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Mean square Hyers-Ulam stability of stochastic differential equations driven by Brownian motion

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Abstract
In this paper, the Hyers-Ulam stability for a class of first order stochastic differential equations is studied by using the Ito formula. Furthermore, the research results are applied to a class of second order stochastic differential equations with constant coefficients by the substitution method. In the end, the Hyers-Ulam stability of general second order stochastic differential equations is considered by the solutions of two deterministic second order differential equation boundary value problems.

Keywords: Hyers-Ulam stability; stochastic differential equations; Brownian motion; substitution method

1 Introduction

The Hyers-Ulam stability of functional equations was introduced with the motivation of studying the stability of approximate solutions [1, 2]. Since then, much attention was given to the stability studies of functional equations; see [3–6] and the references therein. In 1993, Obloza introduced the notion of Hyers-Ulam stability for the studies of differential equations [7, 8]. Furthermore, the stability studies of differential equations have been considered in the recent decade; see [9–20] and the references therein. To the best of the author's knowledge, after the success of the investigations of the Hyers-Ulam stability for deterministic differential equations, there are a few arguments about the Hyers-Ulam stability of stochastic differential equations in the literature. However, uncertainty is involved in all kinds of natural phenomena, and stochastic differential equations are the suitable mathematical models for the natural phenomena. Therefore, it is important to generalize the research results of deterministic differential equations to stochastic differential equations. In the paper, we will consider the Hyers-Ulam stability of the following stochastic differential equations in the mean square which are perturbed by the Brownian motion:

$$dX_t = (a_t X_t + f_t)\,dt + h_t\,dB_t, \tag{1.1}$$

and

$$dX'_t = \left(b_t X'_t + c_t X_t + r_t\right)dt + k_t\,dB_t, \tag{1.2}$$

where $t \geq 0, a, b, c, f, h, r, k : [0, +\infty) \to \mathbb{R}$ are continuous, B_t is a standard one-dimensional Brownian motion, X_t is a stochastic process which is adapted to the same filtration as B_t. If $h_t \equiv 0$, $k_t \equiv 0$, equations (1.1) and (1.2) are deterministic equations, which had been considered by the method of integral factors in [13–15].

2 Preliminary

Now we introduce the fundamental definitions and a lemma, which are used later in the article. Throughout this paper, we consider a filtered probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F}_t$, $t \geq 0$ satisfying the usual conditions, that is, it is right continuous and increasing, while $\mathcal{F}_0$ contains all P-null sets.

Definition 2.1 Assume that for any $\varepsilon \geq 0$ and any stochastic process

$$Y_t \in \mathcal{L}_2\bigl(\Omega, \mathcal{F}, (\mathcal{F}_t), P\bigr)$$

satisfies

$$E\biggl(Y_t - \int_0^t (aY_s + f_s)\,ds - \int_0^t h_s\,dB_s\biggr)^2 < \varepsilon, \quad t \in (0, T),$$

where E is the expectation operator, then there exists a solution X_t of equation (1.1) such that $|Y_t - X_t| \leq K\varepsilon$, $t \in (0, T)$ with K is a positive real constant. We say that equation (1.1) is Hyers-Ulam stable on $(0, T)$ in the mean square.

Definition 2.2 Assume that for any $\varepsilon > 0$ and any stochastic process

$$Y_t \in \mathcal{L}_2\bigl(\Omega, \mathcal{F}, (\mathcal{F}_t), P\bigr)$$

satisfies the following inequality:

$$E\biggl(Y_t' - \int_0^t \bigl(b_s Y_s' + c_s Y_s + r_s\bigr)\,ds - \int_0^t k_s\,dB_s\biggr)^2 < \varepsilon, \quad t \in (0, T),$$

where E is the expectation operator, then there exists a solution X_t of equation (1.2) such that $|Y_t - X_t| \leq K\varepsilon$, $t \in (0, T)$ with K a positive real constant. We say that equation (1.2) is Hyers-Ulam stable on $(0, T)$ in the mean square.

To consider the integration of the stochastic process, we use the Ito formula as follows.

Lemma 2.1 ([21]) *Suppose $dX_t = U_t\,dt + V_t\,dB_t$, where the vector $U = (U_1, \ldots, U_m)$ and the matrix $V = (V_1, \ldots, V_m)$ have $\mathcal{L}_2$ components and B is the vector of m independent Brownian motions. Let F be a twice continuously differentiable function from $\mathbb{R}^m$ into $\mathbb{R}$. Then $Y_t = F(X_t)$ is also an Ito process and*

$$dY_t = \sum_{i=1}^m \frac{\partial F}{\partial x_i}(X_t)\,dX_{i,t} + \frac{1}{2}\sum_{i,j=1}^m \frac{\partial^2 F}{\partial x_i x_j}(X_t)\,dX_{i,t} \cdot dX_{j,t},$$

where $dX_{i,t} \cdot dX_{j,t}$ is computed by using the rules $dt\,dt = dt\,dB_{i,t} = dB_{i,t}\,dt = 0$, $dB_{i,t}\,dB_{j,t} = 0$ for $i \neq j$ and $(dB_{i,t})^2 = dt$.

Let $m = 2$, $F(X_t) = X_{1,t}X_{2,t}$, then from Lemma 2.1, we see

$$dF(X_t) = dX_{1,t}X_{2,t} = X_{2,t}\,dX_{1,t} + X_{1,t}\,dX_{2,t} + dX_{1,t}\,dX_{2,t}$$

and

$$X_{1,t}X_{2,t} = X_{1,0}X_{2,0} + \int_0^t (X_{2,s}\,dX_{1,s} + X_{1,s}\,dX_{2,s} + dX_{1,s}\,dX_{2,s}).$$

3 Hyers-Ulam stability of (1.1)

In this section, we establish some criteria of the Hyers-Ulam stability of equation (1.1), by using the Ito formula.

Theorem 3.1 *Let Y_t be an Ito process, $a, f, h \in \mathcal{L}^2[0, T]$,*

$$dg(t, Y_t) = dY_t - (a_t Y_t + f_t)\,dt - h_t\,dB_t, \tag{3.1}$$

assume that Y_t satisfies $E(g(t, Y_t))^2 \leq \varepsilon$, for $t \in (0, T)$, $\varepsilon \geq 0$. Then there exists a solution X_t of equation (1.1) *such that $X_0 = Y_0$, $E(X_t - Y_t)^2 \leq M\varepsilon$ with*

$$M_1 = 4 \max_{0\leq t\leq T} \bigl(1 + e^{\int_0^t a_s\,ds}\bigr)^2.$$

That means equation (1.1) *is Hyers-Ulam stable in the mean square on the interval* $(0, T)$.

Proof Multiplying two sides of (3.1) by the function $e^{-\int_0^t a_s\,ds}$, we obtain

$$e^{-\int_0^t a_s\,ds}\bigl(dg(t, Y_t) + f_t\,dt + h_t\,dB_t\bigr) = e^{-\int_0^t a_s\,ds}(dY_t - aY_t\,dt). \tag{3.2}$$

Applying Lemma 2.1, we have

$$\begin{aligned} d\bigl(e^{-\int_0^t a_s\,ds}Y_t\bigr) &= Y_t\,de^{-\int_0^t a_s\,ds} + e^{-\int_0^t a_s\,ds}\,dY_t + de^{-\int_0^t a_s\,ds}\,dY_t \\ &= e^{-\int_0^t a_s\,ds}(dY_t - a_t Y_t\,dt). \end{aligned}$$

From (3.2), we have

$$e^{-\int_0^t a_s\,ds}\bigl(dg(t, Y_t) + f_t\,dt + h_t\,dB_t\bigr) = e^{-\int_0^t a_s\,ds}(dY_t - a_t Y_t\,dt) = d\bigl(e^{-\int_0^t a_s\,ds}Y_t\bigr). \tag{3.3}$$

Integrating the two sides of (3.3) from 0 to t and multiplying the two sides of (3.3) by the function $e^{\int_0^t a_s\,ds}$, we get

$$e^{\int_0^t a_s\,ds}Y_0 + e^{\int_0^t a_s\,ds}\int_0^t e^{-\int_0^s a_\tau\,d\tau}(f_s\,ds + h_s\,dB_s) + e^{\int_0^t a_s\,ds}\int_0^t e^{-\int_0^s a_\tau\,d\tau}\,dg(s, Y_s) = Y_t. \tag{3.4}$$

Define

$$X_t := e^{\int_0^t a_s\,ds}Y_0 + e^{\int_0^t a_s\,ds}\int_0^t e^{-\int_0^s a_\tau\,d\tau}(f_s\,ds + h_s\,dB_s),$$

then we have $X_0 = Y_0$ and

$$\begin{aligned} dX_t &= Y_0 a_t e^{\int_0^t a_s\,ds}\,dt + \left(\int_0^t e^{-\int_0^s a_\tau\,d\tau}(f_s\,ds + h_s\,dB_s)\right) de^{\int_0^t a_s\,ds} \\ &\quad + e^{\int_0^t a_s\,ds}\,d\int_0^t e^{-\int_0^s a_\tau\,d\tau}(f_s\,ds + h_s\,dB_s) \\ &\quad + de^{\int_0^t a_s ds} d\int_0^t e^{-\int_0^s a_\tau\,d\tau}(f_s\,ds + h_s\,dB_s) \\ &= (a_t X_t + f_t)\,dt + h_t\,dB_t. \end{aligned}$$

Hence X_t is a solution of equation (1.1). We rewrite (3.4) as

$$X_t - Y_t = -e^{\int_0^t a_s\,ds}\int_0^t e^{-\int_0^s a_\tau\,d\tau}\,dg(s, Y_s). \tag{3.5}$$

Applying Lemma 2.1, we have

$$\int_0^t e^{-\int_0^s a_\tau\,d\tau}\,dg(s, Y_s) = e^{-\int_0^t a_s\,ds} g(t, Y_t) - g(0, Y_0) + \int_0^t g(s, Y_s)\,de^{-\int_0^s a_\tau\,d\tau}, \tag{3.6}$$

where $\int_0^t g(s, Y_s)\,de^{-\int_0^s a_\tau\,d\tau}$ is a Stieltjes integral. Taking expectations on the two sides of (3.5), we see

$$E(X_t - Y_t)^2 \le \left(1 + e^{\int_0^t a_s\,ds} + \left|e^{\int_0^t a_s\,ds}\int_0^t a_s e^{-\int_0^s a_\tau\,d\tau}\,ds\right|\right)^2 \varepsilon \le 4\left(1 + e^{\int_0^t a_s\,ds}\right)^2 \varepsilon \le M_1 \varepsilon$$

on the interval $[0, T]$ by (3.6). Hence equation (1.1) is Hyers-Ulam stable in the mean square on the interval $[0, T]$. The proof is completed. □

4 Hyers-Ulam stability of (1.2)

First of all, we consider the Hyers-Ulam stability of equation (1.2) by using the substitution method for a special case. We assume that b_t and c_t are both constant functions and write b and c instead of b_t and c_t.

Theorem 4.1 *Let Y_t' be an Ito process,*

$$dG(t, Y_t) = dY_t' - \left(bY_t' + cY_t + r_t\right) dt - k_t\,dB_t. \tag{4.1}$$

Assume that $E(G(t, Y_t))^2 \le \varepsilon$ for $t \in (0, T)$, $\varepsilon \ge 0$. Then there exists a solution X_t of equation (1.2) *such that*

$$E(X_t - Y_t)^2 \le M_2 \varepsilon, \quad t \in (0, T),$$

with

$$\begin{aligned} &X_0 = Y_0, \qquad X_0' = Y_0', \\ &M_2 = \left(1 + \left(1 + |b|\right)\theta + |c|\left(1 + |b|\right)T\theta^2 + \left(1 + |b|\right)\left(b^2 + c + |b|\right)T\theta^2\right)^2, \end{aligned}$$

$$\theta = \frac{e^{\frac{T(|b|+\sqrt{b^2+4c})}{2}}}{\sqrt{b^2+4c}} \max\{1, |b| + \sqrt{b^2+4c}\} \quad \textit{when } b^2+4c > 0;$$

$$\theta = \frac{2e^{\frac{T|b|}{2}}}{\sqrt{b^2+4c}} \max\{1, \sqrt{-c}\} \quad \textit{when } b^2+4c < 0;$$

$$\theta = e^{\frac{T|b|}{2}} \max\left\{\frac{|b|}{2}, \left|1 - \frac{b^2}{4}\right|\right\} \quad \textit{when } b^2+4c = 0.$$

That means equation (1.2) *is Hyers-Ulam stable in the mean square on the interval* $[0, T]$.

Proof Let

$$Z_t = \begin{pmatrix} X_t \\ \dot{X}_t \end{pmatrix}, \qquad U_t = \begin{pmatrix} Y_t \\ \dot{Y}_t \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then (1.2) can be rewritten as

$$dZ_t = (AZ_t + Br_t)\,dt + Bk_t\,dB_t. \tag{4.2}$$

We write

$$B\,dG(t, Y_t) = dU_t - (AU_t + Br_t)\,dt - Bk_t\,dB_t \tag{4.3}$$

instead of (4.1). Multiplying two sides of (4.3) by the matrix function e^{-At}, we get

$$e^{-At}B\,dG(t, Y_t) = e^{-At}(dU_t - AU_t) - e^{-At}(Br_t\,dt + Bk_t\,dB_t). \tag{4.4}$$

Since Y_t' is an Ito process, without loss of generality, we can define

$$dY_t' := U_t^{\dot{y}}\,dt + V_t^{\dot{y}}\,dB_t.$$

By computing, we have

$$dY_t = \left(\int_0^t U_s^{\dot{y}}\,ds\right)dt + \left(\int_0^t V_s^{\dot{y}}\,dB_s\right)dt.$$

By Lemma 2.1, we see $dt\,dU_t = dt(dY_t', dY_t) = 0$. Hence

$$\begin{aligned} de^{-At}U_t &= \left(de^{-At}\right)U_t + e^{-At}(dU_t) + de^{-At}\,dU_t \\ &= e^{-At}(dU_t - AU_t\,dt). \end{aligned}$$

From (4.4), we have

$$e^{-At}B\,dG(t, Y_t) = de^{-At}U_t - e^{-At}(Br_t\,dt + Bk_t\,dB_t). \tag{4.5}$$

Integrating two sides of (4.5) from 0 to t and multiplying (4.5) by the matrix function e^{At}, we see

$$e^{At}U_0 + e^{At}\int_0^t e^{-As}(Br_s\,ds + Bk_s\,dB_s) = U_t - e^{At}\int_0^t e^{-As}B\,dG(s, Y_s). \tag{4.6}$$

Define

$$Z_t := e^{At}U_0 + e^{At}\int_0^t e^{-As}(Br_s\,ds + Bk_s\,dB_s).$$

Then we have

$$\begin{aligned} dZ_t &= Ae^{At}U_0\,dt + Br_t\,dt + Bk_t\,dB_t + Ae^{At}\,dt\int_0^t e^{-As}(Br_s\,ds + Bk_s\,dB_s) \\ &\quad + de^{At}\,d\int_0^t e^{-As}(Br_s\,ds + Bk_s\,dB_s) \\ &= (AZ_t + Br_t)\,dt + Bk_t\,dB_t. \end{aligned}$$

Therefore $Z_t = (X_t, X_t')^T$ is a solution of (4.2), that is, X_t is a solution of (1.2) with $X_0 = Y_0$, $X_0' = Y_0'$. We rewrite (4.6) as

$$Z_t - U_t = -e^{At}\int_0^t e^{-As}B\,dG(s, Y_s). \tag{4.7}$$

Similar to Theorem 3.1, by Lemma 2.1, we have

$$\int_0^t e^{-As}B\,dG(s, Y_s) = e^{-At}BG(t, Y_t) - BG(0, Y_0) + \int_0^t Ae^{-As}BG(s, Y_s)\,ds. \tag{4.8}$$

By (4.7) and (4.8), we have

$$Z_t - U_t = -BG(t, U_t) + e^{At}BG(0, Y_0) - e^{At}\int_0^t Ae^{-As}BG(s, Y_s)\,ds. \tag{4.9}$$

Assume

$$e^{At} = \alpha(t)A + \beta(t)E \tag{4.10}$$

with E the identity matrix. Hence

$$e^{At} = \alpha(t)A + \beta(t)E = \begin{pmatrix} \beta(t) & \alpha(t) \\ \alpha(t)c & \alpha(t)b + \beta(t) \end{pmatrix}, \tag{4.11}$$

$$e^{At}B = \bigl(\alpha(t)A + \beta(t)E\bigr)B = \begin{pmatrix} \alpha(t) \\ \alpha(t)b + \beta(t) \end{pmatrix}, \tag{4.12}$$

$$\int_0^t Ae^{-As}BG(s, Y_s)\,ds = \begin{pmatrix} \int_0^t (\alpha(-s)b + \beta(-s))G(s, Y_s)\,ds \\ \int_0^t (\alpha(-s)(b^2 + c) + \beta(-s)b)G(s, Y_s)\,ds \end{pmatrix}. \tag{4.13}$$

By (4.8), (4.9), (4.10), (4.11), (4.12), (4.13), we have

$$\begin{aligned} (X_t - Y_t)^2 = \biggl(&-G(t, Y_t) + \bigl(\alpha(t)b + \beta(t)\bigr)G(0, Y_0) + \alpha(t)c\int_0^t \bigl(\alpha(-s)b + \beta(-s)\bigr)G(s, Y_s)\,ds \\ &+ \bigl(\alpha(t)b + \beta(t)\bigr)\int_0^t \bigl(\alpha(-s)\bigl(b^2 + c\bigr) + \beta(-s)b\bigr)G(s, Y_s)\,ds\biggr)^2. \end{aligned} \tag{4.14}$$

We consider three possibilities for computing $\alpha(t)$, $\beta(t)$.

(i) If $b^2 + 4c > 0$, we see that

$$\lambda_1 = \frac{b + \sqrt{b^2 + 4c}}{2}, \qquad \lambda_2 = \frac{b - \sqrt{b^2 + 4c}}{2}$$

are different real eigenvalues of the matrix A. By (4.10), we have

$$\begin{cases} e^{\lambda_1 t} = \alpha(t)\lambda_1 + \beta(t), \\ e^{\lambda_2 t} = \alpha(t)\lambda_2 + \beta(t). \end{cases}$$

Hence

$$\alpha(t) = \frac{e^{\frac{b}{2}t}(e^{\frac{t\sqrt{b^2+4c}}{2}} - e^{-\frac{t\sqrt{b^2+4c}}{2}})}{\sqrt{b^2 + 4c}},$$

$$\beta(t) = \frac{(\frac{b+\sqrt{b^2+4c}}{2})e^{(\frac{b-\sqrt{b^2+4c}}{2})t} - (\frac{b-\sqrt{b^2+4c}}{2})e^{(\frac{b+\sqrt{b^2+4c}}{2})t}}{\sqrt{b^2 + 4c}}.$$

(ii) If $b^2 + 4c < 0$, we see that

$$\lambda_1 = \frac{b + i\sqrt{|b^2 + 4c|}}{2}, \qquad \lambda_2 = \frac{b - i\sqrt{|b^2 + 4c|}}{2},$$

are two different complex eigenvalues. By (4.10), we have

$$\begin{cases} e^{\lambda_1 t} = e^{\frac{b}{2}t}(\cos \frac{t\sqrt{|b^2+4c|}}{2} + i \sin \frac{t\sqrt{|b^2+4c|}}{2}) = \alpha(t)\lambda_1 + \beta(t), \\ e^{\lambda_2 t} = e^{\frac{b}{2}t}(\cos \frac{t\sqrt{|b^2+4c|}}{2} - i \sin \frac{t\sqrt{|b^2+4c|}}{2}) = \alpha(t)\lambda_2 + \beta(t). \end{cases}$$

Hence

$$\alpha(t) = \frac{2e^{\frac{b}{2}t} \sin(\frac{t\sqrt{|b^2+4c|}}{2})}{\sqrt{|b^2 + 4c|}},$$

$$\beta(t) = \frac{2e^{\frac{b}{2}t}(\frac{\sqrt{|b^2+4c|}}{2} \cos \frac{t\sqrt{|b^2+4c|}}{2} - \frac{b}{2} \sin \frac{t\sqrt{|b^2+4c|}}{2})}{\sqrt{|b^2 + 4c|}}.$$

(iii) If $b^2 + 4c = 0$, we see that $\lambda_1 = \lambda_2 = \frac{b}{2}$. By (4.10), we have

$$\begin{cases} e^{\lambda_1 t} = \alpha(t)\lambda_1 + \beta(t), \\ \lambda_1 e^{\lambda_1 t} = \alpha(t). \end{cases}$$

Hence

$$\alpha(t) = \frac{b}{2}e^{\frac{b}{2}t}, \qquad \beta(t) = \left(1 - \frac{b^2}{4}\right)e^{\frac{b}{2}t}.$$

Taking expectations on the two sides of (4.14), we have

$$E(X_t - Y_t)^2 \le \bigl(1 + \bigl(1 + |b|\bigr)\theta + |c|\bigl(1 + |b|\bigr)T\theta^2 + \bigl(1 + |b|\bigr)\bigl(b^2 + c + |b|\bigr)T\theta^2\bigr)^2 \varepsilon = M_2\varepsilon$$

with

$$\theta = \frac{e^{\frac{T(|b|+\sqrt{b^2+4c})}{2}}}{\sqrt{b^2+4c}} \max\{1, |b| + \sqrt{b^2+4c}\} \quad \text{when } b^2+4c>0;$$

$$\theta = \frac{2e^{\frac{T|b|}{2}}}{\sqrt{b^2+4c}} \max\{1, \sqrt{-c}\} \quad \text{when } b^2+4c>0;$$

$$\theta = e^{\frac{T|b|}{2}} \max\left\{\frac{|b|}{2}, \left|1-\frac{b^2}{4}\right|\right\} \quad \text{when } b^2+4c=0.$$

Hence equation (1.2) is Hyers-Ulam stable in the mean square on the interval $[0, T]$. The proof is completed. □

Since matrix multiplication is, in general, not commutative, Theorem 4.1 is not suitable for equation (1.2), when b_t is not a constant function or c_t is not a constant function. Now, we consider equation (1.2) by the solutions of two deterministic boundary value problems.

Let u and v be the solutions of the boundary value problems

$$\begin{cases} x''_t - b_t x'_t - c_t x_t = 0, & t \in (0,T), \\ u_0 = 0, & u_T = 1, \end{cases} \tag{4.15}$$

and

$$\begin{cases} x''_t - b_t x'_t - c_t x_t = 0, & t \in (0,T), \\ u_0 = 1, & u_T = 0, \end{cases} \tag{4.16}$$

respectively. Define

$$\begin{aligned} &p := e^{-\int_0^t b_s\,ds}, \qquad \rho := u'_0, \\ &\Lambda_{t,s} = \begin{cases} u'_s v_t, & 0 \le s \le t \le T, \\ u_t v'_s, & 0 \le t \le s \le T. \end{cases} \end{aligned} \tag{4.17}$$

Lemma 4.2 *Let X'_t be an Ito process, B_t is a standard one-dimensional Brownian motion. Assume that $p, b, c \in L^2[0,T]$, then*

$$X_t = \frac{1}{\rho}\int_0^t \left(\Lambda_{t,s}\int_0^s p_\tau \left(dX'_\tau - \left(b_\tau X'_\tau + c_\tau X_\tau\right)d\tau\right)\right) ds + v_t X_0 + u_t X_T.$$

Proof Let

$$Y_s = \int_0^s p_\tau \left(dX'_\tau - \left(b_\tau X'_\tau + c_\tau X_\tau\right)d\tau\right).$$

By Lemma 2.1, we have

$$Y_s = \int_0^s \left(p_\tau\,dX'_\tau + X'_\tau\,dp_\tau + dX'_\tau\,dp_\tau\right) - \int_0^s p_\tau c_\tau X_\tau\,d\tau = p_s X'_s - X'_0 - \int_0^s p_\tau c_\tau X_\tau\,d\tau. \tag{4.18}$$

Multiplying by the function u'_s, integrating two sides of (4.18) from 0 to t, we see

$$\begin{aligned}\int_0^t u'_s Y_s\,ds &= u'_t p_t X_t - u'_0 X'_0 - \int_0^t p_s\big(u''_s - b_s u'_s\big)X_s\,ds - \int_0^t d\big(u'_s p_s\big)\,dX'_s - u_t X'_0 \\ &\quad - u_t\int_0^t p_s c_s X_s\,ds + \int_0^t u_s p_s c_s X_s\,ds + \int_0^t du'_s\,d\int_0^s p_\tau c_\tau X_\tau\,d\tau \\ &= u'_t p_t X_t - u'_0 X_0 - u_t X'_0 - u_t\int_0^t p_s c_s X_s\,ds.\end{aligned}$$

Similarly, multiplying by the function v'_s, integrating two sides of (4.18) from t to T, we see

$$\int_t^T v'_s Y_s\,ds = -v'_t p_t X_t + v'_T p_T X_T + v_t X'_0 + v_t\int_0^t p_s c_s Y_s\,ds.$$

Therefore, by Abel's differential equation identity, we have

$$\begin{aligned}\frac{1}{\rho}\int_0^T \Lambda_{t,s} Y_s\,ds &= \frac{1}{u'_0}\big(u'_t v_t - u_t v'_t\big)p_t X_t - \frac{1}{u'_0}\big(v_t u'_0 X_0 - u_t v'_T p_T X_T\big) \\ &= -\frac{1}{u'_0}\begin{vmatrix} u_t & v_t \\ u'_t & v'_t\end{vmatrix} p_t X_t - v_t X_0 - u_t X_T = -\frac{1}{u'_0}\begin{vmatrix} u_0 & v_0 \\ u'_0 & v'_0\end{vmatrix} X_t - v_t X_0 - u_t X_T \\ &= X_t - v_t X_0 - u_t X_T.\end{aligned}$$

That is,

$$X_t = \frac{1}{\rho}\int_0^T \left(\Lambda_{t,s}\int_0^s p_\tau\big(dX'_\tau - \big(b_\tau X'_\tau + c_\tau X_\tau\big)\,d\tau\big)\right)ds + v_t X_0 + u_t X_T.$$

The proof is completed. □

Lemma 4.3 *Let B_t is a standard one-dimensional Brownian motion. C, D are two stochastic variables. Assume that $p, r, k \in L^2[0, T]$, then the stochastic process*

$$X_t = \frac{1}{\rho}\int_0^T \left(\Lambda_{t,s}\int_0^t p_s(r_s\,dt + k_s\,dB_s)\right)ds + v_t C + u_t D, \quad 0 < t < T \tag{4.19}$$

is a solution of equation (1.2) *such that $X_0 = C$, $X(T) = D$.*

Proof By Lemma 2.1, we obtain

$$\begin{aligned}& d\int_0^t \left(\Lambda_{t,s}\int_0^t p_s(r_s\,dt + k_s\,dB_s)\right)ds \\ &\quad = v'_t\,dt\int_0^t \left(u'_s\int_0^s p_\tau(g_\tau\,d\tau + k_\tau\,dB_\tau)\right)ds + u'_t v_t\left(\int_0^t p_s(r_s\,d\tau + k_s\,dB_s)\right)dt \\ &\qquad + dv_t\,d\int_0^t \left(u'_s\int_0^s p_\tau(g_\tau\,d\tau + k_\tau\,dB_\tau)\,ds\right) \\ &\quad = v'_t\,dt\int_0^t \left(u'_s\int_0^s p_\tau(g_\tau\,d\tau + k_\tau\,dB_\tau)\right)ds + u'_t v_t\left(\int_0^t p_s(r_s\,ds + k_s\,dB_s)\right)dt,\end{aligned}$$

$$
\begin{aligned}
&d\int_0^t \Lambda_{t,s}\left(\int_0^t p_s(r_s\,dt+k_s\,dB_s)\,ds\right)' \\
&\quad = v_t''\,dt\int_0^t\left(u_s'\int_0^s p_\tau(g_\tau\,d\tau+k_\tau\,dB_\tau)\right)ds+u_t'v_t'\left(\int_0^t p_s(r_s\,d\tau+k_s\,dB_s)\right)dt \\
&\qquad + dv_t'\,d\int_0^t\left(u_s'\int_0^s p_\tau(r_\tau\,d\tau+k_\tau\,dB_\tau)\right)ds+\left(u_t'v_t'+u_t''v_t\right)dt\int_0^t p_s(r_s ds+k_s dB_s) \\
&\qquad + u_t'v_tp_t\left(\alpha(t)\,dt+k_t\,dB_t\right)+d\left(u_t'v_t\right)d\left(\int_0^t p_s(r_s\,ds+k_s\,dB_s)\right) \\
&\quad = v_t''\,dt\int_0^t\left(u_s'\int_0^s p_\tau(r_\tau\,d\tau+k_\tau\,dB_\tau)\right)ds+u_t'v_t'\left(\int_0^t p_s(r_s\,d\tau+k_s\,dB_s)\right)dt \\
&\qquad + \left(u_t'v_t'+u_t''v_t\right)dt\int_0^t p_s(r_s\,ds+k_s\,dB_s)+u_t'v_tp_t\left(\alpha(t)\,dt+h_t\,dB_t\right).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&d\int_t^T\left(\Lambda_{t,s}\int_0^t p_s(r_s\,dt+k_s\,dB_s)\right)ds \\
&\quad = u_t'\,dt\int_t^T\left(v_s'\int_0^s p_\tau(r_\tau\,d\tau+k_\tau\,dB_\tau)\right)ds-u_tv_t'\left(\int_0^t p_s(r_s\,ds+k_s\,dB_s)\right)dt, \\
&d\int_t^T \Lambda_{t,s}\left(\int_0^t p_s(r_s\,dt+k_s\,dB_s)\,ds\right)' \\
&\quad = u_t''\,dt\int_t^T\left(v_s'\int_0^s p_\tau(r_\tau\,d\tau+k_\tau\,dB_\tau)\right)ds-u_t'v_t'\left(\int_0^t p_s(r_s\,d\tau+k_s\,dB_s)\right)dt \\
&\qquad - \left(u_t'v_t'+u_tv_t''\right)dt\int_0^t p_s(r_\tau\,d\tau+k_\tau\,dB_\tau)-u_tv_t'p_t\left(\alpha(t)\,dt+k_t\,dB_t\right).
\end{aligned}
$$

Hence, by Abel's differential equation identity, we have

$$
\begin{aligned}
dX_t'-\left(b_tX_t'+c_tX_t\right)dt &= \frac{1}{\rho}\left(u_t'v_t-u_tv_t'\right)p_t(r_t\,dt+k_t\,dB_t) \\
&\quad + \frac{1}{\rho}\left(v_t\left(u_t''-b_tu_t'\right)-u_t\left(v_t''-b_tv_t'\right)\right)dt\int_0^t p_s(r_\tau\,d\tau+k_\tau\,dB_\tau) \\
&\quad + C\left(u_t''-b_tu_t'-c_tu_t'\right)dt \\
&\quad + D\left(v_t''-b_tv_t'-c_tv_t'\right)dt \\
&= -\frac{1}{\rho}\begin{vmatrix} u_t & v_t \\ u_t' & v_t' \end{vmatrix} p_t(r_t\,dt+k_t\,dB_t) \\
&\quad + \frac{1}{\rho}(c_tu_tv_t-c_tu_tv_t)\,dt\int_0^t p_s(r_\tau\,d\tau+k_\tau\,dB_\tau) \\
&= -\frac{1}{\rho}\begin{vmatrix} u_0 & v_0 \\ u_0' & v_0' \end{vmatrix} e^{\int_0^t b_s\,ds}p_t(r_t\,dt+k_t\,dB_t) \\
&= r_t\,dt+k_t\,dB_t.
\end{aligned}
$$

Therefore (4.19) is a solution of equation (1.2). □

Theorem 4.4 *Let Y'_t be an Ito process,*

$$dG(t, Y_t) = dY'_t - \big(b_t Y'_t + c_t Y_t + r_t\big)\,dt - k_t\,dB_t. \tag{4.20}$$

Assume that $E(G(t, Y_t))^2 \le \varepsilon$ for $t \in (0, T)$, $\varepsilon \ge 0$, $b, c, r, k \in \mathcal{L}^2(0, T)$. Then there exists a solution X_t of equation (1.2) *such that*

$$E(X_t - Y_t)^2 \le M_3 \varepsilon, \quad t \in (0, T),$$

with

$$X_0 = Y_0, \qquad X_T = Y_T,$$
$$M_3 = \frac{4}{\rho^2} \max_{t\in[0,T]} \left\{ \left(\int_0^T |\Lambda_{t,s}|(p_s + 1)\,ds \right)^2 \right\}.$$

That means equation (1.2) *is Hyers-Ulam stable in the mean square on the interval* $(0, T)$.

Proof By Lemma 4.2, we have

$$Y_t = \frac{1}{\rho} \int_0^T \left(\Lambda_{t,s} \int_0^s p_\tau \big(dY'_t - \big(b_t Y'_t\,dt + c_t Y_t\,dt\big)\big) \right) ds - v_t Y_0 - u_t Y_T. \tag{4.21}$$

Let

$$X_t = \frac{1}{\rho} \int_0^T \left(\Lambda_{t,s} \int_0^s p_\tau (r_\tau\,d\tau + k_\tau\,dB_\tau) \right) ds + v_t Y_0 + u_t Y_T, \tag{4.22}$$

by Lemma 4.3, we obtain X_t as a solution of equation (1.2) such that $X_0 = Y_0$, $X_T = Y_T$. By (4.20), (4.21), (4.22), we get

$$\frac{1}{\rho} \int_0^T \left(\Lambda_{t,s} \int_0^s p_\tau\,dG(\tau, Y_\tau) \right) ds = Y_t - X_t. \tag{4.23}$$

By computing, we have

$$\int_0^s p_\tau\,dG(\tau, Y_\tau) = p_s G(s, Y_s) - G(0, Y_0) - \int_0^s p_\tau b_\tau G(\tau, Y_\tau)\,d\tau. \tag{4.24}$$

Taking expectations on the two sides of (4.23), we have

$$\begin{aligned}
E(Y_t - Y_t)^2 &= E\left(\frac{1}{\rho} \int_0^T \left(\Lambda_{t,s} \left(p_s G(s, Y_s) - G(0, Y_0) - \int_0^s p_\tau b_\tau G(\tau, Y_\tau)\,d\tau \right) \right) ds \right)^2 \\
&\le \frac{1}{\rho^2} \left(\int_0^T |\Lambda_{t,s}| \left(p_s + 1 + \left| \int_0^s p_\tau b_\tau\,d\tau \right| \right) ds \right)^2 \varepsilon \\
&\le \frac{4}{\rho^2} \left(\int_0^T |\Lambda_{t,s}|(p_s + 1)\,ds \right)^2 \varepsilon \\
&\le M_3 \varepsilon
\end{aligned}$$

by (4.23). Hence equation (1.2) is Hyers-Ulam stable in the mean square on the interval $[0, T]$. The proof is completed. □

Competing interests

The author declares to have no conflict of interests regarding the publication of this paper.

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