ there exists a sufficiently small $\varepsilon > 0$ such that $\langle r - 0.5(\sigma_1 + \frac{\sigma_2}{b})^2\rangle_* - \langle\frac{\omega}{b}\rangle^* - \varepsilon > 0$. In addition, by (A.4) for this $\varepsilon > 0$, there exists $T > 0$ such that

$$\left\langle r - 0.5\left(\sigma_1 + \frac{\sigma_2}{b}\right)^2\right\rangle > \left\langle r - 0.5\left(\sigma_1 + \frac{\sigma_2}{b}\right)^2\right\rangle_* - \frac{\varepsilon}{3}, \qquad \left\langle\frac{\omega}{b}\right\rangle < \left\langle\frac{\omega}{b}\right\rangle^* + \frac{\varepsilon}{3}, \qquad \frac{M_1(t)}{t} > -\frac{\varepsilon}{3}$$

for all $t > T$. Then, we get

$$\frac{\ln x(t) - \ln x_0}{t} \ge \left\langle r - 0.5\left(\sigma_1 + \frac{\sigma_2}{b}\right)^2\right\rangle_* - \left\langle\frac{\omega}{b}\right\rangle^* - \varepsilon - k^u\langle x\rangle.$$

According to Lemma 4 in [8] and the arbitrariness of ε, we obtain

$$\langle x(t)\rangle_* \geq \frac{\langle r - 0.5(\sigma_1 + \frac{\sigma_2}{b})^2\rangle_* - \langle\frac{\omega}{b}\rangle^*}{k^u} \triangleq m_x > 0.$$

The proof is completed.

(5) From the first equation of (A.2) we have that

$$\mathrm{d}\ln x \leq \left(\left(r(t) - 0.5\sigma_1^2(t)\right) - k(t)x\right)\mathrm{d}t + \left(\sigma_1(t) + \frac{\sigma_2(t)y}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_1(t).$$

Thus,

$$\frac{\ln x(t) - \ln x_0}{t} \leq \langle r - 0.5\sigma_1^2\rangle^* - k^l\langle x(t)\rangle + \frac{M_1(t)}{t}.$$

In addition, from the property of the superior limit and (A.4), for the given positive number ε, there is $T_1 > 0$ satisfying

$$\langle r - 0.5\sigma_1^2\rangle < \langle r - 0.5\sigma_1^2\rangle^* + \frac{\varepsilon}{2}, \qquad \frac{M_1(t)}{t} < \frac{\varepsilon}{2}$$

for all $t > T_1$. According to Lemma 4 in [8] and the arbitrariness of ε, we get that

$$\langle x(t)\rangle^* \leq \frac{\langle r - 0.5\sigma_1^2\rangle^*}{k^l} \triangleq M_x. \qquad \square$$

Proof of Theorem 4 (1) Case I. If $\langle r_1\rangle^* \leq 0$, then by Theorem 3 we have $\langle x(t)\rangle^* = 0$. Thus, for arbitrary sufficiently small $\varepsilon > 0$, there exists $T > 0$ such that $\langle -g - 0.5\delta_1^2\rangle < \langle -g - 0.5\delta_1^2\rangle^* + \frac{\varepsilon}{2}$ and $M_2(t) < \frac{\varepsilon t}{2}$ for all $t > T$. Therefore,

$$\left(\frac{\ln y(t) - \ln y_0}{t}\right)^* \leq \langle -g - 0.5\delta_1^2\rangle^* + f^u\langle x(t)\rangle^* + \varepsilon = \langle -g - 0.5\delta_1^2\rangle^* + \varepsilon < 0,$$

and then $\lim_{t\to\infty} y(t) = 0$.

Case II. If $\langle r_1\rangle^* > 0$, then by (A.4), for sufficiently small $\varepsilon > 0$, there exists $T > 0$ such that

$$\frac{\ln x(t) - \ln x_0}{t} \leq \langle r_1\rangle^* - k_*\langle x(t)\rangle + \varepsilon$$

for all $t > T$. By virtue of Lemma 4 in [8] and the arbitrariness of ε, we get

$$\langle x(t)\rangle^* \leq \frac{\langle r_1\rangle^*}{k_*}. \tag{A.6}$$

Thus, we get

$$\begin{aligned}\left(\frac{\ln y(t) - \ln y_0}{t}\right)^* &\leq \langle -g - 0.5\delta_1^2\rangle^* + f^*\langle x(t)\rangle^* \\ &\leq \frac{k_*\langle -g - 0.5\delta_1^2\rangle^* + f^*\langle r_1\rangle^*}{k_*} < 0.\end{aligned} \tag{A.7}$$

Then, $\lim_{t\to\infty} y(t) = 0$.

(2) In (1), we have already shown that if $\langle r_1\rangle^* \le 0$, then $\lim_{t\to\infty} y(t) = 0$, and, as a result, $\langle y(t)\rangle^* = 0$. Now, we show that if $\langle r_1\rangle^* > 0$, then $\langle y(t)\rangle^* = 0$ is still valid. Otherwise, $\langle y(t)\rangle^* > 0$, and by Lemma 1 we show that $[\frac{\ln y(t)}{t}]^* = 0$. According to (A.7), we get

$$0 = \left[\frac{\ln y(t) - \ln y_0}{t}\right]^* \le \left\langle -g - 0.5\delta_1^2\right\rangle^* + f^*\left\langle x(t)\right\rangle^*.$$

Meanwhile, for an arbitrary constant $\varepsilon > 0$, there exists $T > 0$ such that

$$\left\langle -g - 0.5\delta_1^2\right\rangle \le \left\langle -g - 0.5\delta_1^2\right\rangle^* + \frac{\varepsilon}{3}, \qquad \left\langle f(t)x(t)\right\rangle \le f^*\left\langle x(t)\right\rangle^* + \frac{\varepsilon}{3}, \quad \text{and} \quad M_2(t) \le \frac{\varepsilon}{3}t$$

for all $t > T$. Thus, we have

$$\begin{aligned}\frac{\ln y(t) - \ln y_0}{t} &\le \left\langle -g - 0.5\delta_1^2\right\rangle + \left\langle f(t)x(t)\right\rangle - \left\langle h(t)y(t)\right\rangle + \frac{M_2(t)}{t} \\ &\le \left\langle -g - 0.5\delta_1^2\right\rangle^* + f^*\left\langle x(t)\right\rangle^* + \varepsilon - h_*\left\langle y(t)\right\rangle.\end{aligned}$$

Using Lemma 4 in [8], we obtain

$$\left\langle y(t)\right\rangle^* \le \frac{\langle -g - 0.5\delta_1^2\rangle^* + f^*\langle x(t)\rangle^* + \varepsilon}{h_*},$$

which indicates that $\langle y(t)\rangle^* \le \frac{\langle -g-0.5\delta_1^2\rangle^* + f^*\langle x(t)\rangle^*}{h_*}$. Applying (A.6), we have

$$\left\langle y(t)\right\rangle^* \le \frac{k_*\langle -g - 0.5\delta_1^2\rangle^* + f^*\langle r_1\rangle^*}{h_*k_*} = 0.$$

This is a contradiction. Therefore, $\langle y(t)\rangle^* = 0$ a.s.

(3) In this part, we need to prove that $\langle y(t)\rangle^* > 0$ a.s. If not, for arbitrary $\varepsilon_1 > 0$, there exist a solution $(\check{x}(t), \check{y}(t))$ with initial value $(x_0, y_0) \in \mathbb{R}_+^2$ such that $P\{\langle \check{y}(t)\rangle^* < \varepsilon_1\} > 0$. Let ε_1 be sufficiently small such that

$$\begin{aligned}&\left\langle -g - 0.5\left(\delta_1 + \frac{\delta_2}{a}\right)^2\right\rangle^* + \left\langle \frac{f\bar{x}}{1 + a\bar{x} + b\bar{y} + ab\bar{x}\bar{y}}\right\rangle^* - f^u\frac{4b^l\langle\sigma_1\sigma_2\rangle^* + \langle\sigma_2^2\rangle^*}{(b^l)^2k^l} \\ &\quad > 2\left(\frac{f^u\omega^u}{k^l} + h^u + 1\right)\varepsilon_1.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\frac{\ln \check{y}(t) - \ln y_0}{t} \ge{}& \left\langle -g(t) - 0.5\left(\delta_1(t) + \frac{\delta_2(t)\check{x}}{1 + a(t)\check{x} + b(t)\check{y} + a(t)b(t)\check{x}\check{y}}\right)^2\right\rangle - \left\langle h(t)\check{y}(t)\right\rangle \\ &+ \frac{\int_0^t (\delta_1(s) + \frac{\delta_2(s)\check{x}}{1+a(s)\check{x}+b(s)\check{y}+a(s)b(s)\check{x}\check{y}})\,\mathrm{d}B_2(s)}{t} \\ &+ \left\langle \frac{f(t)\check{x}}{1 + a(t)\check{x} + b(t)\check{y} + a(t)b(t)\check{x}\check{y}} - \frac{f(t)\bar{x}}{1 + a(t)\bar{x} + b(t)\bar{y} + a(t)b(t)\bar{x}\bar{y}}\right\rangle \\ &+ \left\langle \frac{f(t)\bar{x}}{1 + a(t)\bar{x} + b(t)\bar{y} + a(t)b(t)\bar{x}\bar{y}}\right\rangle.\end{aligned}$$

Here, $\check{x}(t) \le \bar{x}(t)$ and $\check{y}(t) \le \bar{y}(t)$ a.s. for $t \in [0, +\infty)$. Notice that

$$\begin{aligned}
&\frac{f(t)\check{x}}{1 + a(t)\check{x} + b(t)\check{y} + a(t)b(t)\check{x}\check{y}} - \frac{f(t)\bar{x}}{1 + a(t)\bar{x} + b(t)\bar{y} + a(t)b(t)\bar{x}\bar{y}} \\
&\quad = f(t)\frac{-(\bar{x} - \check{x}) + a(t)b(t)\bar{x}\check{x}(\bar{y} - \check{y}) + b(t)\check{x}(\bar{y} - \check{y}) - b(t)\check{y}(\bar{x} - \check{x})}{(1 + a(t)\check{x} + b(t)\check{y} + a(t)b(t)\check{x}\check{y})(1 + a(t)\bar{x} + b(t)\bar{y} + a(t)b(t)\bar{x}\bar{y})} \\
&\quad \ge -2f(t)(\bar{x} - \check{x}),
\end{aligned}$$

and thus,

$$\begin{aligned}
\frac{\ln\check{y}(t) - \ln y_0}{t} &\ge \left\langle -g(t) - 0.5\left(\delta_1(t) + \frac{\delta_2(t)}{a(t)}\right)^2 \right\rangle - \left\langle h(t)\check{y}(t)\right\rangle \\
&\quad + \frac{\int_0^t (\delta_1(s) + \frac{\delta_2(s)\check{x}}{1+a(s)\check{x}+b(s)\check{y}+a(s)b(s)\check{x}\check{y}})\,\mathrm{d}B_2(s)}{t} \\
&\quad + \left\langle \frac{f(t)\bar{x}}{1 + a(t)\bar{x} + b(t)\bar{y} + a(t)b(t)\bar{x}\bar{y}} \right\rangle - 2\left\langle f(t)(\bar{x} - \check{x})\right\rangle.
\end{aligned} \tag{A.8}$$

Define the Lyapunov function $V_3(t) = |\ln\bar{x}(t) - \ln\check{x}(t)|$, which is a positive function on $\mathbb{R}_+$. Then

$$\begin{aligned}
D^+V_3(t) \le \operatorname{sgn}(\bar{x} - \check{x}) \Bigg\{ \Bigg[& -k(\bar{x} - \check{x}) + \frac{\omega(t)\check{y}}{1 + a(t)\check{x} + b(t)\check{y} + a(t)b(t)\check{x}\check{y}} \\
& - 0.5\left(\sigma_1(t) + \frac{\sigma_2(t)\bar{y}}{1 + a(t)\bar{x} + b(t)\bar{y} + a(t)b(t)\bar{x}\bar{y}}\right)^2 \\
& + 0.5\left(\sigma_1(t) + \frac{\sigma_2(t)\check{y}}{1 + a(t)\check{x} + b(t)\check{y} + a(t)b(t)\check{x}\check{y}}\right)^2 \Bigg] \mathrm{d}t \\
& + \left(\frac{\sigma_2(t)\bar{y}}{1 + a(t)\bar{x} + b(t)\bar{y} + a(t)b(t)\bar{x}\bar{y}} - \frac{\sigma_2(t)\check{y}}{1 + a(t)\check{x} + b(t)\check{y} + a(t)b(t)\check{x}\check{y}}\right) \mathrm{d}B_1(t) \Bigg\}.
\end{aligned}$$

Setting $M_3(t) = \int_0^t (\frac{\sigma_2(s)\bar{y}}{1+a(s)\bar{x}+b(s)\bar{y}+a(s)b(s)\bar{x}\bar{y}} - \frac{\sigma_2(s)\check{y}}{1+a(s)\check{x}+b(s)\check{y}+a(s)b(s)\check{x}\check{y}})\,\mathrm{d}B_1(s)$, by the strong law of large numbers for martingales we get

$$\limsup_{t\to\infty} \frac{M_3(t)}{t} = 0 \quad \text{a.s.}$$

Thus, for the given constant $\varepsilon_1 > 0$, there exists $T > 0$ such that $M_3(t) < \varepsilon_1 t$ for all $t \ge T$. Therefore, we get that

$$\frac{V_3(t) - V_3(0)}{t} \le \omega^u\left\langle\check{y}(t)\right\rangle - k^l\langle\bar{x} - \check{x}\rangle + \frac{4b^l\langle\sigma_1\sigma_2\rangle^* + \langle\sigma_2^2\rangle^*}{2(b^l)^2} + \frac{M_3(t)}{t}.$$

Then, we obtain

$$\langle\bar{x} - \check{x}\rangle \le \frac{\omega^u}{k^l}\left\langle\check{y}(t)\right\rangle + \frac{4b^l\langle\sigma_1\sigma_2\rangle^* + \langle\sigma_2^2\rangle^*}{2k^l(b^l)^2}.$$

Substituting this inequality into (A.8) and taking the superior limit of the inequality, we get

$$\begin{aligned}\left(\frac{\ln\check{y}(t)-\ln y_0}{t}\right)^* &\geq \left\langle -g(t)-0.5\left(\delta_1(t)+\frac{\delta_2(t)}{a(t)}\right)^2\right\rangle^* - h^u\langle\check{y}(t)\rangle_* \\ &\quad + \frac{\int_0^t(\delta_1(s)+\frac{\delta_2(s)\check{x}}{1+a(t)\check{x}+b(t)\check{y}+a(t)b(t)\check{x}\check{y}})\,\mathrm{d}B_2(s)}{t} \\ &\quad + \left\langle\frac{f(t)\bar{x}}{1+a(t)\bar{x}+b(t)\bar{y}+a(t)b(t)\bar{x}\bar{y}}\right\rangle^* \\ &\quad - 2f^u\frac{\omega^u}{k^l}\langle\check{y}(t)\rangle^* - f^u\frac{4b^l\langle\sigma_1\sigma_2\rangle^*+\langle\sigma_2^2\rangle^*}{k^l(b^l)^2} \\ &\geq \left\langle -g(t)-0.5\left(\delta_1(t)+\frac{\delta_2(t)}{a(t)}\right)^2\right\rangle^* + \left\langle\frac{f(t)\bar{x}}{1+a(t)\bar{x}+b(t)\bar{y}+a(t)b(t)\bar{x}\bar{y}}\right\rangle^* \\ &\quad - f^u\frac{4b^l\langle\sigma_1\sigma_2\rangle^*+\langle\sigma_2^2\rangle^*}{k^l(b^l)^2} - 2\left(\frac{f^u\omega^u}{k^l}+h^u+1\right)\varepsilon_1 > 0,\end{aligned}$$

which contradicts with Lemma 1, and thus $\langle y(t)\rangle^* > 0$ a.s.

(4) The proof is motivated by Liu and Bai [22]. We have

$$\begin{aligned}\frac{f^l}{k^u}\ln\frac{x(t)}{x(0)}+\ln\frac{y(t)}{y(0)} &\geq \frac{f^l}{k^u}\left\langle r(t)-0.5\left(\sigma_1(t)+\frac{\sigma_2(t)}{b(t)}\right)^2\right\rangle_* t \\ &\quad + \left\langle -g(t)-0.5\left(\delta_1(t)+\frac{\delta_2(t)}{a(t)}\right)^2\right\rangle^* t - \left(\frac{f^l w^u}{k^u}+h^u\right)\int_0^t y(s)\,\mathrm{d}s \\ &\quad + \frac{f^l}{k^u}\int_0^t\left(\sigma_1(s)+\frac{\sigma_2(s)y}{1+a(s)x+b(s)y+a(s)b(s)xy}\right)\mathrm{d}B_1(s) \\ &\quad + \int_0^t\left(\delta_1(s)+\frac{\delta_2(s)y}{1+a(s)x+b(s)y+a(s)b(s)xy}\right)\mathrm{d}B_2(s).\end{aligned} \tag{A.9}$$

By (6), for arbitrary $0<\varepsilon<\frac{f^l}{k^u}\langle r(t)-0.5(\sigma_1(t)+\frac{\sigma_2(t)}{b(t)})^2\rangle_* + \langle -g(t)-0.5(\delta_1(t)+\frac{\delta_2(t)}{b(t)})^2\rangle^*$, there exists a random time $T=T(\omega)$ satisfying

$$\frac{1}{t}\frac{f^l}{k^u}\ln\frac{x(t)}{x(0)}-\frac{1}{t}\ln y(0)<\varepsilon \quad \text{a.s.}$$

for all $t\geq T$. Substituting this inequality into (A.9), we obtain that

$$\begin{aligned}\ln y(t) &\geq \frac{f^l}{k^u}\left\langle r(t)-0.5\left(\sigma_1(t)+\frac{\sigma_2(t)}{b(t)}\right)^2\right\rangle_* t \\ &\quad + \left\langle -g(t)-0.5\left(\delta_1(t)+\frac{\delta_2(t)}{a(t)}\right)^2\right\rangle^* t - \varepsilon t - \left(\frac{f^l w^u}{k^u}+h^u\right)\int_0^t y(s)\,\mathrm{d}s \\ &\quad + \frac{f^l}{k^u}\int_0^t\left(\sigma_1(s)+\frac{\sigma_2(s)y}{1+a(s)x+b(s)y+a(s)b(s)xy}\right)\mathrm{d}B_1(s) \\ &\quad + \int_0^t\left(\delta_1(s)+\frac{\delta_2(s)y}{1+a(s)x+b(s)y+a(s)b(s)xy}\right)\mathrm{d}B_2(s).\end{aligned}$$

According to Lemma 4 in [8], we get that

$$\liminf_{t\to\infty} y(t) \geq \frac{f^l\langle r(t) - 0.5(\sigma_1(t) + \frac{\sigma_2(t)}{b(t)})^2\rangle_* + k^u\langle -g(t) - 0.5(\delta_1(t) + \frac{\delta_2(t)}{a(t)})^2\rangle^* - k^u\varepsilon}{f^l w^u + k^u h^u} > 0.$$

The proof is completed.

(5) From the second equation of (A.2) we have

$$\begin{aligned} \mathrm{d}\ln y &\leq \left(\left(-g(t) - 0.5\delta_1^2(t)\right) + \frac{f(t)}{a(t)} - h(t)y\right)\mathrm{d}t \\ &\quad + \left(\delta_1(t) + \frac{\delta_2(t)x}{1 + a(t)x + b(t)y + a(t)b(t)xy}\right)\mathrm{d}B_2(t). \end{aligned}$$

Thus,

$$\frac{\ln y(t) - \ln y_0}{t} \leq \left\langle -g - 0.5\delta_1^2\right\rangle^* + \left\langle \frac{f}{a}\right\rangle^* - h^l\left\langle y(t)\right\rangle + \frac{M_2(t)}{t}.$$

In addition, from the property of the superior limit and (A.4) we have that, for the given positive number ε, there exists $T_2 > 0$ such that

$$\left\langle -g - 0.5\delta_1^2\right\rangle < \left\langle -g - 0.5\delta_1^2\right\rangle^* + \frac{\varepsilon}{3}, \qquad \left\langle \frac{f}{a}\right\rangle < \left\langle \frac{f}{a}\right\rangle^* + \frac{\varepsilon}{3}, \quad \text{and} \quad \frac{M_2(t)}{t} < \frac{\varepsilon}{3}$$

for all $t > T_2$. According to Lemma 4 in [8], we have

$$\left\langle y(t)\right\rangle^* \leq \frac{\langle -g - 0.5\delta_1^2\rangle_* + \langle \frac{f}{a}\rangle^*}{h^l} \triangleq M_y. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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