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Existence and upper semicontinuity of attractors for non-autonomous stochastic lattice FitzHugh-Nagumo systems in weighted spaces

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Abstract

In this paper, we first consider the existence of random attractors for the non-autonomous stochastic lattice FitzHugh-Nagumo system with random coupled coefficients and additive white noise in a weighted space $l^2_\rho \times l^2_\rho$; then we establish the upper semicontinuity of random attractors as the intensity of noise approaches zero.

MSC: 37L55; 60H15

Keywords: stochastic lattice dynamical system; random attractor; weighted spaces; white noise

1 Introduction

Stochastic lattice dynamical systems (SLDSs) arise in a variety of applications where the spatial structure has a discrete character, and uncertainties or random influences are taken into account. In recent years, some work has been done regarding the existence of random attractors for SLDSs (see *e.g.* [1–12]). In this work, Bates, Lu and Wang [2] considered the existence of random attractors for first-order non-autonomous SLDS driven by multiplicative white noise. Han, Shen and Zhou [5] considered the existence of random attractors for first-order SLDSs with random coupled coefficients and multiplicative/additive white noise. Wang and Zhou [11] considered non-autonomous stochastic lattice FitzHugh-Nagumo system with random coupled coefficients and multiplicative white noise.

Motivated by [2, 5, 11], we will study the asymptotic behavior of solutions of the following non-autonomous stochastic lattice FitzHugh-Nagumo system with random coupled coefficients and additive white noise: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{cases} du_i = (\sum_{j=-q}^{q} \eta_{i,j}(\theta_t\omega)u_{i+j} - v_i + f_i(u_i) + g_i(t))\,dt + ca_i\,dw_i(t), & i \in \mathbb{Z}, \\ dv_i = (\sigma u_i - \delta v_i + h_i(t))\,dt + cb_i\,dw_i(t), & i \in \mathbb{Z}, \end{cases} \tag{1.1}$$

with the initial data

$$u_i(\tau) = u_{i,\tau}, \qquad v_i(\tau) = v_{i,\tau}, \quad i \in \mathbb{Z},$$

where c, $\sigma > 0$ and $\delta > 0$ are constants; $u_i, v_i \in \mathbb{R}$; $f_i(u_i)$, $g_i(t)$, $h_i(t) \in \mathbb{R}$; $\eta_{i,j}(\omega)$ ($j \in \{-q, \dots, 0, \dots, q\}, q \in \mathbb{N}$) are random variables; $a = (a_i)_{i\in\mathbb{Z}}, b = (b_i)_{i\in\mathbb{Z}} \in l^2$; $(\theta_t)_{t\in\mathbb{R}}$ is a metric dynamical system defined on proper probability space $(\Omega, \mathcal{F}, \mathcal{P})$; $\{w_i(t) : i \in \mathbb{Z}\}$ are independent two-sided real-valued Wiener processes on $(\Omega, \mathcal{F}, \mathcal{P})$. If g_i and h_i do not depend on t for all $i \in Z$, then we say (1.1) is an autonomous stochastic lattice FitzHugh-Nagumo system.

The lattice FitzHugh-Nagumo system is used to stimulate the propagation of action potentials in myelinated nerve axons (see [13]). The attractor of stochastic lattice FitzHugh-Nagumo system has been investigated in [4, 14, 15] in the autonomous case. In practice, the coupled mode between two nodes (say, adjacent nodes) is usually random. It is then of great importance to investigate SLDSs with random coupled coefficients. To the best of our knowledge, there are no results on non-autonomous stochastic lattice FitzHugh-Nagumo system with random coupled coefficients and additive white noise in a weighted space.

In this paper, we shall transform the stochastic lattice FitzHugh-Nagumo system (1.1) into a deterministic one with random parameters through two Ornstein-Uhlenbeck processes, and prove the existence of a tempered random attractor in a weighted space $l^2_\rho \times l^2_\rho$ (see (2.2)) for the continuous cocycle (see Definition 3.1) generated by system (1.1), which attracts the random tempered bounded sets in pullback sense. Then we consider the dependence of attractors on the parameters c of the system (1.1) and establish the upper semicontinuity of the random attractor as the intensity c of noise approaches zero.

The rest of this paper is organized as follows. In the next section, we present some mathematical setting for system (1.1). In Section 3, we mainly consider the existence of a tempered random attractor in a weighted space of infinite sequences for system (1.1). Then in Section 4, we consider the upper semicontinuity of the tempered random attractor for system (1.1).

2 Mathematical settings

Throughout this paper, a positive weight function $\rho : \mathbb{Z} \to \mathbb{R}^+$ is chosen to satisfy

$$0 < \rho(i) \le M_0, \qquad \rho(i) \le c_0\rho(i \pm 1), \quad \forall i \in \mathbb{Z}, \tag{2.1}$$

where M_0 and c_0 are positive constants. For example, for $i \in \mathbb{Z}$, $\rho(i) = \frac{1}{(1+\gamma^2 i^2)^q}$ ($q > \frac{1}{2}$) [16] and $\rho(i) = e^{-\gamma|i|}$ satisfy condition (2.1), where $\gamma > 0$. Define $\rho_i = \rho(i), \forall i \in \mathbb{Z}$,

$$l^2_\rho = \left\{ u = (u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} \rho_i |u_i|^2 < \infty, u_i \in \mathbb{R} \right\} \tag{2.2}$$

with norm $\|u\|_{\rho,2} = (\sum_{i\in\mathbb{Z}} \rho_i |u_i|^2)^{\frac{1}{2}}$ and inner product $(u, v)_{\rho,2} = \sum_{i\in\mathbb{Z}} \rho_i u_i v_i$ for $u = (u_i)_{i\in\mathbb{Z}}, v = (v_i)_{i\in\mathbb{Z}} \in l^2_\rho$. We write $\|\cdot\|_{\rho,2}$ as $\|\cdot\|_\rho$, $(\cdot,\cdot)_{\rho,2}$ as $(\cdot,\cdot)_\rho$, and $\|\cdot\|_\rho$ as $\|\cdot\|$ if $\rho(i) \equiv 1$. Then l^2_ρ is a separable Hilbert space with the norm $\|\cdot\|_\rho$.

Let $\Omega = \{\omega \in C(\mathbb{R}, l^2) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathcal{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$ (see [17]). The infinite sequence $\mathbb{e}^i$ ($i \in \mathbb{Z}$) denote the element having 1 at position i and 0 for all other components.

Consider the following non-autonomous stochastic lattice FitzHugh-Nagumo system with random coupled coefficients and additive white noise: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{cases} du_i = (\sum_{j=-q}^{q} \eta_{i,j}(\theta_t\omega)u_{i+j} - v_i + f_i(u_i) + g_i(t))\,dt + ca_i\,dw_i(t), & i \in \mathbb{Z}, \\ dv_i = (\sigma u_i - \delta v_i + h_i(t))\,dt + cb_i\,dw_i(t), & i \in \mathbb{Z}, \\ u_i(\tau) = u_{i,\tau}, \qquad v_i(\tau) = v_{i,\tau}, & i \in \mathbb{Z}, \end{cases} \tag{2.3}$$

where c, $\sigma > 0$ and $\delta > 0$ are constants, $u_i, v_i \in \mathbb{R}$; $f_i(u_i), g_i(t), h_i(t) \in \mathbb{R}$; $\eta_{i,j}(\omega)$ ($j \in \{-q, \ldots, 0, \ldots, q\}, q \in \mathbb{N}$) are random variables on probability space $(\Omega, \mathcal{F}, \mathcal{P})$; $a = (a_i)_{i\in\mathbb{Z}}$, $b = (b_i)_{i\in\mathbb{Z}} \in l^2$; $\{w_i(t) : i \in \mathbb{Z}\}$ are independent two-sided real-valued Wiener processes on $(\Omega, \mathcal{F}, \mathcal{P})$; $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$, for all $\omega \in \Omega, t \in \mathbb{R}$. Then $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is an ergodic metric dynamical system.

System (2.3) can be rewritten as abstract ODEs: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{cases} du = (\mathbb{B}(\theta_t\omega)u - v + f(u) + g(t))\,dt + c\,dW_1(t), \\ dv = (\sigma u - \delta v + h(t))\,dt + c\,dW_2(t), \\ u(\tau) = u_\tau, \qquad v(\tau) = v_\tau, \end{cases} \tag{2.4}$$

where $u = (u_i)_{i\in\mathbb{Z}}, v = (v_i)_{i\in\mathbb{Z}}, f(u) = (f_i(u_i))_{i\in\mathbb{Z}}$ is a nonlinearity satisfying certain conditions, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$ and $h(t) = (h_i(t))_{i\in\mathbb{Z}}$ are given time dependent sequences, $W_1(t) = W_1(t,\omega) = \sum_{i\in\mathbb{Z}} a_i w_i(t)\mathbb{e}^i$ and $W_2(t) = W_2(t,\omega) = \sum_{i\in\mathbb{Z}} b_i w_i(t)\mathbb{e}^i$ are Brownian motions on $(\Omega, \mathcal{F}, \mathcal{P})$, $\mathbb{B}(\omega)$ is a linear operator defined by

$$\big(\mathbb{B}(\omega)u\big)_i = \sum_{j=-q}^{q} \eta_{i,j}(\omega)u_{i+j}. \tag{2.5}$$

To convert the problem (2.4) into a random differential equation, let $z(\theta_t\omega) := -\lambda \times \int_{-\infty}^0 e^{\lambda s}(\theta_t\omega)(s)\,ds$ and $y(\theta_t\omega) := -\mu\int_{-\infty}^0 e^{\mu s}(\theta_t\omega)(s)\,ds$, $t \in \mathbb{R}$, $\omega \in \Omega$, which are Ornstein-Uhlenbeck processes on $(\Omega, \mathcal{F}, \mathcal{P})$ and solve the Ornstein-Uhlenbeck equations $dz + \lambda z\,dt = dW_1(t)$ and $dy + \mu y\,dt = dW_2(t)$, respectively, where $z(\theta_t\omega) = (z_i(\theta_t\omega))_{i\in\mathbb{Z}}$ and $y(\theta_t\omega) = (y_i(\theta_t\omega))_{i\in\mathbb{Z}}$. From [17–19], it is known that the random variables $\|z(\omega)\|$ and $\|y(\omega)\|$ are tempered, and there is a θ_t-invariant set $\widetilde{\Omega} \subset \Omega$ of full $\mathcal{P}$ measure such that $t \mapsto z(\theta_t\omega)$ and $t \mapsto y(\theta_t\omega)$ are continuous in t for every $\omega \in \widetilde{\Omega}$.

Let $\widetilde{u}(t,\omega) = u(t,\omega) - cz(\theta_t\omega)$, $\widetilde{v}(t,\omega) = v(t,\omega) - cy(\theta_t\omega)$, $\omega \in \Omega$, $t \in \mathbb{R}$, then (2.4) can be written as the following equivalent random system with random coefficients: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{cases} \frac{d\widetilde{u}}{dt} = \mathbb{B}(\theta_t\omega)\widetilde{u} - \widetilde{v} + f(\widetilde{u} + cz(\theta_t\omega)) + c\mathbb{B}(\theta_t\omega)z(\theta_t\omega) + g(t) - cy(\theta_t\omega) - c\lambda z(\theta_t\omega), \\ \frac{d\widetilde{v}}{dt} = \sigma\widetilde{u} - \delta\widetilde{v} + h(t) + \sigma cz(\theta_t\omega) + c(\mu - \delta)y(\theta_t\omega), \\ \widetilde{u}(\tau) = \widetilde{u}_\tau, \qquad \widetilde{v}(\tau) = \widetilde{v}_\tau. \end{cases} \tag{2.6}$$

We will consider (2.6) for $\omega \in \widetilde{\Omega}$ and write $\widetilde{\Omega}$ as Ω from now on. In order to obtain the existence and uniqueness of solutions to problem (2.6), we make the following assumptions on g_i, h_i, f_i and the coefficients $\eta_{i,j}(\omega)$, $j \in -q, \ldots, 0, \ldots, q$, for $i \in \mathbb{Z}$:

(A1) Letting $\boldsymbol{\beta}(\omega) = \sup\{|\eta_{i,j}(\omega)| : j \in \{-q, \ldots, 0, \ldots, q\}, q \in \mathbb{N} \text{ and } i \in \mathbb{Z}\} \geq 0$, $\boldsymbol{\beta}(\theta_t\omega)$ belongs to $L^1_{\text{loc}}(\mathbb{R})$ with respect to $t \in \mathbb{R}$ for each $\omega \in \Omega$,

$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t \boldsymbol{\beta}(\theta_s\omega)\, ds = 0; \tag{2.7}$$

and $\boldsymbol{\beta}(\omega)$ is tempered.

(A2) For some positive constants α, β and κ,

$$f_i(0) = 0, \qquad f_i(u)u \leq -\alpha u^2 + \beta, \qquad f_i'(u) \leq \kappa, \quad \forall i \in \mathbb{Z}, u \in \mathbb{R}.$$

(A3) $g = (g_i)_{i\in\mathbb{Z}} \in L^2_{\text{loc}}(\mathbb{R}, l^2_\rho)$, $h = (h_i)_{i\in\mathbb{Z}} \in L^2_{\text{loc}}(\mathbb{R}, l^2_\rho)$.

(A4) Let $\boldsymbol{\lambda} = \min\{\frac{\alpha}{2}, \delta\}$. There exists a positive constant $\mathbb{a} \in (0, \boldsymbol{\lambda})$ such that

$$\int_{-\infty}^0 e^{\mathbb{a}s}\left(\left\|g(s+\tau)\right\|_\rho^2 + \left\|h(s+\tau)\right\|_\rho^2\right) ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{2.8}$$

We call $v : [\tau, \tau + T) \to l^2_\rho$ a mild solution of the following random lattice differential equations:

$$\frac{dv}{dt} = G(v, t, \theta_t\omega), \quad v = (v_i)_{i\in\mathbb{Z}}, G = (G_i)_{i\in\mathbb{Z}}, t \in [\tau, \tau + T), \tau \in \mathbb{R}, \tag{2.9}$$

where $\omega \in \Omega$, if $v \in C([\tau, \tau + T), l^2_\rho)$ and

$$v_i(t) = v_i(\tau) + \int_\tau^t G_i\bigl(v(s), s, \theta_s\omega\bigr)\, ds, \quad i \in \mathbb{Z}, t \in [\tau, \tau + T), \tau \in \mathbb{R}. \tag{2.10}$$

By Theorem 6.1.7 in [20] and Definition 3.1, we have the following theorem.

Theorem 2.1 *Let $T > 0$ and* (A1)-(A3) *hold. Then, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and any initial data $(\widetilde{u}_\tau, \widetilde{v}_\tau) \in l^2_\rho \times l^2_\rho$, problem* (2.6) *admits a unique mild solution $(\widetilde{u}(\cdot, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau), \widetilde{v}(\cdot, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau)) \in C([\tau, \tau + T), l^2_\rho \times l^2_\rho)$ with $(\widetilde{u}(\tau, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau), \widetilde{v}(\tau, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau)) = (\widetilde{u}_\tau, \widetilde{v}_\tau)$, $(\widetilde{u}(t, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau), \widetilde{v}(t, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau))$ being continuous in $(\widetilde{u}_\tau, \widetilde{v}_\tau) \in l^2_\rho \times l^2_\rho$; $(\widetilde{u}(t, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau), \widetilde{v}(t, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau)) \in l^2 \times l^2$ if $(\widetilde{u}_\tau, \widetilde{v}_\tau) \in l^2 \times l^2$. Moreover,* (2.6) *generates a continuous cocycle Ψ_c over $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ with state space $l^2_\rho \times l^2_\rho$: for $(\widetilde{u}_\tau, \widetilde{v}_\tau) \in l^2_\rho \times l^2_\rho, t \in \mathbb{R}^+, \tau \in \mathbb{R}$, and $\omega \in \Omega$,*

$$\Psi_c(t, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau) := \bigl(\widetilde{u}(t + \tau, \tau, \theta_{-\tau}\omega, \widetilde{u}_\tau, \widetilde{v}_\tau), \widetilde{v}(t + \tau, \tau, \theta_{-\tau}\omega, \widetilde{u}_\tau, \widetilde{v}_\tau)\bigr). \tag{2.11}$$

3 Existence of random attractors

We first provide some sufficient conditions for the existence of random attractors for a continuous cocycle (or non-autonomous random dynamical system) in weighted spaces of infinite sequences in [2]. The theory of random attractors for autonomous random dynamical system can be found in [21–26].

In the following, let $(X, \|\cdot\|_X)$ be a separable Banach space, and $\mathcal{D}(X)$ be the collection of all tempered families of nonempty bounded subsets of X.

Definition 3.1 A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions (1)-(4) are satisfied:

(1) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(2) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;

(3) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \Phi(s, \tau, \omega, \cdot))$;

(4) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Definition 3.2 Let Φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

(1) A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$ is called a random absorbing set for Φ if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and for every $D \in \mathcal{D}(X)$, there exists $T = T(D, \tau, \omega) > 0$ such that

$$\Phi\bigl(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)\bigr) \subseteq K(\tau, \omega) \quad \textit{for all } t \geq T. \tag{3.1}$$

(2) A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$ is called a random attractor for Φ if for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, (i) $\mathcal{A}(\tau, \omega)$ is compact in X and is measurable in ω with respect to $\mathcal{F}$; (ii) $\mathcal{A}$ is invariant, that is, $\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t\omega)$; (iii) For every $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$,

$$\lim_{t\to\infty} d_H\bigl(\Phi\bigl(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)\bigr), \mathcal{A}(\tau, \omega)\bigr) = 0, \tag{3.2}$$

where d_H is the Hausdorff semi-distance given by $d_H(F, G) = \sup_{u\in F} \inf_{v\in G} \|u - v\|_X$, for any $F, G \subset X$.

Theorem 3.3 *Let Φ be a continuous cocycle on $l^2_\rho \times l^2_\rho$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$. Suppose that*

(a) *there exists a bounded closed random absorbing set $B_0 = \{B_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l^2_\rho \times l^2_\rho)$ such that, for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l^2_\rho \times l^2_\rho)$, there exists $T_1 = T_1(\tau, \omega, B) > 0$ yielding $\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)) \subset B_0(\tau, \omega)$, $\forall t \geq T_1$;*

(b) *for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for any $\varepsilon > 0$, there exist $T_2 = T_2(\tau, \varepsilon, \omega, B_0) > 0$ and $I_0 = I_0(\tau, \varepsilon, \omega, B_0) \in \mathbb{N}$ such that*

$$\sum_{|i|>I_0} \rho_i \bigl|\Phi_i(t, \tau - t, \theta_{-t}\omega, u_{\tau-t})\bigr|^2 \leq \varepsilon, \quad \forall t \geq T_2, u_{\tau-t} \in B_0(\tau - t, \theta_{-t}\omega). \tag{3.3}$$

Then Φ possesses a unique random attractor $\mathcal{A}$ in $\mathcal{D}(l^2_\rho \times l^2_\rho)$ given, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, by

$$\mathcal{A}(\tau, \omega) = \bigcap_{\tau \geq T_1} \overline{\bigcup_{t\geq\tau} \Phi\bigl(t, \tau - t, \theta_{-t}\omega, B_0(\tau - t, \theta_{-t}\omega)\bigr)}. \tag{3.4}$$

Next, we will use Theorem 3.3 to prove the existence of a random attractor for the continuous cocycle Ψ_c in $l^2_\rho \times l^2_\rho$ under conditions (A1)-(A4).

Theorem 3.4 *If* (A1)-(A4) *hold, then, for every* $c > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and for any* $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$, *there exists* $T = T(\tau,\omega,B,c) > 0$ *such that, for all* $t \geq T$ *and* $(\widetilde{u}_{\tau-t}, \widetilde{v}_{\tau-t}) \in B(\tau - t, \theta_{-t}\omega)$, *the solution* $(\widetilde{u}, \widetilde{v})$ *of* (2.6) *satisfies*

$$\begin{aligned}
&\big\|\widetilde{u}(\tau, \tau - t, \theta_{-\tau}\omega, \widetilde{u}_{\tau-t}, \widetilde{v}_{\tau-t})\big\|_\rho^2 + \big\|\widetilde{v}(\tau, \tau - t, \theta_{-\tau}\omega, \widetilde{u}_{\tau-t}, \widetilde{v}_{\tau-t})\big\|_\rho^2 \\
&\quad + \int_{-t}^0 e^{\int_s^0 (-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr} \big\|\widetilde{u}(s + \tau, \tau - t, \theta_{-\tau}\omega, \widetilde{u}_{\tau-t}, \widetilde{v}_{\tau-t})\big\|_\rho^2 \,ds \\
&\leq I(c, \tau, \omega),
\end{aligned} \tag{3.5}$$

where $I(c,\tau,\omega) > 0$ *is given by*

$$\begin{aligned}
&I(c,\tau,\omega) \\
&\quad = \mathbb{c}_0 + \int_{-t}^0 e^{\int_s^0 (-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr} \left(\frac{8}{\alpha}\big\|g(s+\tau)\big\|_\rho^2 + \frac{2}{\delta\sigma}\big\|h(s+\tau)\big\|_\rho^2\right) ds \\
&\qquad + \int_{-t}^0 e^{\int_s^0 (-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr} \big(\big(\mathbb{c}_1 + \mathbb{c}_2\boldsymbol{\beta}^2(\theta_s\omega)\big)\big\|z(\theta_s\omega)\big\|_\rho^2 + \mathbb{c}_3\big\|y(\theta_s\omega)\big\|_\rho^2\big) ds \\
&\qquad + 2\beta\sum_{i\in\mathbb{Z}} \rho_i \int_{-t}^0 e^{\int_s^0 (-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr} \,ds,
\end{aligned} \tag{3.6}$$

where $\mathbb{c}_0, \mathbb{c}_1, \mathbb{c}_2$, *and* $\mathbb{c}_3$ *are positive constants independent of* τ, ω, *and* B.

Proof For each $\omega \in \Omega$, there exists a sequence $\eta_{i,j}^{(m)}(t,\omega)$ of continuous functions in $t \in \mathbb{R}$ (see [27]) such that

$$\begin{aligned}
&\lim_{m\to\infty} \int_\tau^t \big|\eta_{i,j}^{(m)}(s,\omega) - \eta_{i,j}(\theta_s\omega)\big|\,ds = 0, \\
&\forall t > 0, \tau \in \mathbb{R}, j \in \{-q, \dots, 0, \dots, q\}, q \in \mathbb{N} \text{ and } i \in \mathbb{Z},
\end{aligned} \tag{3.7}$$

and $|\eta_{i,j}^{(m)}(t,\omega)| \leq |\eta_{i,j}(\theta_t\omega)| \leq \boldsymbol{\beta}(\theta_t\omega)$ for $t \in \mathbb{R}$. Consider the following random differential equations:

$$\begin{cases}
\frac{d\widetilde{u}^{(m)}}{dt} = \mathbb{B}_m(t,\omega)\widetilde{u}^{(m)} - \widetilde{v}^{(m)} + f(\widetilde{u}^{(m)} + cz(\theta_t\omega)) \\
\qquad + c\mathbb{B}_m(t,\omega)z(\theta_t\omega) + g(t) - c\lambda z(\theta_t\omega) - cy(\theta_t\omega), \\
\frac{d\widetilde{v}^{(m)}}{dt} = \sigma\widetilde{u}^{(m)} - \delta\widetilde{v}^{(m)} + h(t) + \sigma cz(\theta_t\omega) + c(\mu - \delta)y(\theta_t\omega), \\
\widetilde{u}^{(m)}(\tau) = \widetilde{u}_\tau, \qquad \widetilde{v}^{(m)}(\tau) = \widetilde{v}_\tau,
\end{cases} \tag{3.8}$$

where $(\mathbb{B}_m(t,\omega)\widetilde{u}^{(m)})_i = \sum_{j=-q}^q \eta_{i,j}^{(m)}(t,\omega)\widetilde{u}_{i+j}^{(m)}$. It is easy to see that (3.8) has a unique mild solution $(\widetilde{u}^{(m)}(\cdot,\tau,\omega,\widetilde{u}_\tau,\widetilde{v}_\tau), \widetilde{v}^{(m)}(\cdot,\tau,\omega,\widetilde{u}_\tau,\widetilde{v}_\tau)) \in C([\tau,+\infty), l^2 \times l^2) \cap C^1((\tau,+\infty), l^2 \times l^2)$ satisfying (3.8). Taking the inner product of (3.8) with $\widetilde{u}^{(m)}(t)$ and $\widetilde{v}^{(m)}(t)$, respectively, in l_ρ^2, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\big\|\widetilde{u}^{(m)}\big\|_\rho^2 + \frac{1}{\sigma}\big\|\widetilde{v}^{(m)}\big\|_\rho^2\right) \\
&\quad = 2\big(\mathbb{B}_m(t,\omega)\widetilde{u}^{(m)}, \widetilde{u}^{(m)}\big)_\rho + 2\big(f\big(\widetilde{u}^{(m)} + cz(\theta_t\omega)\big), \widetilde{u}^{(m)}\big)_\rho
\end{aligned}$$

$$\begin{aligned}&+2c\big(\mathbb{B}_m(t,\omega)z(\theta_t\omega),\widetilde{u}^{(m)}\big)_\rho+2\big(g(t),\widetilde{u}^{(m)}\big)_\rho-2c\lambda\big(z(\theta_t\omega),\widetilde{u}^{(m)}\big)_\rho\\&-2c\big(y(\theta_t\omega),\widetilde{u}^{(m)}\big)_\rho-\frac{2\delta}{\sigma}\big\|\widetilde{v}^{(m)}\big\|^2+\frac{2}{\sigma}\big(h(t),\widetilde{v}^{(m)}\big)_\rho\\&+2c\big(z(\theta_t\omega),\widetilde{v}^{(m)}\big)_\rho+\frac{2c(\mu-\delta)}{\sigma}\big(y(\theta_t\omega),\widetilde{v}^{(m)}\big)_\rho.\end{aligned}\tag{3.9}$$

Note that

$$2\big(g(t),\widetilde{u}^{(m)}\big)_\rho\le\frac{8}{\alpha}\big\|g(t)\big\|_\rho^2+\frac{\alpha}{8}\big\|\widetilde{u}^{(m)}\big\|_\rho^2,\tag{3.10}$$

$$-2c\lambda\big(z(\theta_t\omega),\widetilde{u}^{(m)}\big)_\rho\le\frac{4c^2\lambda^2}{\alpha}\big\|z(\theta_t\omega)\big\|_\rho^2+\frac{\alpha}{4}\big\|\widetilde{u}^{(m)}\big\|_\rho^2,\tag{3.11}$$

$$-2c\big(y(\theta_t\omega),\widetilde{u}^{(m)}\big)_\rho\le\frac{4c^2}{\alpha}\big\|y(\theta_t\omega)\big\|_\rho^2+\frac{\alpha}{4}\big\|\widetilde{u}^{(m)}\big\|_\rho^2,\tag{3.12}$$

$$\frac{2}{\sigma}\big(h(t),\widetilde{v}^{(m)}\big)_\rho\le\frac{2}{\delta\sigma}\big\|h(t)\big\|_\rho^2+\frac{\delta}{2\sigma}\big\|\widetilde{v}^{(m)}\big\|_\rho^2,\tag{3.13}$$

$$2c\big(z(\theta_t\omega),\widetilde{v}^{(m)}\big)_\rho\le\frac{4c^2\sigma}{\delta}\big\|z(\theta_t\omega)\big\|_\rho^2+\frac{\delta}{4\sigma}\big\|\widetilde{v}^{(m)}\big\|_\rho^2,\tag{3.14}$$

$$\frac{2c(\mu-\delta)}{\sigma}\big(y(\theta_t\omega),\widetilde{v}^{(m)}\big)_\rho\le\frac{4c^2(\mu-\delta)^2}{\delta\sigma}\big\|y(\theta_t\omega)\big\|_\rho^2+\frac{\delta}{4\sigma}\big\|\widetilde{v}^{(m)}\big\|_\rho^2.\tag{3.15}$$

By (2.1), we get

$$\begin{aligned}2\big(\mathbb{B}_m(t,\omega)\widetilde{u}^{(m)},\widetilde{u}^{(m)}\big)_\rho&=2\sum_{i\in\mathbb{Z}}\left(\rho_i\widetilde{u}_i^{(m)}\cdot\sum_{j=-q}^{q}\eta_{i,j}^{(m)}(t,\omega)\widetilde{u}_{i+j}^{(m)}\right)\\&\le2\boldsymbol{\beta}(\theta_t\omega)\sum_{i\in\mathbb{Z}}\left(\rho_i\big|\widetilde{u}_i^{(m)}\big|\cdot\sum_{j=-q}^{q}\big|\widetilde{u}_{i+j}^{(m)}\big|\right)\\&\le2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_t\omega)\big\|\widetilde{u}^{(m)}\big\|_\rho^2,\end{aligned}\tag{3.16}$$

$$\begin{aligned}2c\big(\mathbb{B}_m(t,\omega)z(\theta_t\omega),\widetilde{u}^{(m)}\big)_\rho&=2c\sum_{i\in\mathbb{Z}}\left(\rho_i\widetilde{u}_i^{(m)}\sum_{j=-q}^{q}\eta_{i,j}^{(m)}(t,\omega)z_{i+j}(\theta_t\omega)\right)\\&\le\frac{\alpha}{8}\big\|\widetilde{u}^{(m)}\big\|_\rho^2+\frac{8c^2(1+2q)(1+2\widetilde{q})\boldsymbol{\beta}^2(\theta_t\omega)}{\alpha}\big\|z(\theta_t\omega)\big\|_\rho^2,\end{aligned}\tag{3.17}$$

where $\widetilde{q}=\sum_{k=1}^{q}c_0^k$. By (A2), we have

$$\begin{aligned}&2\big(f\big(\widetilde{u}^{(m)}+cz(\theta_t\omega)\big),\widetilde{u}^{(m)}\big)_\rho\\&\quad=2\sum_{i\in\mathbb{Z}}\rho_if_i\big(\widetilde{u}_i^{(m)}+cz_i(\theta_t\omega)\big)\cdot\widetilde{u}_i^{(m)}\\&\quad=2\sum_{i\in\mathbb{Z}}\rho_if_i\big(\widetilde{u}_i^{(m)}+cz_i(\theta_t\omega)\big)\cdot\big(\widetilde{u}_i^{(m)}+cz_i(\theta_t\omega)\big)\\&\qquad-2\sum_{i\in\mathbb{Z}}\rho_if_i\big(\widetilde{u}_i^{(m)}+cz_i(\theta_t\omega)\big)\cdot cz_i(\theta_t\omega)\\&\quad\le2\sum_{i\in\mathbb{Z}}\rho_i\big(-\alpha\big(\widetilde{u}_i^{(m)}+cz_i(\theta_t\omega)\big)^2+\beta\big)\end{aligned}$$

$$
\begin{aligned}
&+ 2\sum_{i\in\mathbb{Z}} \rho_i \kappa \left|\widetilde{u}_i^{(m)} + cz_i(\theta_t\omega)\right| \cdot \left|cz_i(\theta_t\omega)\right| \\
&\le -2\alpha\left\|\widetilde{u}^{(m)}\right\|_\rho^2 - 2\alpha c^2\left\|z(\theta_t\omega)\right\|_\rho^2 + 4\alpha\sum_{i\in\mathbb{Z}}\rho_i\left|\widetilde{u}_i^{(m)}\right|\cdot\left|cz_i(\theta_t\omega)\right| + 2\beta\sum_{i\in\mathbb{Z}}\rho_i \\
&\quad + 2\kappa\sum_{i\in\mathbb{Z}}\rho_i\left|\widetilde{u}_i^{(m)}\right|\cdot\left|cz_i(\theta_t\omega)\right| + 2\kappa c^2\left\|z(\theta_t\omega)\right\|_\rho^2 \qquad (3.18) \\
&\le -2\alpha\left\|\widetilde{u}^{(m)}\right\|_\rho^2 - 2\alpha c^2\left\|z(\theta_t\omega)\right\|_\rho^2 + 4\alpha\sum_{i\in\mathbb{Z}}\rho_i\left|\widetilde{u}_i^{(m)}\right|\cdot\left|cz_i(\theta_t\omega)\right| + 2\beta\sum_{i\in\mathbb{Z}}\rho_i \\
&\quad + 2\kappa\sum_{i\in\mathbb{Z}}\rho_i\left|\widetilde{u}_i^{(m)}\right|\cdot\left|cz_i(\theta_t\omega)\right| + 2\kappa c^2\left\|z(\theta_t\omega)\right\|_\rho^2 \qquad (3.19) \\
&\le -2\alpha\left\|\widetilde{u}^{(m)}\right\|_\rho^2 - 2\alpha c^2\left\|z(\theta_t\omega)\right\|_\rho^2 + \frac{\alpha}{8}\left\|\widetilde{u}^{(m)}\right\|_\rho^2 + 32\alpha c^2\left\|z(\theta_t\omega)\right\|_\rho^2 + 2\beta\sum_{i\in\mathbb{Z}}\rho_i \\
&\quad + \frac{\alpha}{8}\left\|\widetilde{u}^{(m)}\right\|_\rho^2 + \frac{8\kappa c^2}{\alpha}\left\|z(\theta_t\omega)\right\|_\rho^2 + 2\kappa c^2\left\|z(\theta_t\omega)\right\|_\rho^2 \qquad (3.20) \\
&\le -\frac{7\alpha}{4}\left\|\widetilde{u}^{(m)}\right\|_\rho^2 + c^2\left(30\alpha + \frac{8\kappa}{\alpha} + 2\kappa\right)\left\|z(\theta_t\omega)\right\|_\rho^2 + 2\beta\sum_{i\in\mathbb{Z}}\rho_i. \qquad (3.21)
\end{aligned}
$$

From (3.9)-(3.21), we obtain, for $t > 0$,

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\widetilde{u}^{(m)}\right\|_\rho^2 + \frac{1}{\sigma}\left\|\widetilde{v}^{(m)}\right\|_\rho^2\right) + \frac{\alpha}{2}\left\|\widetilde{u}^{(m)}\right\|_\rho^2 \\
&\le \left(2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_t\omega) - \frac{\alpha}{2}\right)\left\|\widetilde{u}^{(m)}\right\|_\rho^2 - \frac{\delta}{\sigma}\left\|\widetilde{v}^{(m)}\right\|_\rho^2 + \frac{8}{\alpha}\left\|g(t)\right\|_\rho^2 + \frac{2}{\delta\sigma}\left\|h(t)\right\|_\rho^2 \\
&\quad + c^2\left(\frac{4\lambda^2}{\alpha} + \frac{4\sigma}{\delta} + 30\alpha + \frac{8\kappa}{\alpha} + 2\kappa + \frac{8(1+2q)(1+2\widetilde{q})\boldsymbol{\beta}^2(\theta_t\omega)}{\alpha}\right)\left\|z(\theta_t\omega)\right\|_\rho^2 \\
&\quad + 4c^2\left(\frac{1}{\alpha} + \frac{(\mu-\delta)^2}{\delta\sigma}\right)\left\|y(\theta_t\omega)\right\|_\rho^2 + 2\beta\sum_{i\in\mathbb{Z}}\rho_i. \qquad (3.22)
\end{aligned}
$$

Recalling that $\boldsymbol{\lambda} = \min\{\frac{\alpha}{2}, \delta\}$, then we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\widetilde{u}^{(m)}\right\|_\rho^2 + \frac{1}{\sigma}\left\|\widetilde{v}^{(m)}\right\|_\rho^2\right) + \frac{\alpha}{2}\left\|\widetilde{u}^{(m)}\right\|_\rho^2 \\
&\le \left(-\boldsymbol{\lambda} + 2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_t\omega)\right)\left(\left\|\widetilde{u}^{(m)}\right\|_\rho^2 + \frac{1}{\sigma}\left\|\widetilde{v}^{(m)}\right\|_\rho^2\right) \\
&\quad + \frac{8}{\alpha}\left\|g(t)\right\|_\rho^2 + \frac{2}{\delta\sigma}\left\|h(t)\right\|_\rho^2 \\
&\quad + \left(\mathbb{c}_1 + \mathbb{c}_2\boldsymbol{\beta}^2(\theta_t\omega)\right)\left\|z(\theta_t\omega)\right\|_\rho^2 + \mathbb{c}_3\left\|y(\theta_t\omega)\right\|_\rho^2 + 2\beta\sum_{i\in\mathbb{Z}}\rho_i, \qquad (3.23)
\end{aligned}
$$

where $\mathbb{c}_1 = c^2(\frac{4\lambda^2}{\alpha} + \frac{4\sigma}{\delta} + 30\alpha + \frac{8\kappa}{\alpha} + 2\kappa)$, $\mathbb{c}_2 = \frac{8c^2(1+2q)(1+2\widetilde{q})}{\alpha}$ and $\mathbb{c}_3 = 4c^2(\frac{1}{\alpha} + \frac{(\mu-\delta)^2}{\delta\sigma})$. Then we obtain, for $t > 0$,

$$
\begin{aligned}
&\left\|\widetilde{u}^{(m)}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2 + \frac{1}{\sigma}\left\|\widetilde{v}^{(m)}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2 \\
&\quad + \frac{\alpha}{2}\int_{\tau-t}^{\tau} e^{\int_s^\tau(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left\|\widetilde{u}^{(m)}(s,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\le e^{\int_{\tau-t}^{\tau}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\|\widetilde{u}_{\tau-t}\|_\rho^2+\frac{1}{\sigma}\|\widetilde{v}_{\tau-t}\|_\rho^2\right)\\
&\quad+\int_{\tau-t}^{\tau}e^{\int_s^{\tau}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\frac{8}{\alpha}\|g(s)\|_\rho^2+\frac{2}{\delta\sigma}\|h(s)\|_\rho^2\right)ds\\
&\quad+\int_{\tau-t}^{\tau}e^{\int_s^{\tau}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\left(\mathbb{c}_1+\mathbb{c}_2\boldsymbol{\beta}^2(\theta_s\omega)\right)\|z(\theta_s\omega)\|_\rho^2+\mathbb{c}_3\|y(\theta_s\omega)\|_\rho^2\right)ds\\
&\quad+2\beta\sum_{i\in\mathbb{Z}}\rho_i\int_{\tau-t}^{\tau}e^{\int_s^{\tau}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\,ds.
\end{aligned}
\tag{3.24}
$$

From (3.24) and by replacing ω by $\theta_{-\tau}\omega$, we have

$$
\begin{aligned}
&\left\|\widetilde{u}^{(m)}(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2+\frac{1}{\sigma}\left\|\widetilde{v}^{(m)}(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2\\
&\qquad+\frac{\alpha}{2}\int_{\tau-t}^{\tau}e^{\int_s^{\tau}(-\lambda+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_{r-\tau}\omega))\,dr}\left\|\widetilde{u}^{(m)}(s,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2ds\\
&\le e^{\int_{-t}^{0}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\|\widetilde{u}_{\tau-t}\|_\rho^2+\frac{1}{\sigma}\|\widetilde{v}_{\tau-t}\|_\rho^2\right)\\
&\qquad+\int_{-t}^{0}e^{\int_s^{0}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\frac{8}{\alpha}\|g(s+\tau)\|_\rho^2+\frac{2}{\delta\sigma}\|h(s+\tau)\|_\rho^2\right)ds\\
&\qquad+\int_{-t}^{0}e^{\int_s^{0}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\left(\mathbb{c}_1+\mathbb{c}_2\boldsymbol{\beta}^2(\theta_s\omega)\right)\|z(\theta_s\omega)\|_\rho^2+\mathbb{c}_3\|y(\theta_s\omega)\|_\rho^2\right)ds\\
&\qquad+2\beta\sum_{i\in\mathbb{Z}}\rho_i\int_{-t}^{0}e^{\int_s^{0}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\,ds.
\end{aligned}
\tag{3.25}
$$

Note that (3.25) holds with $\widetilde{u}^{(m)}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})$ and $\widetilde{v}^{(m)}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})$ being replaced by $\widetilde{u}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})$ and $\widetilde{v}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})$, then we have

$$
\begin{aligned}
&\left\|\widetilde{u}(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2+\frac{1}{\sigma}\left\|\widetilde{v}(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2\\
&\qquad+\frac{\alpha}{2}\int_{\tau-t}^{\tau}e^{\int_s^{\tau}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_{r-\tau}\omega))\,dr}\left\|\widetilde{u}(s,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2ds\\
&\le e^{\int_{-t}^{0}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\|\widetilde{u}_{\tau-t}\|_\rho^2+\frac{1}{\sigma}\|\widetilde{v}_{\tau-t}\|_\rho^2\right)\\
&\qquad+\int_{-t}^{0}e^{\int_s^{0}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\frac{8}{\alpha}\|g(s+\tau)\|_\rho^2+\frac{2}{\delta\sigma}\|h(s+\tau)\|_\rho^2\right)ds\\
&\qquad+\int_{-t}^{0}e^{\int_s^{0}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\left(\mathbb{c}_1+\mathbb{c}_2\boldsymbol{\beta}^2(\theta_s\omega)\right)\|z(\theta_s\omega)\|_\rho^2+\mathbb{c}_3\|y(\theta_s\omega)\|_\rho^2\right)ds\\
&\qquad+2\beta\sum_{i\in\mathbb{Z}}\rho_i\int_{-t}^{0}e^{\int_s^{0}(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\,ds.
\end{aligned}
\tag{3.26}
$$

By (2.7), we find that there exists $T_1=T_1(\omega)>0$ such that, for $t>T_1$,

$$
\int_{-t}^{0}\boldsymbol{\beta}(\theta_s\omega)\,ds\le\frac{\boldsymbol{\lambda}}{4(1+q+\widetilde{q})}t.
$$

By (A4) and $(\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t}) \in B(\tau - t,\theta_{-t}\omega) \in \mathcal{D}(l^2_\rho \times l^2_\rho)$, we have

$$\begin{aligned}
&\lim_{t\to+\infty} e^{\int_{-t}^0(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\|\widetilde{u}_{\tau-t}\|^2_\rho + \frac{1}{\sigma}\|\widetilde{v}_{\tau-t}\|^2_\rho\right) \\
&\quad\le \limsup_{t\to+\infty} e^{\frac{-\lambda t}{2}}\left\|B(\tau-t,\theta_{-t}\omega)\right\|^2_\rho \\
&\quad\le 0.
\end{aligned} \tag{3.27}$$

Therefore, there exists $T_2 = T_2(\tau,\omega,B,c) > 0$ such that, for all $t \ge T_2$,

$$e^{\int_{-t}^0(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\|\widetilde{u}_{\tau-t}\|^2_\rho + \frac{1}{\sigma}\|\widetilde{v}_{\tau-t}\|^2_\rho\right) \le 1. \tag{3.28}$$

Note that $z(\theta_t\omega)$, $y(\theta_t\omega)$ and $\boldsymbol{\beta}(\theta_t\omega)$ are tempered. Then by (A4), we can verify the following integrals are convergent:

$$\begin{aligned}
&\int_{-t}^0 e^{\int_s^0(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\frac{8}{\alpha}\left\|g(s+\tau)\right\|^2_\rho + \frac{2}{\delta\sigma}\left\|h(s+\tau)\right\|^2_\rho\right)ds \\
&\quad+\int_{-t}^0 e^{\int_s^0(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\left(\left(\mathbb{c}_1+\mathbb{c}_2\boldsymbol{\beta}^2(\theta_s\omega)\right)\left\|z(\theta_s\omega)\right\|^2_\rho + \mathbb{c}_3\left\|y(\theta_s\omega)\right\|^2_\rho\right)ds \\
&\quad+2\beta\sum_{i\in\mathbb{Z}}\rho_i\int_{-t}^0 e^{\int_s^0(-\boldsymbol{\lambda}+2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega))\,dr}\,ds \\
&< \infty.
\end{aligned} \tag{3.29}$$

Thus the theorem follows from (3.26), (3.28), and (3.29). □

Theorem 3.5 *Assume that* (A1)-(A4) *hold. Then the continuous cocycle* Ψ_c *associated with* (2.6) *has a unique random attractor* $\mathscr{A}_c = \{\mathscr{A}_c(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l^2_\rho \times l^2_\rho)$.

Proof By Theorem 3.3, it suffices to prove that, for every $\varepsilon > 0$, $c > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for any $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l^2_\rho \times l^2_\rho)$, there exist $T = T(\tau,\omega,B,c,\varepsilon) > 0$ and $R = R(\tau,\omega,c,\varepsilon) > 1$ such that, for all $t \ge T$ and $(\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t}) \in B(\tau-t,\theta_{-t}\omega)$, the solution $(\widetilde{u},\widetilde{v})$ of (2.6) satisfies

$$\sum_{|i|>R}\rho_i\left(\left|\widetilde{u}_i(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right|^2+\left|\widetilde{v}_i(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right|^2\right) \le \varepsilon. \tag{3.30}$$

Choose a smooth increasing function $\chi : \mathbb{R}^+ \to [0,1]$ such that

$$\chi(s) = \begin{cases} 0, & 0 \le s \le 1, \\ 1, & s \ge 2, \end{cases} \tag{3.31}$$

and there exists a positive constant $\mathbb{c}_\chi$ such that $|\chi'(s)| \le \mathbb{c}_\chi$ for $s \in \mathbb{R}^+$.

Let $(\widetilde{u}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t}),\widetilde{v}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t}))$ be a mild solution of (2.6) with $(\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t}) \in l^2_\rho \times l^2_\rho$. For any given $N > 0$ define $\mathcal{Q}_N : l^2_\rho \times l^2_\rho \to l^2 \times l^2$, $(\widetilde{u},\widetilde{v}) = (\widetilde{u}_i,\widetilde{v}_i)_{i\in\mathbb{Z}} \mapsto \mathcal{Q}_N(\widetilde{u},\widetilde{v}) = ((\mathcal{Q}_N\widetilde{u})_i,(\mathcal{Q}_N\widetilde{v})_i)_{i\in\mathbb{Z}}$ by $((\mathcal{Q}_N\widetilde{u})_j,(\mathcal{Q}_N\widetilde{v})_j) = (\widetilde{u}_j,\widetilde{v}_j)$ if $|j| \le N$ and $((\mathcal{Q}_N\widetilde{u})_j,(\mathcal{Q}_N\widetilde{v})_j) = (0,0)$ otherwise.

For any $n \geq 1$, let $(\widetilde{u}^{(m)}, \widetilde{v}^{(m)}) = (\widetilde{u}^{(m)}(\tau, \tau - t, \omega, \mathcal{Q}_n \widetilde{u}_{\tau-t}, \mathcal{Q}_n \widetilde{v}_{\tau-t}), \widetilde{v}^{(m)}(\tau, \tau - t, \omega, \mathcal{Q}_n \widetilde{u}_{\tau-t}, \mathcal{Q}_n \widetilde{v}_{\tau-t})) = (\widetilde{u}_i^{(m)}, \widetilde{v}_i^{(m)})_{i\in\mathbb{Z}}$ be the solution of (3.8). Then taking the inner product $(\widehat{\chi}(r)\widetilde{u}^{(m)}, \widehat{\chi}(r)\widetilde{v}^{(m)}) = (\chi(\frac{|i|}{r})\widetilde{u}_i^{(m)}, \chi(\frac{|i|}{r})\widetilde{v}_i^{(m)})_{i\in\mathbb{Z}}$ of (3.8) in $l_\rho^2 \times l_\rho^2$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left(\left|\widetilde{u}_i^{(m)}\right|^2 + \frac{1}{\sigma}\left|\widetilde{v}_i^{(m)}\right|^2\right) \\
&\quad = 2\bigl(\mathbb{B}_m(t,\omega)\widetilde{u}^{(m)}, \widehat{\chi}(r)\widetilde{u}^{(m)}\bigr)_\rho + 2\bigl(f\bigl(\widetilde{u}^{(m)} + cz(\theta_t\omega)\bigr), \widehat{\chi}(r)\widetilde{u}^{(m)}\bigr)_\rho \\
&\qquad + 2\bigl(c\mathbb{B}_m(t,\omega)z(\theta_t\omega), \widehat{\chi}(r)\widetilde{v}^{(m)}\bigr)_\rho \\
&\qquad + 2\bigl(g(t), \widehat{\chi}(r)\widetilde{u}^{(m)}\bigr)_\rho - 2\bigl(c\lambda z(\theta_t\omega), \widehat{\chi}(r)\widetilde{u}^{(m)}\bigr)_\rho \\
&\qquad - 2\bigl(cy(\theta_t\omega), \widehat{\chi}(r)\widetilde{u}^{(m)}\bigr)_\rho - \frac{2\delta}{\sigma}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\bigl(\widetilde{v}_i^{(m)}\bigr)^2 \\
&\qquad + \frac{2}{\sigma}\bigl(h(t), \widehat{\chi}(r)\widetilde{v}^{(m)}\bigr)_\rho + 2\bigl(cz(\theta_t\omega), \widehat{\chi}(r)\widetilde{v}^{(m)}\bigr)_\rho \\
&\qquad + \frac{2}{\sigma}\bigl(c(\mu-\delta)y(\theta_t\omega), \widehat{\chi}(r)\widetilde{v}^{(m)}\bigr)_\rho.
\end{aligned} \tag{3.32}$$

For each term of (3.32), it has been checked that

$$\begin{aligned}
&2\bigl(\mathbb{B}_m(t,\omega)\widetilde{u}^{(m)}, \widehat{\chi}(r)\widetilde{u}^{(m)}\bigr)_\rho \\
&\quad = 2\sum_{i\in\mathbb{Z}}\left(\rho_i\chi\left(\frac{|i|}{r}\right)\widetilde{u}_i^{(m)}\sum_{j=-q}^{q}\eta_{i,j}^{(m)}(t,\omega)\widetilde{u}_{i+j}^{(m)}\right) \\
&\quad \leq \boldsymbol{\beta}(\theta_t\omega)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\left((1+2q)\left|\widetilde{u}_i^{(m)}\right|^2 + \sum_{j=-q}^{q}\left|\widetilde{u}_{i+j}^{(m)}\right|^2\right) \\
&\quad \leq \boldsymbol{\beta}(\theta_t\omega)\sum_{i\in\mathbb{Z}}\left[\rho_i\sum_{j=-q}^{q}\left(\left(\chi\left(\frac{|i|}{r}\right) - \chi\left(\frac{|i+j|}{r}\right)\right)\left|\widetilde{u}_{i+j}^{(m)}\right|^2 + \chi\left(\frac{|i+j|}{r}\right)\left|\widetilde{u}_{i+j}^{(m)}\right|^2\right)\right] \\
&\qquad + (1+2q)\boldsymbol{\beta}(\theta_t\omega)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right|^2 \\
&\quad \leq \frac{\mathbb{c}_\chi q(1+2\widetilde{q})\boldsymbol{\beta}(\theta_t\omega)}{r}\left\|\widetilde{u}^{(m)}\right\|_\rho^2 + 2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_t\omega)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right|^2,
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
&2\bigl(f\bigl(\widetilde{u}^{(m)} + cz(\theta_t\omega)\bigr), \widehat{\chi}(r)\widetilde{u}^{(m)}\bigr)_\rho \\
&\quad = 2\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)f_i\bigl(\widetilde{u}_i^{(m)} + cz_i(\theta_t\omega)\bigr)\cdot\bigl(\widetilde{u}_i^{(m)} + cz_i(\theta_t\omega)\bigr) \\
&\qquad - 2\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)f_i\bigl(\widetilde{u}_i^{(m)} + cz_i(\theta_t\omega)\bigr)\cdot cz_i(\theta_t\omega) \\
&\quad \leq 2\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\bigl(-\alpha\bigl(\widetilde{u}_i^{(m)} + cz_i(\theta_t\omega)\bigr)^2 + \beta\bigr) \\
&\qquad + 2\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\kappa\left|\widetilde{u}_i^{(m)} + cz_i(\theta_t\omega)\right|\cdot\left|cz_i(\theta_t\omega)\right|
\end{aligned}$$

$$
\begin{aligned}
&\le -2\alpha \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right|^2 + 2\beta \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) + 4\alpha \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right| \cdot \left|cz_i(\theta_t\omega)\right| \\
&\quad + 2\alpha \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|cz_i(\theta_t\omega)\right|^2 \\
&\quad + 2\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\kappa\left|\widetilde{u}_i^{(m)}\right| \cdot \left|cz_i(\theta_t\omega)\right| + 2\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\kappa\left|cz_i(\theta_t\omega)\right|^2 \\
&\le -2\alpha \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right|^2 + 2\beta \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) \\
&\quad + 2(2\alpha+\kappa) \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right| \cdot \left|cz_i(\theta_t\omega)\right| \\
&\quad + 2(\alpha+\kappa) \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|cz_i(\theta_t\omega)\right|^2 \\
&\le -\frac{3\alpha}{2} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right|^2 + 2\beta \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) \\
&\quad + \left(\frac{(2\alpha+\kappa)^2}{\alpha} + 2(\alpha+\kappa)\right)c^2 \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|z_i(\theta_t\omega)\right|^2,
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
&2\left(c\mathbb{B}_m(t,\omega)z(\theta_t\omega), \widehat{\chi}(r)\widetilde{u}^{(m)}\right)_\rho \\
&\quad = 2c \sum_{i\in\mathbb{Z}} \left(\rho_i \chi\left(\frac{|i|}{r}\right)\widetilde{u}_i^{(m)} \sum_{j=-q}^{q} \eta_{i,j}^{(m)}(t,\omega) z_{i+j}(\theta_t\omega)\right) \\
&\quad \le \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left(\frac{\alpha}{8}\left|\widetilde{u}_i^{(m)}\right|^2 + \frac{8c^2}{\alpha}(1+2q)\boldsymbol{\beta}^2(\theta_t\omega) \sum_{j=-q}^{q}\left|z_{i+j}(\theta_t\omega)\right|^2\right) \\
&\quad \le \frac{8c^2}{\alpha}(1+2q)\boldsymbol{\beta}^2(\theta_t\omega) \sum_{i\in\mathbb{Z}} \rho_i \sum_{j=-q}^{q}\left(\left(\chi\left(\frac{|i|}{r}\right) - \chi\left(\frac{|i+j|}{r}\right)\right)\left|z_{i+j}(\theta_t\omega)\right|^2\right. \\
&\qquad \left. + \chi\left(\frac{|i+j|}{r}\right)\left|z_{i+j}(\theta_t\omega)\right|^2\right) \\
&\qquad + \frac{\alpha}{8} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right|^2 \\
&\quad \le \frac{8c^2}{\alpha}(1+2q)(1+2\widetilde{q})\boldsymbol{\beta}^2(\theta_t\omega)\left(\frac{\mathbb{c}_\chi q}{r}\left\|z(\theta_t\omega)\right\|_\rho^2 + \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|z_i(\theta_t\omega)\right|^2\right) \\
&\qquad + \frac{\alpha}{8} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right|^2,
\end{aligned}
\tag{3.35}
$$

$$
2\left(g(t), \widehat{\chi}(r)\widetilde{u}^{(m)}\right)_\rho \le \frac{2}{\alpha} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|g_i(t)\right|^2 + \frac{\alpha}{2} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right|^2,
\tag{3.36}
$$

$$
\begin{aligned}
&-2\left(c\lambda z(\theta_t\omega), \widehat{\chi}(r)\widetilde{u}^{(m)}\right)_\rho \\
&\quad \le \frac{4c^2\lambda^2}{\alpha} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|z_i(\theta_t\omega)\right|^2 + \frac{\alpha}{4} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right|^2,
\end{aligned}
\tag{3.37}
$$

$$-2\big(cy(\theta_t\omega),\widehat{\chi}(r)\widetilde{u}^{(m)}\big)_\rho \le \frac{4c^2}{\alpha}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|y_i(\theta_t\omega)\big|^2+\frac{\alpha}{4}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|\widetilde{u}_i^{(m)}\big|^2, \quad (3.38)$$

$$\frac{2}{\sigma}\big(h(t),\widehat{\chi}(r)\widetilde{v}^{(m)}\big)_\rho \le \frac{2}{\delta\sigma}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|h_i(t)\big|^2+\frac{\delta}{2\sigma}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|\widetilde{v}_i^{(m)}\big|^2, \quad (3.39)$$

$$\begin{aligned}&2\big(cz(\theta_t\omega),\widehat{\chi}(r)\widetilde{v}^{(m)}\big)_\rho\\&\quad\le \frac{4c^2\sigma}{\delta}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|z_i(\theta_t\omega)\big|^2+\frac{\delta}{4\sigma}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|\widetilde{v}_i^{(m)}\big|^2,\end{aligned} \quad (3.40)$$

$$\begin{aligned}&\frac{2}{\sigma}\big(c(\mu-\delta)y(\theta_t\omega),\widehat{\chi}(r)\widetilde{v}^{(m)}\big)_\rho\\&\quad\le \frac{4c^2(\mu-\delta)^2}{\delta\sigma}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|y_i(\theta_t\omega)\big|^2+\frac{\delta}{4\sigma}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|\widetilde{v}_i^{(m)}\big|^2.\end{aligned} \quad (3.41)$$

By putting (3.33)-(3.41) into (3.32), we have

$$\begin{aligned}&\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\left(\big|\widetilde{u}_i^{(m)}\big|^2+\frac{1}{\sigma}\big|\widetilde{v}_i^{(m)}\big|^2\right)\\
&\quad\le\big(2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_t\omega)-\alpha\big)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|\widetilde{u}_i^{(m)}\big|^2-\frac{\delta}{\sigma}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|\widetilde{v}_i^{(m)}\big|^2\\
&\qquad+\frac{\mathbb{c}_\chi q(1+2\widetilde{q})\boldsymbol{\beta}(\theta_t\omega)}{r}\big\|\widetilde{u}^{(m)}\big\|_\rho^2+\frac{2}{\alpha}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|g_i(t)\big|^2+\frac{2}{\delta\sigma}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|h_i(t)\big|^2\\
&\qquad+2\beta\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)+\frac{8c^2\mathbb{c}_\chi q(1+2q)(1+2\widetilde{q})}{\alpha r}\boldsymbol{\beta}^2(\theta_t\omega)\big\|z(\theta_t\omega)\big\|_\rho^2\\
&\qquad+c^2\left(\frac{4\sigma}{\delta}+\frac{4\lambda^2}{\alpha}+\frac{(2\alpha+\kappa)^2}{\alpha}+2(\alpha+\kappa)\right.\\
&\qquad\left.+\frac{8(1+2q)(1+2\widetilde{q})}{\alpha}\boldsymbol{\beta}^2(\theta_t\omega)\right)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|z_i(\theta_t\omega)\big|^2\\
&\qquad+\left(\frac{4c^2(\mu-\delta)^2}{\delta\sigma}+\frac{4c^2}{\alpha}\right)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\big|y_i(\theta_t\omega)\big|^2.\end{aligned} \quad (3.42)$$

Recalling $\boldsymbol{\lambda}=\min\{\frac{\alpha}{2},\delta\}$, multiplying (3.42) by $e^{\int_0^t(2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr}$ and then integrating over $[\tau-t,\tau]$ with $t>0$, we get

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\left(\big|\widetilde{u}_i^{(m)}(\tau,\tau-t,\omega,\mathcal{Q}_n\widetilde{u}_{\tau-t},\mathcal{Q}_n\widetilde{v}_{\tau-t})\big|^2\right.\\
&\qquad\left.+\frac{1}{\sigma}\big|\widetilde{v}_i^{(m)}(\tau,\tau-t,\omega,\mathcal{Q}_n\widetilde{u}_{\tau-t},\mathcal{Q}_n\widetilde{v}_{\tau-t})\big|^2\right)\\
&\quad\le e^{\int_{\tau-t}^{\tau}(2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr}\left(\|\mathcal{Q}_n\widetilde{u}_{\tau-t}\|_\rho^2+\frac{1}{\sigma}\|\mathcal{Q}_n\widetilde{v}_{\tau-t}\|_\rho^2\right)\\
&\qquad+\frac{\mathbb{c}_\chi q(1+2\widetilde{q})}{r}\int_{\tau-t}^{\tau}e^{\int_s^{\tau}(2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr}\\
&\qquad\times\boldsymbol{\beta}(\theta_s\omega)\big\|\widetilde{u}^{(m)}(s,\tau-t,\omega,\mathcal{Q}_n\widetilde{u}_{\tau-t},\mathcal{Q}_n\widetilde{v}_{\tau-t})\big\|_\rho^2\,ds\end{aligned}$$

$$+ \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) \int_{\tau-t}^{\tau} e^{\int_s^\tau (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr} \left(\frac{2}{\alpha}\left|g_i(s)\right|^2 + \frac{2}{\delta\sigma}\left|h_i(s)\right|^2 + 2\beta\right) ds$$
$$+ \mathbb{c}_4 \int_{\tau-t}^{\tau} e^{\int_s^\tau (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr} \boldsymbol{\beta}^2(\theta_s\omega)\left\|z(\theta_s\omega)\right\|_\rho^2 ds$$
$$+ \int_{\tau-t}^{\tau} e^{\int_s^\tau (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left(\left(\mathbb{c}_5 + \mathbb{c}_6\boldsymbol{\beta}^2(\theta_s\omega)\right)\left|z_i(\theta_s\omega)\right|\right.$$
$$\left. + \mathbb{c}_7\left|y_i(\theta_s\omega)\right|\right) ds, \tag{3.43}$$

where $\mathbb{c}_4 = \frac{8c^2 c_\chi q(1+2q)(1+2\widetilde{q})}{\alpha r}$, $\mathbb{c}_5 = c^2(\frac{4\sigma}{\delta} + \frac{4\lambda^2}{\alpha} + \frac{(2\alpha+\kappa)^2}{\alpha} + 2(\alpha+\kappa))$, $\mathbb{c}_6 = \frac{8c^2(1+2q)(1+2\widetilde{q})}{\alpha}$, and $\mathbb{c}_7 = \frac{4c^2(\mu-\delta)^2}{\delta\sigma} + \frac{4c^2}{\alpha}$. Replacing ω and m in (3.43) by $\theta_{-\tau}\omega$ and m_k, respectively, and letting $k \to \infty$, then we obtain

$$\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left(\left|\widetilde{u}_i(\tau,\tau-t,\theta_{-\tau}\omega,\mathcal{Q}_n\widetilde{u}_{\tau-t},\mathcal{Q}_n\widetilde{v}_{\tau-t})\right|^2\right.$$
$$\left. + \frac{1}{\sigma}\left|\widetilde{v}_i(\tau,\tau-t,\theta_{-\tau}\omega,\mathcal{Q}_n\widetilde{u}_{\tau-t},\mathcal{Q}_n\widetilde{v}_{\tau-t})\right|^2\right)$$
$$\le e^{\int_{-t}^0 (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr}\left(\|\mathcal{Q}_n\widetilde{u}_{\tau-t}\|_\rho^2 + \frac{1}{\sigma}\|\mathcal{Q}_n\widetilde{v}_{\tau-t}\|_\rho^2\right)$$
$$+ \frac{\mathbb{c}_\chi q(1+2\widetilde{q})}{r}\int_{-t}^0 e^{\int_s^0 (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr}$$
$$\times \boldsymbol{\beta}(\theta_s\omega)\left\|\widetilde{u}(s+\tau,\tau-t,\theta_{-\tau}\omega,\mathcal{Q}_n\widetilde{u}_{\tau-t},\mathcal{Q}_n\widetilde{v}_{\tau-t})\right\|_\rho^2 ds$$
$$+ \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\int_{-t}^0 e^{\int_s^0 (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr}\left(\frac{8}{\alpha}\left|g_i(s+\tau)\right|^2 + \frac{2}{\delta\sigma}\left|h_i(s+\tau)\right|^2 + 2\beta\right) ds$$
$$+ \mathbb{c}_4\int_{-t}^0 e^{\int_s^0 (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr}\boldsymbol{\beta}^2(\theta_s\omega)\left\|z(\theta_s\omega)\right\|_\rho^2 ds$$
$$+ \int_{-t}^0 e^{\int_s^0 (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr}$$
$$\times \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left(\left(\mathbb{c}_5 + \mathbb{c}_6\boldsymbol{\beta}^2(\theta_s\omega)\right)\left|z_i(\theta_s\omega)\right| + \mathbb{c}_7\left|y_i(\theta_s\omega)\right|\right) ds. \tag{3.44}$$

We now estimate each term on the right-hand side of (3.44). For the first term on the right-hand side of (3.44), since $(\mathcal{Q}_n\widetilde{u}_{\tau-t},\mathcal{Q}_n\widetilde{v}_{\tau-t}) \in B(\tau-t,\theta_{-t}\omega)$, and B is tempered, then there exists $T_1 = T_1(\tau,\varepsilon,\omega,B) > 0$ such that if $t > T_1$, then

$$e^{\int_{-t}^0 (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr}\left(\|\mathcal{Q}_n\widetilde{u}_{\tau-t}\|_\rho^2 + \frac{1}{\sigma}\|\mathcal{Q}_n\widetilde{v}_{\tau-t}\|_\rho^2\right) < \varepsilon. \tag{3.45}$$

For the second term on the right-hand side of (3.44), by (A1) and Theorem 3.4, there exist $T_2 = T_2(\tau,\varepsilon,\omega,B) > 0$ and $R_1 = R_1(\varepsilon,\omega) > 0$ such that, for all $t > T_2$ and $r > R_1$,

$$\frac{\mathbb{c}_\chi q(1+2\widetilde{q})}{r}\int_{-t}^0 e^{\int_s^0 (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr}$$
$$\times \boldsymbol{\beta}(\theta_s\omega)\left\|\widetilde{u}(s+\tau,\tau-t,\theta_{-\tau}\omega,\mathcal{Q}_n\widetilde{u}_{\tau-t},\mathcal{Q}_n\widetilde{v}_{\tau-t})\right\|_\rho^2 ds < \varepsilon. \tag{3.46}$$

For the third term on the right-hand side of (3.44), by (A4), there exist $R_2 = R_2(\varepsilon, \omega) > 0$ and $T_3 = T_3(\varepsilon, \omega) > 0$, such that if $r > R_2$ and $t > T_3$, then

$$\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) \int_{-t}^{0} e^{\int_s^0 (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr} \times \left(\frac{2}{\alpha}\left|g_i(s+\tau)\right|^2 + \frac{2}{\delta\sigma}\left|h_i(s+\tau)\right|^2 + 2\beta\right) ds < \varepsilon. \tag{3.47}$$

For the fourth and fifth terms on the right-hand side of (3.44), since $z(\theta_t\omega)$, $y(\theta_t\omega)$ and $\boldsymbol{\beta}(\theta_t\omega)$ are tempered, then there exist $R_3 = R_3(\varepsilon, \omega) > 0$ and $T_4 = T_4(\varepsilon, \omega) > 0$, such that if $r > R_3$ and $t > T_4$, then

$$\begin{aligned} &\mathbb{c}_4 \int_{-t}^{0} e^{\int_s^0 (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr} \boldsymbol{\beta}^2(\theta_s\omega)\left\|z(\theta_s\omega)\right\|_\rho^2\,ds \\ &\quad + \int_{-t}^{0} e^{\int_s^0 (2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)-\boldsymbol{\lambda})\,dr} \\ &\quad \times \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left(\left(\mathbb{c}_5 + \mathbb{c}_6\boldsymbol{\beta}^2(\theta_s\omega)\right)\left|z_i(\theta_s\omega)\right| + \mathbb{c}_7\left|y_i(\theta_s\omega)\right|\right) ds \\ &< \varepsilon. \end{aligned} \tag{3.48}$$

Let $T = \max\{T_1, T_2, T_3, T_4\}$ and $R = 2\max\{R_1, R_2, R_3\}$. By (3.45)-(3.48), we have, for all $t > T$ and $r > R$,

$$\sum_{|i|>R} \rho_i \left(\left|\widetilde{u}_i(\tau, \tau-t, \theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})\right|^2 + \frac{1}{\sigma}\left|\widetilde{v}_i(\tau, \tau-t, \theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})\right|^2\right) < 4\varepsilon, \tag{3.49}$$

for any $n \geq 1$. Let $n \to \infty$, we see that (3.30) holds. The proof is completed. □

4 Upper semicontinuity of random attractors

In this section, we first present a criterion concerning upper semicontinuity of non-autonomous random attractors with respect to a parameter in [28]. Similar results can be found in [29, 30] for deterministic equations and in [26, 31] for autonomous stochastic equations.

Theorem 4.1 *Let Φ_c be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$. Suppose that*

(i) *Φ_c has a closed measurable random absorbing set $K_c = \{K_c(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(X)$ and a unique random attractor $\mathcal{A}_c = \{\mathcal{A}_c(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(X)$.*

(ii) *For each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $K_0(\tau, \omega) = \{u \in X : \|u\|_X \leq r_0(\tau, \omega)\}$ and*

$$\limsup_{c\to 0}\left\|K_c(\tau, \omega)\right\|_X = \limsup_{c\to 0} \sup_{x\in K_c(\tau,\omega)} \|x\|_X \leq r_0(\tau, \omega), \tag{4.1}$$

where $r_0(\tau, \omega)$ is a positive valued tempered random variable.

(iii) *There exists* $\varepsilon > 0$ *such that, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, $\bigcup_{|c|\leq\varepsilon} \mathcal{A}_c(\tau,\omega)$ *is precompact in* X.

(iv) *For* $t > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $c_n \to 0$ *when* $n \to \infty$, *and* $x_n, x_0 \in X$ *with* $x_n \to x_0$ *when* $n \to \infty$, *we have*

$$\lim_{n\to\infty} \Phi_{c_n}(t,\tau,\omega,x_n) = \Phi_0(t,\tau,\omega,x_0). \tag{4.2}$$

Then, for $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$d_H\big(\mathcal{A}_c(\tau,\omega), \mathcal{A}_0(\tau,\omega)\big) = \sup_{u\in\mathcal{A}_c(\tau,\omega)} \inf_{v\in\mathcal{A}_0(\tau,\omega)} \|u - v\|_X \to 0 \quad \text{as } c \to 0. \tag{4.3}$$

Next, we use Theorem 4.1 to prove an upper semicontinuity of random attractors $\mathcal{A}_c(\tau,\omega)$ to $\mathcal{A}_0(\tau,\omega)$ as $c \to 0$.

To indicate the dependence of solutions on c, we will write the solution of (2.6) as $\widetilde{\varphi}^{(c)} = (\widetilde{u}^{(c)}, \widetilde{v}^{(c)})$. When $c = 0$, the system (2.4) reduces to the limiting system:

$$\begin{cases} du = (\mathbb{B}(\theta_t\omega)u - v + f(u) + g(t))\,dt, \\ dv = (\sigma u - \delta v + h(t))\,dt, \\ u(\tau) = u_\tau, \qquad v(\tau) = v_\tau. \end{cases} \tag{4.4}$$

Let $\varphi = (u, v)$ be a mild solution of (4.4) with initial data (u_τ, v_τ).

Theorem 4.2 *Assume that* (A1)-(A4) *hold. Then, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *we have*

$$\begin{aligned} &d_H\big(\mathcal{A}_c(\tau,\omega), \mathcal{A}_0(\tau,\omega)\big) \\ &\quad= \sup_{(\widetilde{u}^{(c)},\widetilde{v}^{(c)})\in\mathcal{A}_c(\tau,\omega)} \inf_{(u,v)\in\mathcal{A}_0(\tau,\omega)} \big(\|\widetilde{u}^{(c)} - u\|_\rho^2 + \|\widetilde{v}^{(c)} - v\|_\rho^2\big)^{\frac{1}{2}} \to 0 \quad \text{as } c \to 0. \end{aligned} \tag{4.5}$$

Proof Let $I(c,\tau,\omega)$ be as in Theorem 3.4. (i) By Theorems 3.4 and 3.5, Φ_c has a closed measurable random absorbing set $B_c = \{B_c(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$, where $B_c(\tau,\omega) = \{(\widetilde{u}^{(c)}, \widetilde{v}^{(c)}) \in l_\rho^2 \times l_\rho^2 : \|\widetilde{u}^{(c)}\|_\rho^2 + \|\widetilde{v}^{(c)}\|_\rho^2 \leq I(c,\tau,\omega)\}$, and a unique random attractor $\mathcal{A}_c = \{\mathcal{A}_c(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(l_\rho^2 \times l_\rho^2)$, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\mathcal{A}_c(\tau,\omega) \subseteq B_c(\tau,\omega)$.

(ii) Given $|c| < 1$. By (3.6), we have

$$I(c,\tau,\omega) \leq I(1,\tau,\omega) < \infty \tag{4.6}$$

and

$$\limsup_{c\to 0} I(c,\tau,\omega) \leq I(1,\tau,\omega). \tag{4.7}$$

So, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\limsup_{c\to 0} \big\|B_c(\tau,\omega)\big\|_\rho = \limsup_{c\to 0} \sup_{x\in B_c(\tau,\omega)} \|x\|_{l_\rho^2\times l_\rho^2} \leq I^{\frac{1}{2}}(1,\tau,\omega). \tag{4.8}$$

Moreover, $B_0(\tau,\omega) = \{(u,v) \in l^2_\rho \times l^2_\rho : \|u\|^2_\rho + \|v\|^2_\rho \leq I(1,\tau,\omega)\}$ is a closed tempered random absorbing set for the continuous cocycle Φ_0 associated with the limiting system (4.4), and

$$\bigcup_{|c|\leq 1} \mathcal{A}_c(\tau,\omega) \subseteq \bigcup_{|c|\leq 1} B_c(\tau,\omega) \subseteq B_0(\tau,\omega). \tag{4.9}$$

(iii) Given $|c| < 1$. Let us prove the precompactness of $\bigcup_{|c|\leq 1} \mathcal{A}_c(\tau,\omega)$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$. For one thing, by Theorem 3.5, for every $\varepsilon > 0$, $c > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exist $T = T(\tau,\omega,B_0,c,\varepsilon) > 0$ and $R = R(\tau,\omega,c,\varepsilon) > 1$ such that, for all $t \geq T$ and $(\widetilde{u}^{(c)}_{\tau-t},\widetilde{v}^{(c)}_{\tau-t}) \in B_0(\tau-t,\theta_{-t}\omega)$, the solution $(\widetilde{u}^{(c)},\widetilde{v}^{(c)})$ of (2.6) satisfies

$$\sum_{|i|>R} \rho_i\big(\big|\widetilde{u}^{(c)}_i\big(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}^{(c)}_{\tau-t},\widetilde{v}^{(c)}_{\tau-t}\big)\big|^2 + \big|\widetilde{v}^{(c)}_i\big(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}^{(c)}_{\tau-t},\widetilde{v}^{(c)}_{\tau-t}\big)\big|^2\big) \leq \varepsilon, \tag{4.10}$$

which, along with (4.9) and the invariance of $\mathcal{A}_c(\tau,\omega)$, we have, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq T$,

$$\sup_{(\widetilde{u}^{(c)},\widetilde{v}^{(c)})\in\bigcup_{|c|\leq 1}\mathcal{A}_c(\tau,\omega)} \sum_{|i|>R} \rho_i\big(\big|\widetilde{u}^{(c)}_i\big|^2 + \big|\widetilde{v}^{(c)}_i\big|^2\big) \leq \varepsilon.$$

By (4.9) we find that the set $\{(\widetilde{u}^{(c)}_i,\widetilde{v}^{(c)}_i)_{|i|\leq R} : (\widetilde{u}^{(c)},\widetilde{v}^{(c)}) \in \bigcup_{|c|\leq 1}\mathcal{A}_c(\tau,\omega)\}$ is bounded in a finite-dimensional space and hence $\bigcup_{|c|\leq 1}\mathcal{A}_c(\tau,\omega)$ is precompact in $l^2_\rho \times l^2_\rho$.

(iv) Let $U = \widetilde{u}^{(c)} - u$, $V = \widetilde{v}^{(c)} - v$, $(U,V) = \widetilde{\varphi}^{(c)} - \varphi$. Let $(\widetilde{u}^{(c,m)}(t,\tau,\omega,\widetilde{u}^{(c)}_\tau,\widetilde{v}^{(c)}_\tau), \widetilde{v}^{(c,m)}(t,\tau,\omega,\widetilde{u}^{(c)}_\tau,\widetilde{v}^{(c)}_\tau))$ and $(u^{(m)}(t,\tau,\omega,u_\tau,v_\tau), v^{(m)}(t,\tau,\omega,u_\tau,v_\tau))$ be the solutions of the following random differential equations with initial data:

$$\begin{cases} \frac{d\widetilde{u}^{(c,m)}}{dt} = \mathbb{B}_m(t,\omega)\widetilde{u}^{(c,m)} - \widetilde{v}^{(c,m)} + f(\widetilde{u}^{(c,m)} + cz(\theta_t\omega)) \\ \qquad\qquad + c\mathbb{B}_m(t,\omega)z(\theta_t\omega) + g(t) - c\lambda z(\theta_t\omega) - cy(\theta_t\omega), \\ \frac{d\widetilde{v}^{(c,m)}}{dt} = \sigma\widetilde{u}^{(c,m)} - \delta\widetilde{v}^{(c,m)} + h(t) + \sigma cz(\theta_t\omega) + c(\mu-\delta)y(\theta_t\omega), \\ \widetilde{u}^{(c,m)}(\tau) = \widetilde{u}_\tau, \qquad \widetilde{v}^{(c,m)}(\tau) = \widetilde{v}_\tau, \end{cases} \tag{4.11}$$

and

$$\begin{cases} \frac{du^{(m)}}{dt} = \mathbb{B}_m(t,\omega)u^{(m)} - v^{(m)} + f(u^{(m)}) + g(t), \\ \frac{dv^{(m)}}{dt} = \sigma u^{(m)} - \delta v^{(m)} + h(t), \\ u^{(m)}(\tau) = u_\tau, \qquad v^{(m)}(\tau) = v_\tau, \end{cases} \tag{4.12}$$

respectively. Then $\widetilde{\varphi}^{(c,m)}(\cdot,\tau,\omega,\widetilde{u}^{(c)}_\tau,\widetilde{v}^{(c)}_\tau)$, $\varphi^{(m)}(\cdot,\tau,\omega,u_\tau,v_\tau) \in C([\tau,+\infty), l^2_\rho \times l^2_\rho)$ and satisfy the differential equations (4.11) and (4.12), respectively. Moreover, $\widetilde{\varphi}^{(c)}(t,\tau,\omega,\widetilde{u}^{(c)}_\tau,\widetilde{v}^{(c)}_\tau)$ and $\varphi(t,\tau,\omega,u_\tau,v_\tau)$ are limit functions of subsequences of $\{\widetilde{\varphi}^{(c,m)}(t,\tau,\omega,\widetilde{u}^{(c)}_\tau,\widetilde{v}^{(c)}_\tau)\}$ and $\{\varphi^{(m)}(t,\tau,\omega,u_\tau,v_\tau)\} \in l^2_\rho \times l^2_\rho$. So $\widetilde{\varphi}^{(c)}(t,\tau,\omega,\widetilde{u}^{(c)}_\tau,\widetilde{v}^{(c)}_\tau) - \varphi(t,\tau,\omega,u_\tau,v_\tau)$ is a limit function of a subsequence of $\{\widetilde{\varphi}^{(c,m)}(t,\tau,\omega,\widetilde{u}^{(c)}_\tau,\widetilde{v}^{(c)}_\tau) - \varphi^{(m)}(t,\tau,\omega,u_\tau,v_\tau)\}$ in $l^2_\rho \times l^2_\rho$, and $(U^{(m)}(t,\tau,\omega,\widetilde{u}^{(c)}_\tau,\widetilde{v}^{(c)}_\tau,u_\tau,v_\tau), V^{(m)}(t,\tau,\omega,\widetilde{u}^{(c)}_\tau,\widetilde{v}^{(c)}_\tau,u_\tau,v_\tau))$ satisfies

$$\begin{cases} \frac{dU^{(m)}}{dt} = \mathbb{B}_m(t,\omega)U^{(m)} - V^{(m)} + f(\widetilde{u}^{(c,m)} + cz(\theta_t\omega)) - f(u^{(m)}) \\ \qquad\qquad + c\mathbb{B}_m(t,\omega)z(\theta_t\omega) - c\lambda z(\theta_t\omega) - cy(\theta_t\omega), \\ \frac{dV^{(m)}}{dt} = \sigma U^{(m)} - \delta V^{(m)} + \sigma cz(\theta_t\omega) + c(\mu-\delta)y(\theta_t\omega), \\ U^{(m)}(\tau) = \widetilde{u}^{(c)}_\tau - u_\tau, \qquad V^{(m)}(\tau) = \widetilde{v}^{(c)}_\tau - v_\tau. \end{cases} \tag{4.13}$$

By taking the inner product of (4.13) with $(U^{(m)}, V^{(m)})$ in $l_\rho^2 \times l_\rho^2$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|U^{(m)}\|_\rho^2 + \frac{1}{\sigma}\|V^{(m)}\|_\rho^2\right) \\
&\quad= \left(\mathbb{B}_m(t,\omega)U^{(m)}, U^{(m)}\right)_\rho + \left(f\left(\widetilde{u}^{(c,m)} + cz(\theta_t\omega)\right) - f\left(u^{(m)}\right), U^{(m)}\right)_\rho \\
&\qquad+ \left(c\mathbb{B}_m(t,\omega)z(\theta_t\omega), U^{(m)}\right)_\rho - c\left(\lambda z(\theta_t\omega) + y(\theta_t\omega), U^{(m)}\right)_\rho \\
&\qquad- \frac{\delta}{\sigma}\|V^{(m)}\|_\rho^2 + \left(cz(\theta_t\omega) + \frac{c(\mu-\delta)}{\sigma}y(\theta_t\omega), V^{(m)}\right)_\rho.
\end{aligned} \tag{4.14}$$

Now let us estimate the terms in (4.14):

$$\left(\mathbb{B}_m(t,\omega)U^{(m)}, U^{(m)}\right)_\rho \le (1+q+\widetilde{q})\boldsymbol{\beta}(\theta_t\omega)\|U^{(m)}\|_\rho^2, \tag{4.15}$$

$$\begin{aligned}
&\left(f\left(\widetilde{u}^{(c,m)} + cz(\theta_t\omega)\right) - f\left(u^{(m)}\right), U^{(m)}\right)_\rho \\
&\quad= \sum_{i\in\mathbb{Z}} \rho_i\left(f_i\left(\widetilde{u}_i^{(c,m)} + cz_i(\theta_t\omega)\right) - f_i\left(u_i^{(m)} + cz_i(\theta_t\omega)\right)\right)\cdot U_i^{(m)} \\
&\qquad+ \sum_{i\in\mathbb{Z}} \rho_i\left(f_i\left(u_i^{(m)} + cz_i(\theta_t\omega)\right) - f_i\left(u_i^{(m)}\right)\right)\cdot U_i^{(m)} \\
&\quad\le \kappa\sum_{i\in\mathbb{Z}}\rho_i\left|\widetilde{u}_i^{(c,m)} - u_i^{(m)}\right|\cdot\left|U_i^{(m)}\right| + \kappa\sum_{i\in\mathbb{Z}}\rho_i\left|cz_i(\theta_t\omega)\right|\cdot\left|U_i^{(m)}\right| \\
&\quad\le \frac{5\kappa}{4}\|U^{(m)}\|_\rho^2 + \kappa c^2\|z(\theta_t\omega)\|_\rho^2,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&\left(c\mathbb{B}_m(t,\omega)z(\theta_t\omega), U^{(m)}\right)_\rho \\
&\quad\le \frac{\kappa}{4}\|U^{(m)}\|_\rho^2 + \frac{c^2(1+2q)(1+2\widetilde{q})\boldsymbol{\beta}^2(\theta_t\omega)}{\kappa}\|z(\theta_t\omega)\|_\rho^2,
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
&-c\left(\lambda z(\theta_t\omega) + y(\theta_t\omega), U^{(m)}\right)_\rho \\
&\quad\le \frac{c^2\lambda^2}{\kappa}\|z(\theta_t\omega)\|_\rho^2 + \frac{c^2}{\kappa}\|y(\theta_t\omega)\|_\rho^2 + \frac{\kappa}{2}\|U^{(m)}\|_\rho^2,
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
&\left(cz(\theta_t\omega) + \frac{c(\mu-\delta)}{\sigma}y(\theta_t\omega), V^{(m)}\right)_\rho \\
&\quad\le \frac{\sigma c^2}{2\delta}\|z(\theta_t\omega)\|_\rho^2 + \frac{c^2(\mu-\delta)^2}{2\delta\sigma}\|y(\theta_t\omega)\|_\rho^2 + \frac{\delta}{\sigma}\|V^{(m)}\|_\rho^2.
\end{aligned} \tag{4.19}$$

It follows from (4.15)-(4.19) that

$$\begin{aligned}
&\frac{d}{dt}\left(\|U^{(m)}\|_\rho^2 + \frac{1}{\sigma}\|V^{(m)}\|_\rho^2\right) \\
&\quad\le \left(2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_t\omega) + 2\kappa\right)\|U^{(m)}\|_\rho^2 \\
&\qquad+ c^2\left(\frac{(1+2q)(1+2\widetilde{q})\boldsymbol{\beta}^2(\theta_t\omega)}{\kappa} + \frac{\sigma}{2\delta} + \kappa\right)\|z(\theta_t\omega)\|_\rho^2 \\
&\qquad+ \frac{c^2(\mu-\delta)^2}{2\delta\sigma}\|y(\theta_t\omega)\|_\rho^2.
\end{aligned} \tag{4.20}$$

Applying the Gronwall inequality to (4.20) from τ to $t+\tau$, we have

$$\begin{aligned} &\|U^{(m)}(t+\tau,\tau,\omega,\widetilde{u}_\tau^{(c)},\widetilde{v}_\tau^{(c)},u_\tau,v_\tau)\|_\rho^2+\frac{1}{\sigma}\|V^{(m)}(t+\tau,\tau,\omega,\widetilde{u}_\tau^{(c)},\widetilde{v}_\tau^{(c)},u_\tau,v_\tau)\|_\rho^2 \\ &\quad\le e^{\int_\tau^{t+\tau}(2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)+2\kappa)\,dr}\left(\|\widetilde{u}_\tau^{(c)}-u_\tau\|_\rho^2+\frac{1}{\sigma}\|\widetilde{v}_\tau^{(c)}-v_\tau\|_\rho^2\right) \\ &\qquad+\int_\tau^{t+\tau}e^{\int_s^{t+\tau}(2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)+2\kappa)\,dr}\big((\mathbb{c}_8+\mathbb{c}_9\boldsymbol{\beta}^2(\theta_s\omega))\|z(\theta_s\omega)\|_\rho^2+\mathbb{c}_{10}\|y(\theta_s\omega)\|_\rho^2\big)\,ds, \end{aligned} \tag{4.21}$$

where $\mathbb{c}_8=\frac{\sigma c^2}{2\delta}+\kappa c^2$, $\mathbb{c}_9=\frac{c^2(1+2q)(1+2\widetilde{q})}{\kappa}$ and $\mathbb{c}_{10}=\frac{c^2(\mu-\delta)^2}{2\delta\sigma}$. Replacing ω in the above by $\theta_{-\tau}\omega$, we have

$$\begin{aligned} &\|U^{(m)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(c)},\widetilde{v}_\tau^{(c)},u_\tau,v_\tau)\|_\rho^2+\frac{1}{\sigma}\|V^{(m)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(c)},\widetilde{v}_\tau^{(c)},u_\tau,v_\tau)\|_\rho^2 \\ &\quad\le e^{\int_0^{t}(2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)+2\kappa)\,dr}\left(\|\widetilde{u}_\tau^{(c)}-u_\tau\|_\rho^2+\frac{1}{\sigma}\|\widetilde{v}_\tau^{(c)}-v_\tau\|_\rho^2\right) \\ &\qquad+\int_0^{t}e^{\int_s^{t}(2(1+q+\widetilde{q})\boldsymbol{\beta}(\theta_r\omega)+2\kappa)\,dr}\big((\mathbb{c}_8+\mathbb{c}_9\boldsymbol{\beta}^2(\theta_s\omega))\|z(\theta_s\omega)\|_\rho^2+\mathbb{c}_{10}\|y(\theta_s\omega)\|_\rho^2\big)\,ds. \end{aligned} \tag{4.22}$$

From (4.22), we find that, for $\tau\in\mathbb{R}$, $t\in\mathbb{R}^+$, $\omega\in\Omega$, $c\to 0$ and $(\widetilde{u}_\tau^{(c)},\widetilde{v}_\tau^{(c)})$, $(u_\tau,v_\tau)\in l_\rho^2\times l_\rho^2$ with $(\widetilde{u}_\tau^{(c)},\widetilde{v}_\tau^{(c)})\to(u_\tau,v_\tau)$,

$$\lim_{c\to 0}\widetilde{\varphi}^{(c,m)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(c)},\widetilde{v}_\tau^{(c)})=\varphi^{(m)}(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)\quad\text{in } l_\rho^2\times l_\rho^2. \tag{4.23}$$

Let $\{c_n\}\subset[-1,1]$ be a sequence of numbers with $c_n\to 0$ when $n\to+\infty$. Then $\widetilde{\varphi}^{(c)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(c)},\widetilde{v}_\tau^{(c)})$ and $\varphi(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)$ being limit functions of subsequences of $\{\widetilde{\varphi}^{(c,m)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(c)},\widetilde{v}_\tau^{(c)})\}$ and $\{\varphi^{(m)}(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)\}$ in $l_\rho^2\times l_\rho^2$ implies that, for $\tau\in\mathbb{R}$, $t\in\mathbb{R}^+$, $\omega\in\Omega$, $c_n\to 0$ and $(\widetilde{u}_\tau^{(c_n)},\widetilde{v}_\tau^{(c_n)})$, $(u_\tau,v_\tau)\in l_\rho^2\times l_\rho^2$ with $(\widetilde{u}_\tau^{(c_n)},\widetilde{v}_\tau^{(c_n)})\to(u_\tau,v_\tau)$, the following holds:

$$\lim_{n\to\infty}\widetilde{\varphi}^{(c_n)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(c_n)},\widetilde{v}_\tau^{(c_n)})=\varphi(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)\quad\text{in } l_\rho^2\times l_\rho^2. \tag{4.24}$$

We completed the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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