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Existence and exponential stability of pseudo almost periodic solutions for impulsive nonautonomous partial stochastic evolution equations

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Abstract

In this paper, the concept of p-mean piecewise pseudo almost periodic for stochastic processes is first introduced. Using the exponential dichotomy techniques and a fixed point strategy with stochastic analysis theory, we establish the existence of p-mean piecewise pseudo almost periodic mild solutions for a class of impulsive nonautonomous partial stochastic evolution equations in Hilbert spaces. Moreover, the exponential stability of p-mean piecewise pseudo almost periodic mild solutions is investigated. Finally, an example is provided to illustrate the obtained theory.

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Keywords: impulsive nonautonomous partial stochastic evolution equations; p-mean piecewise pseudo almost periodic functions; exponential dichotomy; exponential stability; fixed point theorems

1 Introduction

The almost periodic functions plays an important role in describing the phenomena that are similar to the periodic oscillations which can be observed frequently in many fields, such as celestial mechanics, nonlinear vibration, electromagnetic theory, and so on (see [1]). The concept of pseudo almost periodic functions is a natural generalization of almost periodic functions. The study of the existence of pseudo almost periodic solutions is one of the most interesting topics in the qualitative theory of differential equations both due to its mathematical interest as well as due to their applications in physics, mathematical biology, and other areas [2–6]. In the real world, stochastic perturbation is unavoidable. Therefore, we must move from deterministic problems to stochastic ones. The stochastic differential equations with delays and without delays have been extensively studied in the last decades (see [7–12]). Particularly, some authors focused on the existence of almost periodic or pseudo almost periodic solutions to stochastic differential equations in Hilbert spaces [13–21]. Among them, Bezandry and Diagana [22, 23] studied the existence of square-mean almost periodic solutions nonautonomous stochastic differential equations. In [24, 25], the authors introduced the concepts of p-mean pseudo almost periodicity, and studied the existence of p-mean pseudo almost periodic mild solutions to a nonautonomous semilinear stochastic differential equations. Diop *et al.* [26] obtained the existence, uniqueness and global attractiveness of an p-mean pseudo almost periodic solution for stochastic evolution equation driven by a fractional Brownian motion.

The theory of impulsive partial differential equations has become an active area of investigation due to their applications in fields such as mechanics, electrical engineering, medicine biology (see [27–29]). The existence, uniqueness and stability of almost periodic solutions for impulsive differential equations have been considered in abstract spaces by many authors. For example, Henríquez *et al.* [30], Stamov *et al.* [31, 32] discussed the existence and uniqueness of piecewise almost periodic solutions for a class of abstract impulsive semilinear differential equations. Stamov [33] established the existence and asymptotic stability of piecewise almost periodic solutions of impulsive differential equations with time-varying delay. Liu and Zhang [34] studied the existence and exponential stability of piecewise almost periodic solutions to abstract impulsive differential equation. The authors in [35, 36] introduced the concept of piecewise pseudo almost periodic functions on a Banach space and established the existence, uniqueness and exponential stability of piecewise pseudo almost periodic solutions to impulsive differential equations. Bainov and Simeonov [37] concerned with the asymptotic equivalence of impulsive differential equations. However, besides impulse effects and delays, stochastic effects likewise exist in real systems. In recent years, several interesting results on impulsive partial stochastic systems have been reported in many publications [38–41] and the references therein. Further, Zhang *et al.* [42] obtained the existence and uniqueness of almost periodic solutions for a class of impulsive stochastic differential equations with delay by mean of the Banach contraction principle. In [43], the authors investigated the existence and stability of square-mean piecewise almost periodic solutions for nonlinear impulsive stochastic differential equations by using Schauder fixed point theorem. In this paper, we consider the existence and exponential stability of p-mean piecewise pseudo almost periodic mild solutions to the following impulsive nonautonomous partial stochastic evolution equations:

$$dx(t) = \big[A(t)x(t) + g\big(t, x(t)\big)\big]\,dt + f\big(t, x(t)\big)\,dW(t), \quad t \in \mathbb{R}, t \neq t_i, i \in \mathbb{Z}, \tag{1.1}$$

$$\Delta x(t_i) = x\big(t_i^+\big) - x\big(t_i^-\big) = I_i\big(x(t_i)\big), \quad i \in \mathbb{Z}, \tag{1.2}$$

where $A(t) : D(A(t)) \subseteq L^p(\mathbb{P}, \mathbb{H}) \to L^p(\mathbb{P}, \mathbb{H})$ is a family of densely defined closed linear operators satisfying the so-called 'Acquistapace-Terrani' conditions, and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \le t\}$. g, f, I_i, t_i satisfy suitable conditions which will be established later. $x(t_i^+)$, $x(t_i^-)$ represent the right-hand side and the left-hand side limits of $x(\cdot)$ at t_i, respectively.

The study of the asymptotic properties of mild solutions to partial differential equations is one of the fundamental tasks of the analysis theory and finds its application in various fields, such as almost periodicity, asymptotically almost periodic, pseudo almost periodicity, almost automorphy, stability, and so on. There are several papers on the pseudo almost periodicity of mild solutions for partial differential systems, stochastic partial differential systems and impulsive partial differential systems in abstract spaces; see [2–6, 20, 21, 35, 36] and the references therein. On the other hand, the stochastic systems with impulse deserve a study because the system is a more general hybrid system, and that of can be more accurate description of the actual phenomenon in the real world. So it is natural to extend the concept of pseudo almost periodicity of mild solutions to dynamical systems represented by these impulsive systems. To the authors knowledge, no results are available for the existence and exponential stability of p-mean piecewise pseudo almost periodic mild solutions for nonlinear impulsive stochastic system (1.1)-(1.2). The systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, *etc.* Therefore, the system (1.1)-(1.2) involves a wide area of applications in physics and mathematics. Motivated by the above consideration, we will study these interesting problems, which are natural generalizations of the concept of pseudo almost periodicity for stochastic differential equations well known in the theory of infinite dimensional systems.

In this paper, we introduce and develop the notion of p-mean piecewise pseudo almost periodic for stochastic processes, which generalizes in a natural fashion the concept of piecewise almost periodic and p-mean almost periodic stochastic processes. As an application, we study the existence and exponential stability of p-mean piecewise pseudo almost periodic mild solution for the impulsive stochastic evolution equation (1.1)-(1.2) with pseudo almost periodic coefficients. In order to obtain the existence of pseudo almost periodic mild solutions for differential equations, most of the previous research on composition theorems for pseudo almost periodic functions was based upon a Lipschitz condition. It is obvious that the conditions for contraction mapping principle are too strong. In this paper we establish a new composition theorem for p-mean pseudo almost periodic functions under conditions which are different from Lipschitz conditions. Then, using this new composition theorem together with the Leray-Schauder nonlinear alternative and the exponential dichotomy techniques with stochastic analysis theory, we get new existence and exponential stability results. The well-known results that appeared in [2–6, 20, 21, 35, 36] are generalized to the impulsive stochastic systems settings and the case of piecewise pseudo almost periodicity. Moreover, the results are also new for deterministic systems with impulse.

The paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give the existence of p-mean piecewise pseudo almost periodic mild solutions for (1.1)-(1.2). In Section 4, we establish the exponential stability of p-mean piecewise pseudo almost periodic mild solutions for (1.1)-(1.2). In Section 5, an interesting example is given to illustrate our results. Finally, concluding remarks are given in Section 6.

2 Preliminaries

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{R}^+$ stand for the set of natural numbers, integers, real numbers, positive real numbers, respectively. We assume that $(\mathbb{H}, \|\cdot\|)$, $(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ are real separable Hilbert spaces and $(\Omega, \mathcal{F}, \mathbb{P})$ is supposed to be a filtered complete probability space. Define $L^p(\mathbb{P}, \mathbb{H})$, for $p \geq 1$, to be the space of all $\mathbb{H}$-valued random variables x such that $E\|x\|^p = \int_\Omega \|x\|^p \, d\mathbb{P} < \infty$. Then $L^p(\mathbb{P}, \mathbb{H})$ is a Banach space when it is equipped with its natural norm $\|\cdot\|_p$ defined by $\|x\|_p = (\int_\Omega E\|x\|^p \, d\mathbb{P})^{1/p} < \infty$ for each $x \in L^p(\mathbb{P}, \mathbb{H})$. We let $L(\mathbb{K}, \mathbb{H})$ be the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|\cdot\|_{L(\mathbb{K},\mathbb{H})}$; in particular, this is simply denoted by $L(\mathbb{H})$ when $\mathbb{K} = \mathbb{H}$. Furthermore, $L_2^0(\mathbb{K}, \mathbb{H})$ denotes the space of all Q-Hilbert-Schmidt operators from

$\mathbb{K}$ to $\mathbb{H}$ with the norm $\|\psi\|^2_{L_2^0} = \mathrm{Tr}(\psi Q\psi^*) < \infty$ for any $\psi \in L(\mathbb{K}, \mathbb{H})$. Let $C(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$, $BC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ stand for the collection of all continuous functions from $\mathbb{R}$ into $L^p(\mathbb{P}, \mathbb{H})$, the Banach space of all bounded continuous functions from $\mathbb{R}$ into $L^p(\mathbb{P}, \mathbb{H})$, equipped with the sup norm, respectively.

Definition 2.1 ([24]) A stochastic process $x : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ is said to be continuous provided that, for any $s \in \mathbb{R}$,

$$\lim_{t\to s} E\|x(t) - x(s)\|^p = 0.$$

Definition 2.2 ([24]) A stochastic process $x : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ is said to be stochastically bounded provided that

$$\lim_{N\to\infty} \limsup_{t\in\mathbb{R}} \{\mathbb{P}\|x(t)\| > N\} = 0.$$

Let $\mathbb{T}$ be the set consisting of all real sequences $\{t_i\}_{i\in\mathbb{Z}}$ such that $\alpha = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$, $\lim_{i\to\infty} t_i = \infty$, and $\lim_{i\to-\infty} t_i = -\infty$. For $\{t_i\}_{i\in\mathbb{Z}} \in \mathbb{T}$, let $PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ be the space consisting of all stochastically bounded piecewise continuous functions $f : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ such that $f(\cdot)$ is stochastically continuous at t for any $t \notin \{t_i\}_{i\in\mathbb{Z}}$ and $f(t_i) = f(t_i^-)$ for all $i \in \mathbb{Z}$; let $PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ be the space formed by all stochastically piecewise continuous functions $f : \mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}) \to L^p(\mathbb{P}, \mathbb{H})$ such that, for any $x \in L^p(\mathbb{P}, \mathbb{K})$, $f(\cdot, x) \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ and for any $t \in \mathbb{R}$, $f(t, \cdot)$ is stochastically continuous at $x \in L^p(\mathbb{P}, \mathbb{K})$.

Definition 2.3 ([24]) A function $f \in C(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean almost periodic if, for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$, such that every interval J of length $l(\varepsilon)$ contains a number τ with the property that $E\|f(t + \tau) - f(t)\|^p < \varepsilon$ for all $t \in R$. Denote by $AP(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ the set of such functions.

Definition 2.4 (Compare with [28]) A sequence $\{x_n\}$ is called p-mean almost periodic if, for any $\varepsilon > 0$, there exists a relatively dense set of its ε-periods, *i.e.*, there exists a natural number $l = l(\varepsilon)$, such that, for $k \in \mathbb{Z}$, there is at least one number q in $[k, k + l]$, for which inequality $E\|x_{n+q} - x_n\|^p < \varepsilon$ holds for all $n \in \mathbb{N}$. Denote by $AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$ the set of such sequences.

Define $l^\infty(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) = \{x : \mathbb{Z} \to L^p(\mathbb{P}, \mathbb{H}) : \|x\| = \sup_{n\in\mathbb{Z}}(E\|x(n)\|^p)^{1/p} < \infty\}$, and

$$PAP_0(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) = \left\{x \in l^\infty(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{n\to\infty} \frac{1}{2n} \sum_{j=-n}^{n} E\|x(n)\|^p \, dt = 0\right\}.$$

Definition 2.5 A sequence $\{x_n\}_{n\in Z} \in l^\infty(\mathbb{Z}, X)$ is called p-mean pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, $x_n^2 \in PAP_0(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$. Denote by $PAP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$ the set of such sequences.

Definition 2.6 (Compare with [28]) For $\{t_i\}_{i\in\mathbb{Z}} \in \mathbb{T}$, the function $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise almost periodic if the following conditions are fulfilled:

(i) $\{t_i^j = t_{i+j} - t_i\}$, $j \in \mathbb{Z}$, is equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set Q_ε of $\mathbb{R}$ such that for each $\tau \in Q_\varepsilon$ there is an integer $q \in \mathbb{Z}$ such that $|t_{i+q} - t_i - \tau| < \varepsilon$ for all $i \in \mathbb{Z}$.

(ii) For any $\varepsilon > 0$, there exists a positive number $\tilde{\delta} = \tilde{\delta}(\varepsilon)$ such that if the points t' and t'' belong to a same interval of continuity of φ and $|t' - t''| < \tilde{\delta}$, then $E\|f(t') - f(t'')\|^p < \varepsilon$.

(iii) For every $\varepsilon > 0$, there exists a relatively dense set $\tilde{\Omega}(\varepsilon)$ in $\mathbb{R}$ such that if $\tau \in \tilde{\Omega}(\varepsilon)$, then

$$E\|f(t+\tau) - f(t)\|^p < \varepsilon$$

for all $t \in \mathbb{R}$ satisfying the condition $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$. The number τ is called ε-translation number of f.

We denote by $AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ the collection of all the p-mean piecewise almost periodic functions. Obviously, the space $AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ endowed with the sup norm defined by $\|f\|_\infty = \sup_{t\in\mathbb{R}}(E\|f(t)\|^p)^{1/p}$ for any $f \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is a Banach space. Let $UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ be the space of all stochastic functions $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ such that f satisfies the condition (ii) in Definition 2.6.

Definition 2.7 (Compare with [28]) The function $f \in PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise almost periodic in $t \in \mathbb{R}$ uniform in $x \in L^p(\mathbb{P}, \mathbb{K})$ if, for every compact subset $K \subseteq L^p(\mathbb{P}, \mathbb{K})$, $\{f(\cdot, x) : x \in K\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense subset Ω_ε such that

$$E\|f(t+\tau, x) - f(t, x)\|^p < \varepsilon$$

for all $x \in K$, $\tau \in \Omega_\varepsilon$, and $t \in \mathbb{R}$ satisfying $|t - t_i| > \varepsilon$. Denote by $AP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ the set of all such functions.

Denote

$$PC_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) = \Bigl\{f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{t\to\infty} E\|f(t)\|^p = 0\Bigr\},$$

$$PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) = \Bigl\{f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} E\|f(t)\|^p\,dt = 0\Bigr\},$$

$$\begin{aligned} &PAP_T^0(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H})) \\ &\quad = \Bigl\{f \in PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H})) : \\ &\qquad \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} E\|f(t, x)\|^p\,dt = 0 \text{ uniformly with respect to } x \in K, \\ &\qquad \text{where } K \text{ is an arbitrary compact subset of } L^p(\mathbb{P}, \mathbb{K})\Bigr\}. \end{aligned}$$

Definition 2.8 A function $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = h + \varphi$, where $h \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ and $\varphi \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$. Denoted by $PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ the set of all such functions.

$PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is a Banach space with the sup norm $\|\cdot\|_\infty$.

Similar to [2, 35], one has the following.

Remark 2.1

(i) $PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is a translation invariant set of $PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

(ii) $PC_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) \subset PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

Lemma 2.1 *Let* $\{f_n\}_{n\in\mathbb{N}} \subset PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ *be a sequence of functions. If* f_n *converges uniformly to* f, *then* $f \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

One can refer to Lemma 2.5 in [5] for the proof of Lemma 2.1.

Definition 2.9 A function $f \in PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = h + \varphi$, where $h \in AP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ and $\varphi \in PAP_T^0(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$. Denoted by $PAP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ the set of all such functions.

We need the following composition of p-mean pseudo almost periodic processes.

Lemma 2.2 *Assume* $f \in PAP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$. *Assume that the following conditions hold*:

(i) $\{f(t,x) : t \in \mathbb{R}, x \in K\}$ *is bounded for every bounded subset* $K \subset L^p(\mathbb{P}, \mathbb{K})$.

(ii) $f(t,\cdot)$ *is uniformly continuous in each bounded subset of* $L^p(\mathbb{P}, \mathbb{K})$ *uniformly in* $t \in \mathbb{R}$.

If $\phi(\cdot) \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{K}))$ *such that* $\mathbb{R}(\phi) \subset L^p(\mathbb{P}, \mathbb{K})$, *then* $f(\cdot, \phi(\cdot)) \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

Lemma 2.3 *Assume the sequence of vector-valued functions* $\{I_i\}_{i\in\mathbb{Z}}$ *is pseudo almost periodic, i.e., for any* $x \in L^p(\mathbb{P}, \mathbb{H})$, $\{I_i(x), i \in \mathbb{Z}\}$ *is a pseudo almost periodic sequence. Suppose* $\{I_i(x) : i \in \mathbb{Z}, x \in K\}$ *is bounded for every bounded subset* $K \subset L^p(\mathbb{P}, \mathbb{H})$, $I_i(x)$ *is uniformly continuous in* $x \in L^p(\mathbb{P}, \mathbb{H})$ *uniformly in* $\in \mathbb{Z}$. *If* $\phi \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) \cap UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ *such that* $\mathbb{R}(\phi) \subset L^p(\mathbb{P}, \mathbb{K})$, *then* $I_i(\phi(t_i))$ *is pseudo almost periodic*.

One can refer to Lemmas 3.1, 3.4 in [35] for the proof of Lemmas 2.2 and 2.3.

Next, we introduce a useful compactness criterion on $PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$. Let $h : \mathbb{R} \to \mathbb{R}^+$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathbb{R}$ and $h(t) \to \infty$ as $|t| \to \infty$. Define

$$PC_h^0\big(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})\big) = \left\{f \in PC\big(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})\big) : \lim_{|t|\to\infty} \frac{E\|f(t)\|^p}{h(t)} = 0\right\}$$

endowed with the norm $\|f\|_h = \sup_{t\in\mathbb{R}} \frac{E\|f(t)\|^p}{h(t)}$, it is a Banach space.

Similarly as the proof of Lemma 4.2 in [35], one has the following.

Lemma 2.4 *A set* $B \subseteq PC_h^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ *is relatively compact if and only if it verifies the following conditions*:

(i) $\lim_{|t|\to\infty} \frac{E\|f(t)\|^p}{h(t)} = 0$ *uniformly for* $f \in B$.

(ii) $B(t) = \{f(t) : f \in B\}$ *is relatively compact in* $L^p(\mathbb{P}, \mathbb{H})$ *for every* $t \in \mathbb{R}$.

(iii) *The set* B *is equicontinuous on each interval* (t_i, t_{i+1}) $(i \in Z)$.

We also need the following concepts concerning evolution family and exponential dichotomy.

Definition 2.10 ([44]) A family of bounded linear operators $\{U(t,s) : t \geq s, t, s \in \mathbb{R}\}$ on $L^p(\mathbb{P}, \mathbb{H})$ associated with $A(t)$ is said to be an evolution family of operators if the following conditions hold:

(a) $U(s,s) = I$, $U(t,s) = U(t,\tau)U(\tau,s)$ for $t \geq \tau \geq s$ and $t, \tau, s \in \mathbb{R}$.
(b) $(t,s) \to U(t,s) \in L(L^p(\mathbb{P}, \mathbb{H}))$ is strongly continuous for $t > s$.

Definition 2.11 ([44]) An evolution family U is called hyperbolic (or has exponential dichotomy) if there are projections $P(t)$, $t \in \mathbb{R}$, uniformly bounded and strongly continuous in t, and constants $M, \delta > 0$ such that

(a) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$;
(b) the restriction $U_Q(t,s) : Q(s)L^p(\mathbb{P}, \mathbb{H}) \to Q(t)L^p(\mathbb{P}, \mathbb{H})$ is invertible for all $t \geq s$ (and we set $U_Q(s,t) = U_Q(t,s)^{-1}$);
(c) $\|U(t,s)P(s)\| \leq Me^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq Me^{-\delta(t-s)}$ for all $t \geq s$.

Here and below $Q := I - P$. If $P(t) = I$ for $t \in \mathbb{R}$, then $(U(t,s))_{t \geq s}$ is exponentially stable (see [44, 45]).

Definition 2.12 ([44]) If U is a hyperbolic evolution family, then

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s) & \text{if } t \geq s, t, s \in \mathbb{R}, \\ -U_Q(t,s)Q(s) & \text{if } t < s, t, s \in \mathbb{R}, \end{cases}$$

is called Green's function corresponding to U and $P(\cdot)$.

Lemma 2.5 (Leray-Schauder nonlinear alternative [46]) *Let X be a Banach space with $D \subset X$ closed and convex. Assume V is a relatively open subset of D with $0 \in V$ and $\Psi : \overline{V} \to D$ is a compact map, then either*

(i) *Ψ has a fixed point in $\overline{V}$, or*
(ii) *there is a point $x \in \partial V$ and $\lambda \in (0,1)$ with $x \in \lambda\Psi(x)$.*

3 Existence

In this section, we investigate the existence of p-mean piecewise pseudo almost periodic mild solution for system (1.1)-(1.2). We first introduce the notion of mild solution to system (1.1)-(1.2).

Definition 3.1 An $\mathcal{F}_t$-progressively measurable process $\{x(t)\}_{t \in \mathbb{R}}$ is called a mild solution of system (1.1)-(1.2) if, for any $t \in \mathbb{R}$, $t > s$, $s \neq t_i$, $i \in \mathbb{Z}$,

$$\begin{aligned} x(t) = {} & U(t,s)x(s) + \int_s^t U(t,\tau)g\bigl(\tau, x(\tau)\bigr)\,d\tau \\ & + \int_s^t U(t,\tau)f\bigl(\tau, x(\tau)\bigr)\,dW(\tau) + \sum_{s<t_i<t} U(t,t_i)I_i\bigl(x(t_i)\bigr). \end{aligned} \tag{3.1}$$

In order to obtain our main results, we make the following hypotheses:

(H1) There exist constants $\lambda_0 > 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $K_0, K_1 \geq 0$, and $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 + \alpha_2 > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho\bigl(A(t) - \lambda_0\bigr), \qquad \bigl\|R\bigl(\lambda, A(t) - \lambda_0\bigr)\bigr\| \leq \frac{K_1}{1 + |\lambda|},$$

and

$$\left\| \left(A(t) - \lambda_0\right) R\left(\lambda, A(t) - \lambda_0\right) \left[R\left(\lambda_0, A(t)\right) - R\left(\lambda_0, A(s)\right)\right] \right\| \le K_0 |t - s|^{\alpha_1} |\lambda|^{-\alpha_2}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \le \theta\}$.

(H2) $R(\lambda_0, A(\cdot)) \in AP(L(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})))$.

(H3) The evolution family $U(t,s)$ generated by $A(t)$ has an exponential dichotomy with constants $M, \delta > 0$, dichotomy projections $P(t)$, $t \in \mathbb{R}$, and Green's function $\Gamma(t,s)$. Moreover, $U(t,s)$ is compact for $t > s$.

(H4) For each $x \in \mathbb{H}$, $U(t+h,t)x \to x$ as $h \to 0^+$ uniformly for $t \in \mathbb{R}$.

(H5) The functions $g \in PAP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$, $f \in PAP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, L_2^0))$, and $g(t,\cdot), f(t,\cdot)$ are uniformly continuous in each bounded subset of $L^p(\mathbb{P}, \mathbb{K})$ uniformly in $t \in \mathbb{R}$; I_i is a pseudo almost periodic sequence, $I_i(x)$ is uniformly continuous in $x \in L^p(\mathbb{P}, \mathbb{K})$ uniformly in $i \in \mathbb{Z}$.

(H6) There exists a continuous nondecreasing function $\Theta : [0,\infty) \to (0,\infty)$ such that

$$\sup_{t \in \mathbb{R}} \left[E\left\|g(t,x)\right\|^p + E\left\|f(t,x)\right\|_{L_2^0}^p\right] \le \Theta\left(\|x\|^p\right), \quad x \in L^p(\mathbb{P}, \mathbb{K}),$$

and there exist continuous nondecreasing functions $\tilde{\Theta}_i : [0,\infty) \to (0,\infty)$, $i \in \mathbb{Z}$, such that

$$E\left\|I_i(x)\right\|^p \le \tilde{\Theta}_i\left(E\|x\|^p\right), \quad x \in L^p(\mathbb{P}, \mathbb{K}).$$

(H7) There exists a constant $M^* > 0$ such that

$$\frac{M^*}{N_1 \Theta(M^*) + N_2 \sup_{i \in \mathbb{Z}} \tilde{\Theta}_i(M^*)} > 1,$$

where for $p > 2$, $N_1 = 6^{p-1} M^p [\frac{2}{\delta^p} + C_p (\frac{p-2}{p\delta})^{\frac{p-2}{p}} \frac{4}{p\delta}]$, $N_2 = 6^{p-1} M^p \frac{2}{(1-e^{-\delta\alpha})^p}$, and for $p = 2$, $N_1 = 12M^2 [\frac{1}{\delta^2} + \frac{1}{2\delta}]$, $N_2 = 12M^2 \frac{1}{(1-e^{-\delta\alpha})^2}$.

Remark 3.1 Assumption (H1) is usually called the 'Acquistapace-Terreni' condition, which was first introduced in [44] and widely used to investigate nonautonomous evolution equations in [4–6, 45]. If (H1) holds, then there exists a unique evolution family $\{U(t,s), t \ge s > -\infty\}$ on $L^p(\mathbb{P}, \mathbb{H})$.

Lemma 3.1 ([47]) *Assume that* (H1)-(H2) *hold. If* $U(\cdot,\cdot)$ *has an exponential dichotomy with constants* $M, \delta > 0$, *then for each* $\varepsilon > 0$ *and* $h > 0$ *there is a relatively dense set* $\Omega_{\varepsilon,h}$ *such that*

$$\left\|\Gamma(t+\tau, s+\tau) - \Gamma(t,s)\right\| \le \varepsilon e^{-\frac{\delta}{2}|t-s|}, \quad |t-s| > h, t, s \in \mathbb{R}, \tau \in \Omega_{\varepsilon,h}.$$

We abbreviate this property by writing $\Gamma \in AP(L(\mathbb{H}))$.

Lemma 3.2 (Compare with [28]) *Assume that* $f \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$, *the sequence* $\{x_i\}_{i \in \mathbb{Z}} \in AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, *and* $\{t_i^j\}$, $j \in \mathbb{Z}$ *are equipotentially almost periodic. Then, for each* $\varepsilon > 0$, *there exist relatively dense sets* Ω_ε *of* $\mathbb{R}$ *and* Ω_ε *of* $\mathbb{Z}$ *such that*

(i) $E\|f(t+\tau)-f(t)\|^p < \varepsilon$ *for all* $t \in \mathbb{R}$, $|t-t_i| > \varepsilon$, $\tau \in \Omega_\varepsilon$ *and* $i \in \mathbb{Z}$.
(ii) $\|\Gamma(t+\tau,s+\tau)-\Gamma(t,s)\|^p < \varepsilon$ *for all* $t,s \in \mathbb{R}$, $|t-s| > 0$, $|s-t_i| > \varepsilon$, $|t-t_i| > \varepsilon$, $\tau \in \Omega_\varepsilon$ *and* $i \in \mathbb{Z}$.
(iii) $E\|x_{i+q}-x_i\|^p < \varepsilon$ *for all* $q \in \Omega_\varepsilon$ *and* $i \in \mathbb{Z}$.
(iv) $E\|x_i^q-\tau\|^p < \varepsilon$ *for all* $q,\tau \in \Omega_\varepsilon$ *and* $i \in \mathbb{Z}$.

Also, we need to introduce a few preliminary and important results.

Lemma 3.3 *Assume that* (H1)-(H4) *hold. If* $g \in PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ *and if* G *is the function defined by*

$$G(t) := \int_{-\infty}^{t} U(t,\tau)P(\tau)g(\tau)\,d\tau - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)g(\tau)\,d\tau \tag{3.2}$$

for each $t \in \mathbb{R}$, *then* $G \in PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$.

Proof Since $g \in PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$, there exist $g_1 \in AP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ and $g_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$, such that $g = g_1 + g_2$, then $G(t)$ can be decomposed as

$$\begin{aligned} G(t) &= \left[\int_{-\infty}^{t} U(t,\tau)P(\tau)g_1(\tau)\,d\tau - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)g_1(\tau)\,d\tau\right] \\ &\quad + \left[\int_{-\infty}^{t} U(t,\tau)P(\tau)g_2(\tau)\,d\tau - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)g_2(\tau)\,d\tau\right] \\ &=: G_1(t) + G_2(t). \end{aligned}$$

Next we show that $G_1(t) \in AP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ and $G_2(t) \in PAP_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$. Thus, the following verification procedure is divided into three steps.

Step 1. $G_1 \in UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$.

Let $t',t'' \in (t_i,t_{i+1})$, $i \in \mathbb{Z}$, $t'' < t'$. By (H4), for any $\varepsilon > 0$, there exists $0 < \xi < (\frac{\varepsilon}{4\tilde{g}_1})^{1/p}$ such that $0 < t'-t'' < \xi$, we have

$$\bigl\|U(t',t'')-I\bigr\|^p \le \frac{\delta^p\varepsilon}{4\tilde{g}_1}, \qquad \bigl\|U_Q(t',t'')-I\bigr\|^p \le \frac{\delta^p\varepsilon}{4\tilde{g}_1},$$

where $\tilde{g}_1 = 4^{p-1}M^p\|g_1\|_\infty^p$. Using Hölder's inequality, we have

$$\begin{aligned} &E\bigl\|G_1(t')-G_1(t'')\bigr\|^p \\ &\quad\le 4^{p-1}E\left\|\int_{-\infty}^{t''}\bigl[U(t',t'')-I\bigr]U(t'',\tau)P(\tau)g_1(\tau)\,d\tau\right\|^p \\ &\qquad + 4^{p-1}E\left\|\int_{t''}^{t'} U(t',\tau)P(\tau)g_1(\tau)\,d\tau\right\|^p \\ &\qquad + 4^{p-1}E\left\|\int_{t''}^{+\infty}\bigl[U_Q(t',t'')-I\bigr]U_Q(t'',\tau)Q(\tau)g_1(\tau)\,d\tau\right\|^p \\ &\qquad + 4^{p-1}E\left\|\int_{t''}^{t'} U_Q(t',\tau)Q(\tau)g_1(\tau)\,d\tau\right\|^p \\ &\quad\le 4^{p-1}M^p\bigl\|U(t',t'')-I\bigr\|^p\left(\int_{-\infty}^{t''} e^{-\delta(t''-\tau)}\,d\tau\right)^{p-1}\left(\int_{-\infty}^{t''} e^{-\delta(t''-\tau)}E\bigl\|g_1(\tau)\bigr\|^p\,d\tau\right) \end{aligned}$$

$$
\begin{aligned}
&+4^{p-1}M^p\left(\int_{t''}^{t'} e^{-\delta(t'-\tau)}\,d\tau\right)^{p-1}\left(\int_{t''}^{t'} e^{-\delta(t'-\tau)}E\big\|g_1(\tau)\big\|^p\,d\tau\right)\\
&+4^{p-1}M^p\big\|U_Q\big(t',t''\big)-I\big\|^p\left(\int_{t''}^{+\infty} e^{-\delta(t''-\tau)}\,d\tau\right)^{p-1}\left(\int_{t''}^{+\infty} e^{-\delta(t''-\tau)}E\big\|g_1(\tau)\big\|^p\,d\tau\right)\\
&+4^{p-1}M^p\left(\int_{t''}^{t'} e^{-\delta(t'-\tau)}\,d\tau\right)^{p-1}\left(\int_{t''}^{t'} e^{-\delta(t'-\tau)}E\big\|g_1(\tau)\big\|^p\,d\tau\right)\\
\le{}& 4^{p-1}M^p\big\|U\big(t',t''\big)-I\big\|^p\left(\int_{-\infty}^{t''} e^{-\delta(t''-\tau)}\,d\tau\right)^{p}\sup_{\tau\in\mathbb{R}}\big\|g_1(\tau)\big\|^p\\
&+4^{p-1}M^p\left(\int_{t''}^{t'} e^{-\delta(t-\tau)}\,d\tau\right)^{p}\sup_{\tau\in\mathbb{R}}\big\|g_1(\tau)\big\|^p\\
&+4^{p-1}M^p\big\|U_Q\big(t',t''\big)-I\big\|^p\left(\int_{t''}^{+\infty} e^{\delta(t''-\tau)}\,d\tau\right)^{p}\sup_{\tau\in\mathbb{R}}\big\|g_1(\tau)\big\|^p\\
&+4^{p-1}M^p\left(\int_{t''}^{t'} e^{\delta(t-\tau)}\,d\tau\right)^{p}\sup_{\tau\in\mathbb{R}}\big\|g_1(\tau)\big\|^p\\
\le{}& 4^{p-1}M^p\|g_1\|_\infty^p\frac{\delta^p\varepsilon}{4\tilde{g}_1}\left(\int_{-\infty}^{t''} e^{-\delta(t''-\tau)}\,d\tau\right)^{p}+4^{p-1}M^p\|g_1\|_\infty^p\left[\left(\frac{\varepsilon}{4\tilde{g}_1}\right)^{p}\right]^{1/p}\\
&+4^{p-1}M^p\|g_1\|_\infty^p\frac{\delta^p\varepsilon}{4\tilde{g}_1}\left(\int_{t''}^{+\infty} e^{\delta(t''-\tau)}\,d\tau\right)^{p}+4^{p-1}M^p\|g_1\|_\infty^p\left[\left(\frac{\varepsilon}{4\tilde{g}_1}\right)^{p}\right]^{1/p}\\
<{}&\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon.
\end{aligned}
$$

Step 2. $G_1\in AP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$.

Let $t_i<t\le t_{i+1}$. For $\varepsilon>0$, let Ω_ε be a relatively dense set of $\mathbb{R}$ formed by ε-periods of G_1. For $\tau'\in\Omega_\varepsilon$ and $0<\eta<\min\{\varepsilon,\alpha/2\}$, we have

$$
\begin{aligned}
&E\big\|G_1\big(t+\tau'\big)-G_1(t)\big\|^p\\
&\quad\le 4^{p-1}E\left\|\int_{-\infty}^{t}U\big(t+\tau',\tau+\tau'\big)P\big(\tau+\tau'\big)\big[g_1\big(\tau+\tau'\big)-g_1(\tau)\big]\,d\tau\right\|^p\\
&\qquad+4^{p-1}E\left\|\int_{-\infty}^{t}\big[U\big(t+\tau',\tau+\tau'\big)P\big(\tau+\tau'\big)-U(t,\tau)P(\tau)\big]g_1(\tau)\,d\tau\right\|^p\\
&\qquad+4^{p-1}E\left\|\int_{t}^{+\infty}U_Q\big(t+\tau',\tau+\tau'\big)Q\big(\tau+\tau'\big)\big[g_1\big(\tau+\tau'\big)-g_1(\tau)\big]\,d\tau\right\|^p\\
&\qquad+4^{p-1}E\left\|\int_{t}^{+\infty}\big[U_Q\big(t+\tau',\tau+\tau'\big)Q\big(\tau+\tau'\big)-U_Q(t,\tau)Q(\tau)\big]g_1(\tau)\,d\tau\right\|^p\\
&\quad=\sum_{k=1}^{4}J_k.
\end{aligned}
$$

Using Hölder's inequality, it follows that

$$
\begin{aligned}
J_1\le{}&4^{p-1}M^p\left(\int_{-\infty}^{t}e^{-\delta(t-\tau)}\,d\tau\right)^{p-1}\left(\int_{-\infty}^{t}e^{-\delta(t-\tau)}E\big\|g_1\big(\tau+\tau'\big)-g_1(\tau)\big\|^p\,d\tau\right)\\
\le{}&4^{p-1}M^p\left(\int_{-\infty}^{t}e^{-\delta(t-\tau)}\,d\tau\right)^{p-1}\left[\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta}e^{-\delta(t-\tau)}E\big\|g_1\big(\tau+\tau'\big)-g_1(\tau)\big\|^p\,d\tau\right.
\end{aligned}
$$

$$+ \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-\tau)} E\|g_1(\tau+\tau') - g_1(\tau)\|^p \, d\tau$$
$$+ \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-\tau)} E\|g_1(\tau+\tau') - g_1(\tau)\|^p \, d\tau$$
$$+ \int_{t_i}^{t} e^{-\delta(t-\tau)} E\|g_1(\tau+\tau') - g_1(\tau)\|^p \, d\tau \Bigg].$$

Since $g_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$, one has

$$E\|g_1(t+\tau) - g_1(t)\|^p < \varepsilon$$

for all $t \in [t_j + \eta, t_{j+1} - \eta]$, $j \in \mathbb{Z}$, $j \le i$, and $t - \tau \ge t - t_i + t_i - (t_{j+1} - \eta) \ge t - t_i + \alpha(i - 1 - j) + \eta$. Then

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-\tau)} E\|g_1(\tau+\tau') - g_1(\tau)\|^p \, d\tau \\
&\quad \le \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-\tau)} \, d\tau \\
&\quad \le \frac{\varepsilon}{\delta} \sum_{j=-\infty}^{i-1} e^{-\delta(t-t_{j+1}+\eta)} \\
&\quad \le \frac{\varepsilon}{\delta} \sum_{j=-\infty}^{i-1} e^{-\delta\alpha(i-j-1)} \\
&\quad \le \frac{\varepsilon}{\delta(1-e^{-\delta\alpha})}, \\
&\sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-\tau)} E\|g_1(\tau+\tau') - g_1(\tau)\|^p \, d\tau \\
&\quad \le 2^{p-1} \sup_{s\in\mathbb{R}} E\|g_1(\tau)\|^p \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-\tau)} \, d\tau \\
&\quad \le 2^{p-1} \|g_1\|_\infty^p \varepsilon e^{\delta\eta} \sum_{j=-\infty}^{i-1} e^{-\delta(t-t_j)} \\
&\quad \le 2^{p-1} \|g_1\|_\infty^p \varepsilon e^{\delta\eta} e^{-\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\delta\alpha(i-j)} \\
&\quad \le \frac{2^{p-1} \|g_1\|_\infty^p e^{\delta\alpha/2} \varepsilon}{1-e^{-\delta\alpha}}.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-\tau)} E\|g_1(\tau+\tau') - g_1(\tau)\|^p \, d\tau \le \tilde{M}_1 \varepsilon, \\
&\int_{t_i}^{t} e^{-\delta(t-\tau)} E\|g_1(\tau+\tau') - g_1(\tau)\|^p \, d\tau \le \tilde{M}_2 \varepsilon,
\end{aligned}$$

where $\tilde{M}_1$, $\tilde{M}_2$ are some positive constants. Therefore, we get $J_1 \leq \bar{N}_1\varepsilon$ for $\bar{N}_1 > 0$. Using Lemma 3.1, we have

$$\begin{aligned} J_2 &\leq 8^{p-1}E\biggl[\int_{-\infty}^{t-\eta}\bigl\|\bigl[U\bigl(t+\tau',\tau+\tau'\bigr)P\bigl(\tau+\tau'\bigr)-U(t,\tau)P(\tau)\bigr]g_1(\tau)\bigr\|\,d\tau\biggr]^p \\ &\quad + 8^{p-1}E\biggl[\int_{t-\eta}^{t}\bigl\|\bigl[U\bigl(t+\tau',\tau+\tau'\bigr)P\bigl(\tau+\tau'\bigr)-U(t,\tau)P(\tau)\bigr]g_1(\tau)\bigr\|\,d\tau\biggr]^p \\ &\leq 8^{p-1}\varepsilon^p\biggl(\int_{-\infty}^{t-\eta}e^{-\frac{\delta}{2}(t-\tau)}\,d\tau\biggr)^{p-1}\biggl(\int_{-\infty}^{t-\eta}e^{-\frac{\delta}{2}(t-\tau)}E\bigl\|g_1(\tau)\bigr\|^p\,d\tau\biggr) \\ &\quad + 8^{p-1}(2M)^p\eta^p\sup_{\tau\in\mathbb{R}}E\bigl\|g_1(\tau)\bigr\|^p \\ &\leq 8^{p-1}\biggl[\biggl(\frac{2}{\delta}\biggr)^p+(2M)^p\biggr]\|g_1\|_\infty^p\varepsilon^p. \end{aligned}$$

Similar to the proof of J_1, J_2 we have $J_3 \leq \bar{N}_2\varepsilon$, $J_4 \leq \bar{N}_3\varepsilon^p$ for $\bar{N}_2, \bar{N}_3 > 0$. Hence, $G_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$.

Step 3. $G_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$.

For $r > 0$, by Hölder's inequality, we have

$$\begin{aligned} &\frac{1}{2r}\int_{-r}^{r}E\bigl\|G_2(t)\bigr\|^p\,dt \\ &\quad\leq 2^{p-1}\frac{1}{2r}\int_{-r}^{r}E\biggl\|\int_{-\infty}^{t}U(t,\tau)P(\tau)g_2(\tau)\,d\tau\biggr\|^p\,dt \\ &\qquad + 2^{p-1}\frac{1}{2r}\int_{-r}^{r}E\biggl\|\int_{t}^{+\infty}U_Q(t,\tau)Q(\tau)g_2(\tau)\,d\tau\biggr\|^p\,dt \\ &\quad= 2^{p-1}\frac{1}{2r}\int_{-r}^{r}E\biggl\|\int_{0}^{+\infty}U(t,t-\tau)P(t-\tau)g_2(t-\tau)\,d\tau\biggr\|^p\,dt \\ &\qquad + 2^{p-1}\frac{1}{2r}\int_{-r}^{r}E\biggl\|\int_{-\infty}^{0}U_Q(t,t-\tau)Q(t-\tau)g_2(t-\tau)\,d\tau\biggr\|^p\,dt \\ &\quad\leq 2^{p-1}M^p\frac{1}{2r}\int_{-r}^{r}\biggl(\int_{0}^{+\infty}e^{-\delta\tau}\,d\tau\biggr)^{p-1}\int_{0}^{\infty}e^{-\delta\tau}E\bigl\|g_2(t-\tau)\bigr\|^p\,d\tau\,dt \\ &\qquad + 2^{p-1}M^p\frac{1}{2r}\int_{-r}^{r}\biggl(\int_{-\infty}^{0}e^{\delta\tau}\,d\tau\biggr)^{p-1}\int_{-\infty}^{0}e^{\delta\tau}E\bigl\|g_2(t-\tau)\bigr\|^p\,d\tau\,dt \\ &\quad= 2^{p-1}M^p\biggl(\int_{0}^{+\infty}e^{-\delta\tau}\,d\tau\biggr)^{p-1}\int_{0}^{\infty}e^{-\delta\tau}\,d\tau\frac{1}{2r}\int_{-r}^{r}E\bigl\|g_2(t-\tau)\bigr\|^p\,dt \\ &\qquad + 2^{p-1}M^p\biggl(\int_{-\infty}^{0}e^{\delta\tau}\,d\tau\biggr)^{p-1}\int_{-\infty}^{0}e^{\delta\tau}\,d\tau\frac{1}{2r}\int_{-r}^{r}E\bigl\|g_2(t-\tau)\bigr\|^p\,dt. \end{aligned}$$

Since $g_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$, it follows that $g_2(\cdot - \tau) \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ for each $s \in \mathbb{R}$ by Remark 2.1, hence

$$\frac{1}{2r}\int_{-r}^{r}E\bigl\|g_2(t-\tau)\bigr\|^p\,dt \to 0 \quad \text{as } r \to \infty$$

for all $s \in \mathbb{R}$. Using Lebesgue's dominated convergence theorem, we have $G_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$. This completes the proof. □

Lemma 3.4 *Assume that* (H1)-(H4) *hold. If* $f \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, L_2^0))$ *and if* F *is the function defined by*

$$F(t) := \int_{-\infty}^{t} U(t,\tau)P(\tau)f(\tau)\,dW(\tau) - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)f(\tau)\,dW(\tau) \tag{3.3}$$

for each $t \in \mathbb{R}$, *then* $F \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, L_2^0))$.

Proof Since $f \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, L_2^0))$, there exist $f_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P}, L_2^0))$ and $f_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, L_2^0))$, such that $f = f_1 + f_2$, then $F(t)$ can be decomposed as

$$\begin{aligned} F(t) &= \left[\int_{-\infty}^{t} U(t,\tau)P(\tau)f_1(\tau)\,dW(\tau) - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)f_1(\tau)\,dW(\tau)\right] \\ &\quad + \left[\int_{-\infty}^{t} U(t,\tau)P(\tau)f_2(\tau)\,dW(\tau) - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)f_2(\tau)\,dW(\tau)\right] \\ &=: F_1(t) + F_2(t). \end{aligned}$$

Next we show that $F_1(t) \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ and $F_2(t) \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$. Thus, the verification procedure is divided into the following three steps.

Step 1. $F_1 \in UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

Let $t', t'' \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$, $t'' < t'$. By (H4), for any $\varepsilon > 0$, there exists $0 < \xi < (\frac{\varepsilon}{4\tilde{f}_1})^{2(p-1)/p}$ such that $0 < t' - t'' < \xi$, we have, for $p > 2$,

$$\|U(t', t'') - I\|^p \le \frac{(\frac{p\delta}{p-2})^{(p-2)/2}\frac{p\delta}{2}\varepsilon}{4\tilde{f}_1}, \qquad \|U_Q(t', t'') - I\|^p \le \frac{(\frac{p\delta}{p-2})^{(p-2)/2}\frac{p\delta}{2}\varepsilon}{4\tilde{f}_1},$$

where $\tilde{f}_1 = 4^{p-1}M^p C_p \|f_1\|_\infty^p$. Using Hölder's inequality and the Ito integral [48], we have

$$\begin{aligned} &E\|F_1(t') - F_1(t'')\|^p \\ &\quad \le 4^{p-1}E\left\|\int_{-\infty}^{t''} [U(t', t'') - I]U(t'', \tau)P(\tau)f_1(\tau)\,dW(\tau)\right\|^p \\ &\qquad + 4^{p-1}E\left\|\int_{t''}^{t'} U(t', \tau)P(\tau)f_1(\tau)\,dW(\tau)\right\|^p \\ &\qquad + 4^{p-1}E\left\|\int_{t''}^{+\infty} [U_Q(t', t'') - I]U_Q(t'', \tau)Q(\tau)f_1(\tau)\,dW(\tau)\right\|^p \\ &\qquad + 4^{p-1}E\left\|\int_{t''}^{t'} U_Q(t', \tau)Q(\tau)f_1(\tau)\,dW(\tau)\right\|^p \\ &\quad \le 4^{p-1}M^p C_p E\left[\int_{-\infty}^{t''} e^{-2\delta(t''-\tau)}\|U(t', t'') - I\|^2 \|f_1(\tau)\|_{L_2^0}^2\,d\tau\right]^{p/2} \\ &\qquad + 4^{p-1}M^p C_p E\left[\int_{t''}^{t'} e^{-2\delta(t'-\tau)}\|f_1(\tau)\|_{L_2^0}^2\,d\tau\right]^{p/2} \\ &\qquad + 4^{p-1}M^p C_p E\left[\int_{t''}^{+\infty} e^{2\delta(t''-\tau)}\|U_Q(t', t'') - I\|^2 \|f_1(\tau)\|_{L_2^0}^2\,d\tau\right]^{p/2} \\ &\qquad + 4^{p-1}M^p C_p E\left[\int_{t''}^{t'} e^{2\delta(t'-\tau)}\|f_1(\tau)\|_{L_2^0}^2\,d\tau\right]^{p/2} \end{aligned}$$

$$
\begin{aligned}
&\le 4^{p-1}M^pC_p\big\|U(t',t'')-I\big\|^p\left(\int_{-\infty}^{t''}e^{-\frac{p}{p-2}\delta(t''-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\\
&\quad\times\left(\int_{-\infty}^{t''}e^{-\frac{p}{2}\delta(t''-\tau)}\,d\tau\right)\sup_{\tau\in\mathbb{R}}E\big\|f_1(\tau)\big\|_{L_2^0}^p\\
&\quad+4^{p-1}M^pC_p\left(\int_{t''}^{t'}e^{-\frac{p}{p-2}\delta(t-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\left(\int_{t''}^{t'}e^{-\frac{p}{2}\delta(t-\tau)}\,d\tau\right)\sup_{\tau\in\mathbb{R}}E\big\|f_1(\tau)\big\|_{L_2^0}^p\\
&\quad+4^{p-1}M^pC_p\big\|U_Q(t',t'')-I\big\|^p\left(\int_{t''}^{+\infty}e^{\frac{p}{p-2}\delta(t''-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\\
&\quad\times\left(\int_{t''}^{+\infty}e^{\frac{p}{2}\delta(t''-\tau)}\,d\tau\right)\sup_{\tau\in\mathbb{R}}E\big\|f_1(\tau)\big\|_{L_2^0}^p\\
&\quad+4^{p-1}M^pC_p\left(\int_{t''}^{t'}e^{\frac{p}{p-2}\delta(t-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\left(\int_{t''}^{t'}e^{\frac{p}{2}\delta(t-\tau)}\,d\tau\right)\sup_{\tau\in\mathbb{R}}E\big\|f_1(\tau)\big\|_{L_2^0}^p\\
&\le 4^{p-1}M^pC_p\|f_1\|_\infty^p\frac{(\frac{p\delta}{p-2})^{(p-2)/p}\frac{p\delta}{2}\varepsilon}{4\tilde{f}_1}\left(\int_{-\infty}^{t''}e^{-\frac{p}{p-2}\delta(t''-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\\
&\quad\times\left(\int_{-\infty}^{t''}e^{-\frac{p}{2}\delta(t''-\tau)}\,d\tau\right)\\
&\quad+4^{p-1}M^pC_p\|f_1\|_\infty^p\left[\left(\frac{\varepsilon}{4\tilde{f}_1}\right)^{\frac{p}{2(p-1)}}\right]^{\frac{2(p-2)}{p}}\\
&\quad+4^{p-1}M^pC_p\|f_1\|_\infty^p\frac{(\frac{p\delta}{p-2})^{(p-2)/p}\frac{p\delta}{2}\varepsilon}{4\tilde{f}_1}\left(\int_{t''}^{+\infty}e^{\frac{p}{p-2}\delta(t''-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\\
&\quad\times\left(\int_{t''}^{+\infty}e^{\frac{p}{2}\delta(t''-\tau)}\,d\tau\right)\\
&\quad+4^{p-1}M^pC_p\|f_1\|_\infty^p\left[\left(\frac{\varepsilon}{4\tilde{f}_1}\right)^{\frac{p}{2(p-1)}}\right]^{\frac{2(p-2)}{p}}\\
&<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon.
\end{aligned}
$$

For $p=2$. Let $\varepsilon>0$, there exists $0<\xi<\frac{\varepsilon}{4\tilde{f}_1}$ such that $0<t'-t''<\xi$, we have

$$
\big\|U(t',t'')-I\big\|^2\le\frac{\delta\varepsilon}{2\tilde{f}_1},\qquad\big\|U_Q(t',t'')-I\big\|^2\le\frac{\delta\varepsilon}{2\tilde{f}_1},
$$

where $\tilde{f}_1=4M^2\|f_1\|_\infty^2$. Similar to the above discussion, one has

$$
\begin{aligned}
&E\big\|F_1(t')-F_1(t'')\big\|^2\\
&\quad\le 4M^2\big\|U(t',t'')-I\big\|^2\left(\int_{-\infty}^{t''}e^{-2\delta(t''-\tau)}\,d\tau\right)\sup_{\tau\in\mathbb{R}}E\big\|f_1(\tau)\big\|_{L_2^0}^2\\
&\qquad+4M^2\left(\int_{t''}^{t'}e^{-2\delta(t-s)}\,ds\right)\sup_{\tau\in\mathbb{R}}E\big\|f_1(\tau)\big\|_{L_2^0}^2\\
&\qquad+4M^2\big\|U_Q(t',t'')-I\big\|^2\left(\int_{t''}^{+\infty}e^{2\delta(t''-\tau)}\,d\tau\right)\sup_{\tau\in\mathbb{R}}E\big\|f_1(\tau)\big\|_{L_2^0}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 4M^2 \left(\int_{t''}^{t'} e^{2\delta(t-\tau)}\, d\tau \right) \sup_{\tau\in\mathbb{R}} E\big\| f_1(\tau)\big\|^2_{L_2^0} \\
&\le 4M^2 \|f_1\|_\infty^2 \frac{\delta\varepsilon}{2\tilde{f}_1} \left(\int_{-\infty}^{t''} e^{-2\delta(t''-\tau)}\, d\tau \right) + 4M^p \|f_1\|_\infty^2 \left(\frac{\varepsilon}{4\tilde{f}_1} \right) \\
&\quad + 4M^2 \|f_1\|_\infty^2 \frac{\delta\varepsilon}{2\tilde{f}_1} \left(\int_{t''}^{+\infty} e^{2\delta(t''-\tau)}\, d\tau \right) + 4M^p \|f_1\|_\infty^2 \left(\frac{\varepsilon}{4\tilde{f}_1} \right) \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}
$$

Consequently, $F_1 \in UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

Step 2. $F_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

Let $t_i < t \le t_{i+1}$. For $\varepsilon > 0$, let Ω_ε be a relatively dense set of $\mathbb{R}$ formed by ε-periods of F. For $\tau' \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \alpha/2\}$, we have

$$
\begin{aligned}
&E\big\| F_1(t+\tau') - F_1(t)\big\|^p \\
&\le 4^{p-1} E\left\| \int_{-\infty}^{t} U(t+\tau', \tau+\tau') P(\tau+\tau') \big[f_1(\tau+\tau') - f_1(\tau)\big]\, dW(\tau) \right\|^p \\
&\quad + 4^{p-1} E\left\| \int_{-\infty}^{t} \big[U(t+\tau', \tau+\tau') P(\tau+\tau') - U(t,\tau)P(\tau)\big] f_1(\tau)\, dW(\tau) \right\|^p \\
&\quad + 4^{p-1} E\left\| \int_{t}^{+\infty} U_Q(t+\tau', \tau+\tau') Q(\tau+\tau') \big[f_1(\tau+\tau') - f_1(\tau)\big]\, dW(\tau) \right\|^p \\
&\quad + 4^{p-1} E\left\| \int_{t}^{+\infty} \big[U_Q(t+\tau', \tau+\tau') Q(\tau+\tau') - U_Q(t,\tau)Q(\tau)\big] f_1(\tau)\, dW(\tau) \right\|^p \\
&= \sum_{k=1}^{4} \tilde{J}_k.
\end{aligned}
$$

Using Hölder's inequality and the Ito integral, we have, for $p > 2$,

$$
\begin{aligned}
\tilde{J}_1 &\le 4^{p-1} C_p M^p E\left[\int_{-\infty}^{t} e^{-2\delta(t-\tau)} \big\| f_1(\tau+\tau') - f_1(\tau)\big\|^2_{L_2^0}\, d\tau \right]^{p/2} \\
&\le 4^{p-1} C_p M^p \left(\int_{-\infty}^{t} e^{-\frac{p}{p-2}\delta(t-\tau)}\, d\tau \right)^{\frac{p-2}{p}} \\
&\quad \times \left[\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-\tau)} E\big\| f_1(\tau+\tau') - f_1(\tau)\big\|^p_{L_2^0}\, d\tau \right. \\
&\quad + \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\frac{p}{2}\delta(t-\tau)} E\big\| f_1(\tau+\tau') - f_1(\tau)\big\|^p_{L_2^0}\, d\tau \\
&\quad + \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\frac{p}{2}\delta(t-\tau)} E\big\| f_1(\tau+\tau') - f_1(\tau)\big\|^p_{L_2^0}\, d\tau \\
&\quad \left. + \int_{t_i}^{t} e^{-\frac{p}{2}\delta(t-\tau)} E\big\| f_1(\tau+\tau') - f_1(\tau)\big\|^p_{L_2^0}\, d\tau \right].
\end{aligned}
$$

Since $f_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P}, L_2^0))$, one has

$$
E\big\| f_1(t+\tau) - f_1(t)\big\|^p_{L_2^0} < \varepsilon
$$

for all $t \in [t_j + \eta, t_{j+1} - \eta]$, $t - \tau \geq t - t_i + t_i - (t_{j+1} - \eta) \geq t - t_i + \alpha(i - 1 - j) + \eta$, and $j \in \mathbb{Z}$, $j \leq i$. Then

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-\tau)} E\big\|f_1\big(\tau+\tau'\big)-f_1(\tau)\big\|_{L_2^0}^p \,d\tau \\
&\quad\leq \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\delta(t-\tau)} \,d\tau \\
&\quad\leq \frac{2}{\delta p} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta(t-t_{j+1}+\eta)} \\
&\quad\leq \frac{2\varepsilon}{\delta p} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j-1)} \\
&\quad\leq \frac{2\varepsilon}{\delta p(1-e^{-\delta\alpha})}, \\
&\sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\frac{p}{2}\delta(t-\tau)} E\big\|f_1\big(\tau+\tau'\big)-f_1(\tau)\big\|_{L_2^0}^p \,d\tau \\
&\quad\leq 2^{p-1} \sup_{s\in\mathbb{R}} E\big\|f_1(\tau)\big\|_{L_2^0}^p \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+1}+\eta} e^{-\frac{p}{2}\delta(t-\tau)} \,d\tau \\
&\quad\leq 2^{p-1} \sup_{s\in\mathbb{R}} E\big\|f_1(\tau)\big\|_{L_2^0}^p \varepsilon e^{\frac{p}{2}\delta\eta} e^{-\frac{p}{2}\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j)} \\
&\quad\leq 2^{p-1} \sup_{s\in\mathbb{R}} E\big\|f_1(s)\big\|_{L_2^0}^p \varepsilon e^{\frac{p}{2}\delta\eta} e^{-\frac{p}{2}\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\delta\alpha(i-j)} \\
&\quad\leq \frac{2^{p-1}\|f_1\|_\infty^p e^{\delta\alpha/4}\varepsilon}{1-e^{-\frac{p}{2}\delta\alpha}}.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\frac{p}{2}\delta(t-s)} E\big\|f_1\big(\tau+\tau'\big)-f_1(\tau)\big\|_{L_2^0}^p \,ds \leq \tilde{M}_3\varepsilon, \\
&\int_{t_i}^{t} e^{-\frac{p}{2}\delta(t-s)} E\big\|f_1\big(\tau+\tau'\big)-f_1(\tau)\big\|_{L_2^0}^p \,ds \leq \tilde{M}_4\varepsilon,
\end{aligned}$$

where $\tilde{M}_3, \tilde{M}_4$ are some positive constants. Therefore, we get $\tilde{J}_1 \leq \tilde{N}_1\varepsilon$ for $\tilde{N}_1 > 0$. For $p = 2$, we have

$$\begin{aligned}
\tilde{J}_1 &\leq 4M^p E \int_{-\infty}^{t} e^{-2\delta(t-\tau)} \big\|f_1\big(\tau+\tau'\big)-f_1(\tau)\big\|_{L_2^0}^2 \,d\tau \\
&\leq 4M^2 \Bigg[\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-2\delta(t-\tau)} E\big\|f_1\big(\tau+\tau'\big)-f_1(\tau)\big\|_{L_2^0}^2 \,d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-2\delta(t-\tau)} E\big\|f_1(\tau+\tau') - f_1(\tau)\big\|_{L_2^0}^2 \, d\tau \\
&+ \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-2\delta(t-\tau)} E\big\|f_1(\tau+\tau') - f_1(\tau)\big\|_{L_2^0}^2 \, d\tau \\
&+ \int_{t_i}^{t} e^{-2\delta(t-\tau)} E\big\|f_1(\tau+\tau') - f_1(\tau)\big\|_{L_2^0}^2 \, d\tau \Bigg].
\end{aligned}
$$

Similar to the above proof, we get $\tilde{J}_1 \leq \tilde{N}_2\varepsilon$ for $\tilde{N}_2 > 0$.

Using Lemma 3.1, we have, for $p > 2$,

$$
\begin{aligned}
\tilde{J}_2 &\leq 8^{p-1} E\bigg[\int_{-\infty}^{t-\eta} \big\|\big[U(t+\tau',\tau+\tau')P(\tau+\tau') - U(t,\tau)P(\tau)\big]f_1(\tau)\big\| \, dW(\tau)\bigg]^p \\
&\quad + 8^{p-1} E\bigg[\int_{t-\eta}^{t} \big\|\big[U(t+\tau',\tau+\tau')P(\tau+\tau') - U(t,\tau)P(\tau)\big]f_1(\tau)\big\| \, dW(\tau)\bigg]^p \\
&\leq 8^{p-1} C_p \varepsilon^p E\bigg[\int_{-\infty}^{t-\eta} e^{-\delta(t-\tau)} \big\|f_1(\tau)\big\|_{L_2^0}^2 \, d\tau\bigg]^{p/2} \\
&\quad + 8^{p-1} C_p E\bigg[\int_{t-\eta}^{t} (2M)^2 \big\|f_1(\tau)\big\|_{L_2^0}^2 \, d\tau\bigg]^{p/2} \\
&\leq 8^{p-1} C_p \varepsilon^p \bigg(\int_{-\infty}^{t-\eta} e^{-\frac{p}{p-2}\frac{\delta}{2}(t-\tau)} \, d\tau\bigg)^{\frac{p-2}{p}} \bigg(\int_{-\infty}^{t-\eta} e^{-\frac{\delta}{4}(t-\tau)} E\big\|f_1(\tau)\big\|_{L_2^0}^p \, d\tau\bigg) \\
&\quad + 8^{p-1} C_p (2M)^p \eta^{\frac{2(p-1)}{p}} \sup_{\tau\in\mathbb{R}} E\big\|f_1(\tau)\big\|_{L_2^0}^p \\
&\leq 8^{p-1} C_p \bigg[\bigg(\frac{2(p-2)}{p\delta}\bigg)^{\frac{p-2}{p}} \frac{4}{p\delta}\varepsilon^{p-1} + (2M)^p \varepsilon^{\frac{p-2}{p}}\bigg] \|f_1\|_\infty^p \varepsilon.
\end{aligned}
$$

For $p = 2$, we have

$$
\begin{aligned}
\tilde{J}_2 &\leq 8E\bigg[\int_{-\infty}^{t-\eta} \big\|\big[U(t+\tau',\tau+\tau')P(\tau+\tau') - U(t,\tau)P(\tau)\big]f_1(\tau)\big\| \, dW(\tau)\bigg]^2 \\
&\quad + 8E\bigg[\int_{t-\eta}^{t} \big\|\big[U(t+\tau',\tau+\tau')P(\tau+\tau') - U(t,\tau)P(\tau)\big]f_1(\tau)\big\| \, dW(\tau)\bigg]^2 \\
&\leq 8\varepsilon^2 \int_{-\infty}^{t-\eta} e^{-\delta(t-\tau)} E\big\|f_1(\tau)\big\|_{L_2^0}^2 \, d\tau + 8\int_{t-\eta}^{t} (2M)^2 E\big\|f_1(\tau)\big\|_{L_2^0}^2 \, d\tau \\
&\leq 8\bigg(\varepsilon^2 \int_{-\infty}^{t-\eta} e^{-\delta(t-\tau)} \, d\tau + (2M)^2\eta\bigg) \sup_{\tau\in\mathbb{R}} E\big\|f_1(\tau)\big\|_{L_2^0}^2 \\
&\leq 8\bigg[\frac{\varepsilon}{\delta} + (2M)^2\bigg] \|f_1\|_\infty^2 \varepsilon.
\end{aligned}
$$

Similar to the proof of $\tilde{J}_1$, $\tilde{J}_2$ we have $\tilde{J}_3 \leq \tilde{N}_3\varepsilon$, $\tilde{J}_4 \leq \tilde{N}_4\varepsilon$ for $\tilde{N}_3, \tilde{N}_4 > 0$. Hence, $F_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$.

Step 3. $F_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$.

For $r > 0$, by Hölder's inequality and the Ito integral, we have, for $p > 2$,

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r} E\|F_2(t)\|^p\,dt \\
&\quad\le 2^{p-1}\frac{1}{2r}\int_{-r}^{r} E\left\|\int_{-\infty}^{t} U(t,\tau)P(\tau)f_2(\tau)\,dW(\tau)\right\|^p dt \\
&\qquad + 2^{p-1}\frac{1}{2r}\int_{-r}^{r} E\left\|\int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)f_2(\tau)\,dW(\tau)\right\|^p dt \\
&\quad = 2^{p-1}\frac{1}{2r}\int_{-r}^{r} E\left\|\int_{0}^{\infty} U(t,t-\tau)P(t-\tau)f_2(t-\tau)\,dW(s)\right\|^p dt \\
&\qquad + 2^{p-1}\frac{1}{2r}\int_{-r}^{r} E\left\|\int_{-\infty}^{0} U_Q(t,t-\tau)Q(t-\tau)f_2(t-\tau)\,dW(s)\right\|^p dt \\
&\quad\le 2^{p-1}C_p\frac{1}{2r}\int_{-r}^{r} E\left[\int_{0}^{\infty} e^{-2\tau}\|f_2(t-\tau)\|_{L_2^0}^2\,d\tau\right]^{p/2} dt \\
&\qquad + 2^{p-1}C_p\frac{1}{2r}\int_{-r}^{r} E\left[\int_{-\infty}^{0} e^{2\tau}\|f_2(t-\tau)\|_{L_2^0}^2\,d\tau\right]^{p/2} dt \\
&\quad\le 2^{p-1}M^pC_p\frac{1}{2r}\int_{-r}^{r}\left(\int_{0}^{\infty} e^{-\frac{p}{p-2}\delta\tau}\,d\tau\right)^{\frac{p-2}{p}}\int_{0}^{\infty} e^{-\frac{p}{2}\delta s}E\|f_2(t-\tau)\|_{L_2^0}^p\,d\tau\,dt \\
&\qquad + 2^{p-1}M^pC_p\frac{1}{2r}\int_{-r}^{r}\left(\int_{-\infty}^{0} e^{-\frac{p}{p-2}\delta\tau}\,d\tau\right)^{\frac{p-2}{p}}\int_{-\infty}^{0} e^{-\frac{p}{2}\delta s}E\|f_2(t-\tau)\|_{L_2^0}^p\,d\tau\,dt \\
&\quad = 2^{p-1}M^pC_p\left(\int_{0}^{\infty} e^{-\frac{p-2}{p}\delta\tau}\,d\tau\right)^{\frac{p-2}{p}}\int_{0}^{\infty} e^{-\frac{p}{2}\delta s}\,ds\frac{1}{2r}\int_{-r}^{r} E\|f_2(t-\tau)\|_{L_2^0}^p\,dt \\
&\qquad + 2^{p-1}M^pC_p\left(\int_{-\infty}^{0} e^{-\frac{p-2}{p}\delta\tau}\,d\tau\right)^{\frac{p-2}{p}}\int_{0}^{\infty} e^{-\frac{p}{2}\delta s}\,ds\frac{1}{2r}\int_{-r}^{r} E\|f_2(t-\tau)\|_{L_2^0}^p\,dt.
\end{aligned}$$

For $p = 2$, we have

$$\begin{aligned}
&\frac{2}{2r}\int_{-r}^{r} E\left\|\int_{-\infty}^{t} U(t,\tau)f_2(\tau)\,dW(\tau)\right\|^2 dt \\
&\qquad + \frac{2}{2r}\int_{-r}^{r} E\left\|\int_{+\infty}^{t} U_Q(t,\tau)f_2(\tau)\,dW(\tau)\right\|^2 dt \\
&\quad\le 2M^2\frac{1}{2r}\int_{-r}^{r}\int_{0}^{\infty} e^{-2\tau}E\|f_2(t-\tau)\|_{L_2^0}^2\,d\tau\,dt \\
&\qquad + 2M^2\frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{0} e^{2\tau}E\|f_2(t-\tau)\|_{L_2^0}^2\,d\tau\,dt \\
&\quad = 2M^2\left(\int_{0}^{\infty} e^{-2\delta\tau}\,d\tau\right)\frac{1}{2r}\int_{-r}^{r} E\|f_2(t-\tau)\|_{L_2^0}^p\,dt \\
&\qquad + 2M^2\left(\int_{-\infty}^{0} e^{2\delta\tau}\,d\tau\right)\frac{1}{2r}\int_{-r}^{r} E\|f_2(t-\tau)\|_{L_2^0}^p\,dt.
\end{aligned}$$

Since $f_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$, it follows that $f_2(\cdot - s) \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ for each $s \in \mathbb{R}$ by Remark 2.1, hence

$$\frac{1}{2r}\int_{-r}^{r} E\left\|\int_{-\infty}^{t} f_2(s)\,dW(s)\right\|^p dt \to 0 \quad \text{as } r \to \infty$$

for all $s \in \mathbb{R}$. Using Lebesgue's dominated convergence theorem, we have $F_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$. This completes the proof. □

Lemma 3.5 *Assume that* (H1)-(H4) *hold. If* $\gamma_i \in PAP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, $i \in \mathbb{Z}$, *and if* $\tilde{\gamma}_i$ *is the function defined by*

$$\tilde{\gamma}_i(t) := \sum_{t_i<t} U(t,t_i)P(t_i)\gamma_i - \sum_{t<t_i} U_Q(t,t_i)Q(t_i)\gamma_i \tag{3.4}$$

for each $t \in \mathbb{R}$, *then* $\tilde{\gamma}_i \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

Proof Since $\gamma_i \in PAP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, there exist $\gamma_{1,i} \in AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$ and $\gamma_{2,i} \in PAP_0(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, such that $\gamma_i = \gamma_{1,i} + \gamma_{2,i}$, then $\tilde{\gamma}_i(t)$ can be decomposed as

$$\begin{aligned} \tilde{\gamma}_i(t) &= \Big[\sum_{t_i<t} U(t,t_i)P(t_i)\gamma_{1,i} - \sum_{t<t_i} U_Q(t,t_i)Q(t_i)\gamma_{1,i}\Big] \\ &\quad + \Big[\sum_{t_i<t} U(t,t_i)P(t_i)\gamma_{2,i} - \sum_{t<t_i} U_Q(t,t_i)Q(t_i)\gamma_{2,i}\Big] \\ &=: \Pi_{1,i}(t) + \Pi_{2,i}(t). \end{aligned}$$

Next we show that $\Pi_{1,i}(t) \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ and $\Pi_{2,i}(t) \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$. Thus, the verification procedure is divided into the following three steps.

Step 1. $\Pi_{1,i} \in UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

Let $t', t'' \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$, $t'' < t'$. By (H4), for any $\varepsilon > 0$, we have

$$\big\|U(t',t'') - I\big\|^p \le \frac{(1-e^{-\delta\alpha})^p \varepsilon}{2\tilde{\gamma}^1}, \qquad \big\|U_Q(t',t'') - I\big\|^p \le \frac{(1-e^{-\delta\alpha})^p \varepsilon}{2\tilde{\gamma}^1},$$

where $\tilde{\gamma}^1 = 2^{p-1}M^p\|\gamma_{1,i}\|_\infty^p$. Using Hölder's inequality, we have

$$\begin{aligned} &E\big\|\Pi_{1,i}(t') - \Pi_{1,i}(t'')\big\|^p \\ &\quad\le 2^{p-1}E\bigg\|\sum_{t_i<t''}\big[U(t',t'') - I\big]U(t'',t_i)P(t_i)\gamma_{1,i}\bigg\|^p \\ &\qquad + 2^{p-1}E\bigg\|\sum_{t'<t_i}\big[U_Q(t',t'') - I\big]U_Q(t'',t_i)Q(t_i)\gamma_{1,i}\bigg\|^p \\ &\quad\le 2^{p-1}M^p\big\|U(t',t'') - I\big\|^p\bigg(\sum_{t_i<t''} e^{-\delta(t''-t_i)}\bigg)^{p-1}\bigg(\sum_{t_i<t''} e^{-\delta(t''-t_i)}E\|\gamma_{1,i}\|^p\bigg) \\ &\qquad + 2^{p-1}M^p\big\|U_Q(t',t'') - I\big\|^p\bigg(\sum_{t'<t_i} e^{\delta(t''-t_i)}\bigg)^{p-1}\bigg(\sum_{t'<t_i} e^{\delta(t''-t_i)}E\|\gamma_{1,i}\|^p\bigg) \\ &\quad\le 2^{p-1}M^p\big\|U(t',t'') - I\big\|^p\bigg(\sum_{t_i<t''} e^{-\delta(t''-t_i)}\bigg)^{p}\sup_{i\in\mathbb{Z}}E\|\gamma_{1,i}\|^p \\ &\qquad + 2^{p-1}M^p\big\|U_Q(t',t'') - I\big\|^p\bigg(\sum_{t''<t_i} e^{\delta(t''-t_i)}\bigg)^{p}\sup_{i\in\mathbb{Z}}E\|\gamma_{1,i}\|^p \end{aligned}$$

$$\begin{aligned}
&\le 2^{p-1}M^p \frac{(1-e^{-\delta\gamma})^p \varepsilon}{2\tilde{\gamma}^1}\Bigl(\sum_{t_i<t''} e^{-\delta(t''-t_i)}\Bigr)^p \|\gamma_{1,i}\|_\infty^p \\
&\quad + 2^{p-1}M^p \frac{(1-e^{-\delta\gamma})^p \varepsilon}{2\tilde{\gamma}^1}\Bigl(\sum_{t'<t_i} e^{\delta(t''-t_i)}\Bigr)^p \|\gamma_{1,i}\|_\infty^p \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Consequently, $\Pi_{1,i} \in UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

Step 2. $\Pi_{1,i} \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

For $\varepsilon > 0$, Ω_ε be a relatively dense set of $\mathbb{R}$ formed by ε-periods of $\Pi_{1,i}$. For $\tau' \in \Omega_\varepsilon$ and $0 < \eta < \min\{\varepsilon, \alpha/2\}$, we have

$$\begin{aligned}
&E\bigl\|\Pi_{1,i}(t+\tau') - \Pi_{1,i}(t)\bigr\|^p \\
&\le 2^{p-1}E\Bigl\|\sum_{t_i<t+\tau'} U(t+\tau', t_i)P(t_i)\gamma_{1,i} - \sum_{t_i<t} U(t,t_i)P(t_i)\gamma_{1,i}\Bigr\|^p \\
&\quad + 2^{p-1}E\Bigl\|\sum_{t+\tau'<t_i} U_Q(t+\tau', t_i)Q(t_i)\gamma_{1,i} - \sum_{t<t_i} U_Q(t,t_i)Q(t_i)\gamma_{1,i}\Bigr\|^p \\
&\le 4^{p-1}E\Bigl\|\sum_{t_i<t} U(t+\tau', t_{i+q})P(t_{i+q})\gamma_{1,i+q} - \sum_{t_i<t} U(t+\tau', t_{i+q})P(t_{i+q})\gamma_{1,i}\Bigr\|^p \\
&\quad + 4^{p-1}E\Bigl\|\sum_{t_i<t} U(t+\tau', t_{i+q})P(t_{i+q})\gamma_{1,i} - \sum_{t_i<t} U(t,t_i)P(t_i)\gamma_{1,i}\Bigr\|^p \\
&\quad + 4^{p-1}E\Bigl\|\sum_{t<t_i} U_Q(t+\tau', t_{i+q})Q(t_{i+q})\gamma_{1,i+q} - \sum_{t<t_i} U_Q(t+\tau', t_{i+q})Q(t_{i+q})\gamma_{1,i}\Bigr\|^p \\
&\quad + 4^{p-1}E\Bigl\|\sum_{t_i<t} U(t+\tau', t_{i+q})P(t_{i+q})\gamma_{1,i} - \sum_{t_i<t} U(t,t_i)P(t_i)\gamma_{1,i}\Bigr\|^p \\
&= \sum_{k=1}^4 \hat{J}_k.
\end{aligned}$$

For any $\varepsilon > 0$, by Lemma 3.2, there exist relative dense sets of real numbers Ω_ε and integers Q_ε, for every $\tau' \in \Omega_\varepsilon$, there exists at least one number $q \in Q_\varepsilon$ such that $t_i < t \le t_{i+1}$, $\tau' \in \Omega_\varepsilon$, $q \in Q_\varepsilon$, $|t - t_i| > \varepsilon$, $|t - t_{i+1}| > \varepsilon$, $j \in \mathbb{Z}$, one has $t + \tau' > t_i + \varepsilon + \tau' > t_{i+q}$ and $t_{i+q+1} > t_{i+1} + \tau' - \varepsilon > t + \tau'$, that is $t_{i+q} < t + \tau' < t_{i+q+1}$, such that $|t^q - \tau| < \varepsilon$, $i \in \mathbb{Z}$ and $E\|\gamma_{1,i+q} - \gamma_{1,i}\|^p < \varepsilon$, $q \in Q_\varepsilon$, $i \in \mathbb{Z}$. Then

$$\begin{aligned}
\hat{J}_1 &\le 4^{p-1}E\Bigl[\sum_{t_i<t}\bigl\|U(t+\tau', t_{i+q})P(t_{i+q})\bigr\|\|\gamma_{1,i+q} - \gamma_{1,i}\|\Bigr]^p \\
&\le 4^{p-1}M^p E\Bigl[\Bigl(\sum_{t_i<t} e^{-\delta(t-t_i)}\Bigr)^{p-1}\Bigl(\sum_{t_i<t} e^{-\delta(t-t_i)}\|\gamma_{1,i+q} - \gamma_{1,i}\|^p\Bigr)\Bigr] \\
&\le 4^{p-1}M^p \Bigl(\sum_{t_i<t} e^{-\delta(t-t_i)}\Bigr)^p E\|\gamma_{1,i+q} - \gamma_{1,i}\|^p \\
&\le \frac{4^{p-1}M^p \varepsilon}{(1-e^{-\delta\gamma})^p}
\end{aligned}$$

and

$$\begin{aligned}\hat{J}_2 &\le 4^{p-1}E\Bigg[\sum_{t_i<t}\big\|U\big(t+\tau',t_{i+q}\big)P(t_{i+q})-U(t,t_i)P(t_i)\big\|\|\gamma_{1,i}\|\Bigg]^p\\ &\le 4^{p-1}\varepsilon^p E\Bigg[\Bigg(\sum_{t_i<t}e^{-\frac{\delta}{2}(t-t_i)}\Bigg)^{p-1}\Bigg(\sum_{t_i<t}e^{-\frac{\delta}{2}(t-t_i)}\|\gamma_{1,i}\|^p\Bigg)\Bigg]\\ &\le 4^{p-1}\varepsilon^p\Bigg(\sum_{t_i<t}e^{-\frac{\delta}{2}(t-t_i)}\Bigg)^p\sup_{i\in\mathbb{Z}}E\|\gamma_{1,i}\|^p\\ &\le \frac{4^{p-1}\varepsilon^p}{(1-e^{-\frac{\delta}{2}\gamma})^p}\sup_{i\in\mathbb{Z}}E\|\gamma_{1,i}\|^p.\end{aligned}$$

Similar to the proof of $\hat{J}_1$, $\hat{J}_2$ we have $\hat{J}_3\le\hat{N}_1\varepsilon$, $\hat{J}_4\le\hat{N}_2\varepsilon$ for $\hat{N}_1,\hat{N}_2>0$. Hence, $\Pi_{1,i}\in AP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$.

Step 3. $\Pi_{2,i}\in PAP_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$.

For $r>0$, we have

$$\begin{aligned}\frac{1}{2r}\int_{-r}^{r}E\big\|\Pi_{2,i}(t)\big\|^p\,dt &\le 2^{p-1}\frac{1}{2r}\int_{-r}^{r}E\Bigg\|\sum_{t_i<t}U(t,t_i)P(t_i)\gamma_{2,i}\Bigg\|^p\,dt\\ &\quad+2^{p-1}\frac{1}{2r}\int_{-r}^{r}E\Bigg\|\sum_{t<t_i}U_Q(t,t_i)Q(t_i)\gamma_{2,i}\Bigg\|^p\,dt.\end{aligned}$$

For a given $i\in\mathbb{Z}$, define the function $\nu(t)$ by $\nu(t)=U(t,t_i)P(t_i)\gamma_{2,i}$, $t_i<t\le t_{i+1}$, then

$$\lim_{t\to\infty}E\big\|\nu(t)\big\|^p=\lim_{t\to\infty}E\big\|U(t,t_i)P(t_i)\gamma_{2,i}\big\|^p\le\lim_{t\to\infty}M^pe^{-p\delta(t-t_i)}\sup_{i\in\mathbb{Z}}E\|\gamma_{2,i}\|^p=0.$$

Thus $\nu\in PC_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\subset PAP_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$. Define $\nu_j:\mathbb{R}\to L^p(\mathbb{P},\mathbb{H})$ by

$$\nu_j(t)=U(t,t_{i-j})P(t_{i-j})\gamma_{2,i-j},\quad t_i<t\le t_{i+1},j\in\mathbb{N}.$$

So $\nu_j\in PAP_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$. Moreover,

$$\begin{aligned}E\big\|\nu_j(t)\big\|^p&=E\big\|U(t,t_{i-j})P(t_{i-j})\gamma_{2,i-j}\big\|^p\\ &\le M^pe^{-p\delta(t-t_{i-j})}\sup_{i\in\mathbb{Z}}E\|\gamma_{2,i}\|^p\\ &\le M^pe^{-p\delta(t-t_i)}e^{-p\delta\alpha j}\sup_{i\in\mathbb{Z}}E\|\gamma_{2,i}\|^p.\end{aligned}$$

Therefore, the series $\sum_{j=0}^{\infty}\nu_j$ is uniformly convergent on $\mathbb{R}$. By Lemma 2.2, one has

$$\sum_{t_i<t}U(t,t_i)P(t_i)\gamma_{2,i}=\sum_{j=0}^{\infty}\nu_j(t)\in PAP_T^0\big(\mathbb{R},L^p(\mathbb{P},\mathbb{H})\big),$$

that is,

$$2^{p-1}\frac{1}{2r}\int_{-r}^{r}E\Bigg\|\sum_{t_i<t}U(t,t_i)P(t_i)\gamma_{2,i}\Bigg\|^p\,dt\to0\quad\text{as }r\to\infty.$$

Similarly, we have

$$2^{p-1}\frac{1}{2r}\int_{-r}^{r}E\left\|\sum_{t_i<t}U_Q(t,t_i)Q(t_i)\gamma_{2,i}\right\|^p dt\to 0 \quad \text{as } r\to\infty.$$

Hence, $\Pi_{2,i}\in PAP_T^0(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ and, we get $\tilde{\gamma}_i\in PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$. This completes the proof. □

Theorem 3.1 *Assume that* (H1)-(H7) *are satisfied. Then system* (1.1)-(1.2) *has at least one p-mean piecewise pseudo almost periodic mild solution on* $\mathbb{R}$.

Proof Consider the operator $\Psi: PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\to PC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ defined by

$$\begin{aligned}(\Psi x)(t) ={}& \biggl[\int_{-\infty}^{t}U(t,\tau)P(\tau)g\bigl(\tau,x(\tau)\bigr)\,d\tau+\int_{-\infty}^{t}U(t,\tau)P(\tau)f\bigl(\tau,x(\tau)\bigr)\,dW(\tau)\\ &+\sum_{t_i<t}U(t,t_i)P(t_i)I_i(x_i)\biggr]+\biggl[-\int_{t}^{+\infty}U_Q(t,\tau)Q(\tau)g\bigl(\tau,x(\tau)\bigr)\,d\tau\\ &-\int_{t}^{+\infty}U_Q(t,\tau)Q(\tau)f\bigl(\tau,x(\tau)\bigr)\,dW(\tau)-\sum_{t<t_i}U_Q(t,t_i)Q(t_i)I_i(x_i)\biggr]\\ ={}&:(\Psi_1x)(t)+(\Psi_2x)(t),\quad t\in\mathbb{R}.\end{aligned}$$

We next show that Ψ has a fixed point in $PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ and divide the proof into several steps.

Step 1. For every $x\in PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$, $\Psi x\in PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$.

Let $x(\cdot)\in PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$, by (H5) and Lemmas 2.2, 2.3, we deduce that $g(\cdot,x(\cdot)),f(\cdot,x(\cdot))\in PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ and $I_i(x(t_i))\in PAP(\mathbb{Z},L^p(\mathbb{P},\mathbb{H}))$ Similarly as the proof of Lemmas 3.3-3.5, one has $\Psi x\in PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$.

Step 2. Ψ maps bounded sets into bounded sets in $PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$.

Indeed, let $r^*>0$ and $x\in B_{r^*}=\{x\in PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H})):E\|x\|^p\le r^*\}$. It is enough to show that there exists a positive constant $\tilde{\mathcal{L}}$ such that for each $x\in B_{r^*}$ one has $E\|\Psi x\|^p\le\tilde{\mathcal{L}}$. Let $x\in B_{r^*}$, and $t\in\mathbb{R}$. By (H6), Hölder's inequality, and the Ito integral, we have, for $p>2$,

$$\begin{aligned}&E\bigl\|(\Psi_1x)(t)\bigr\|^p\\ &\quad\le 3^{p-1}E\left\|\int_{-\infty}^{t}U(t,\tau)P(\tau)g\bigl(\tau,x(\tau)\bigr)\,d\tau\right\|^p\\ &\qquad+3^{p-1}E\left\|\int_{-\infty}^{t}U(t,\tau)P(\tau)f\bigl(\tau,x(\tau)\bigr)\,dW(\tau)\right\|^p\\ &\qquad+3^{p-1}E\left\|\sum_{t_i<t}U(t,t_i)P(t_i)I_i\bigl(x(t_i)\bigr)\right\|^p\\ &\quad\le 3^{p-1}M^p\left(\int_{-\infty}^{t}e^{-\delta(t-\tau)}\,d\tau\right)^{p-1}\left(\int_{-\infty}^{t}e^{-\delta(t-\tau)}E\bigl\|g\bigl(\tau,x(\tau)\bigr)\bigr\|^p\,d\tau\right)\end{aligned}$$

$$
\begin{aligned}
&+3^{p-1}C_pM^pE\left(\int_{-\infty}^{t}e^{-2\delta(t-\tau)}\left\|f(\tau,x(\tau))\right\|_{L_2^0}^2\,d\tau\right)^{p/2}\\
&+3^{p-1}M^pE\left[\left(\sum_{t_i<t}e^{-\delta(t-t_i)}\right)^{p-1}\left(\sum_{t_i<t}e^{-\delta(t-t_i)}\left\|I_i(x(t_i))\right\|^p\right)\right]\\
\leq{}&3^{p-1}M^p\frac{1}{\delta^{p-1}}\left(\int_{-\infty}^{t}e^{-\delta(t-\tau)}\Theta\left(E\left\|x(\tau)\right\|^p\right)d\tau\right)\\
&+3^{p-1}M^p\left(\int_{-\infty}^{t}e^{-\frac{p}{p-2}\delta(t-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\left(\int_{-\infty}^{t}e^{-\frac{p}{2}\delta(t-\tau)}\Theta\left(E\left\|x(\tau)\right\|^p\right)d\tau\right)\\
&+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^{p-1}}\left(\sum_{t_i<t}e^{-\delta(t-t_i)}\tilde{\Theta}_i\left(E\left\|x(t_i)\right\|^p\right)\right)\\
\leq{}&3^{p-1}M^p\frac{1}{\delta^p}\Theta\left(r^*\right)+3^{p-1}M^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{1}{p\delta}\Theta\left(r^*\right)\\
&+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^p}\sup_{i\in\mathbb{Z}}\tilde{\Theta}_i\left(r^*\right):=\mathcal{L}_1.
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
E\left\|(\Psi_1x)(t)\right\|^2\leq{}&6M^2\frac{1}{\delta^2}\Theta\left(r^*\right)+6M^2\frac{1}{\delta}\Theta\left(r^*\right)\\
&+6M^2\frac{1}{(1-e^{-\delta\alpha})^2}\sup_{i\in\mathbb{Z}}\tilde{\Theta}_i\left(r^*\right):=\mathcal{L}_2.
\end{aligned}
$$

Take $\mathcal{L}=\max\{\mathcal{L}_1,\mathcal{L}_2\}$. Then, for each $x\in B_{r^*}$, we have $E\|\Psi_1x\|^p\leq\mathcal{L}$. Similarly, for each $x\in B_{r^*}$, we have $E\|\Psi_2x\|^p\leq\mathcal{L}$. Hence, for each $x\in B_{r^*}$, we get $E\|\Psi x\|^p\leq 2\mathcal{L}=\tilde{\mathcal{L}}$.

Step 3. $\Psi: PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\to PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ is continuous.

Let $\{x^{(n)}\}\subseteq B_{r^*}$ with $x^{(n)}\to x$ $(n\to\infty)$ in $PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$, then there exists a bounded subset $K\subseteq L^p(\mathbb{P},\mathbb{K})$ such that $\mathbb{R}(x)\subseteq K$, $\mathbb{R}(x^n)\subseteq K$, $n\in\mathbb{N}$. By the assumption (H5), for any $\varepsilon>0$, there exists $\xi>0$ such that $x,y\in K$ and $E\|x-y\|^p<\xi$ implies that

$$
\begin{aligned}
&E\left\|g(\tau,x(\tau))-g(\tau,y(\tau))\right\|^p<\varepsilon\quad\text{for all } t\in\mathbb{R},\\
&E\left\|f(\tau,x(\tau))-f(\tau,y(\tau))\right\|_{L_2^0}^p<\varepsilon\quad\text{for all } t\in\mathbb{R},
\end{aligned}
$$

and

$$
E\left\|I_i(x)-I_i(y)\right\|^p<\varepsilon\quad\text{for all } i\in\mathbb{Z}.
$$

For the above ξ there exists n_0 such that $E\|x^{(n)}(t)-x(t)\|^p<\varepsilon$ for $n>n_0$ and $t\in\mathbb{R}$, then, for $n>n_0$, we have

$$
\begin{aligned}
&E\left\|g(\tau,x^{(n)}(\tau))-g(\tau,x(\tau))\right\|^p<\varepsilon\quad\text{for all } t\in\mathbb{R},\\
&E\left\|f(\tau,x^{(n)}(\tau))-f(\tau,x(\tau))\right\|_{L_2^0}^p<\varepsilon\quad\text{for all } t\in\mathbb{R},
\end{aligned}
$$

and

$$E\big\|I_i\big(x^{(n)}\big)-I_i(x)\big\|^p<\varepsilon \quad \text{for all } i\in\mathbb{Z}.$$

Then, by Hölder's inequality, we have, for $p>2$,

$$\begin{aligned}
&E\big\|\big(\Psi_1x^{(n)}\big)(t)-(\Psi_1x)(t)\big\|_H^p\\
&\quad\le 3^{p-1}E\bigg\|\int_{-\infty}^t U(t,\tau)P(\tau)\big[g\big(\tau,x^{(n)}(\tau)\big)-g\big(\tau,x(\tau)\big)\big]\,d\tau\bigg\|^p\\
&\qquad+3^{p-1}E\bigg\|\int_{-\infty}^t U(t,\tau)P(\tau)\big[f\big(\tau,x^{(n)}(\tau)\big)-f\big(\tau,x(\tau)\big)\big]\,dW(\tau)\bigg\|^p\\
&\qquad+3^{p-1}E\bigg\|\sum_{t_i<t}U(t,t_i)P(t_i)\big[I_i\big(x^{(n)}(t_i)\big)-I_i\big(x(t_i)\big)\big]\bigg\|^p\\
&\quad\le 3^{p-1}M^p\bigg(\int_{-\infty}^t e^{-\delta(t-\tau)}\,d\tau\bigg)^{p-1}\bigg(\int_{-\infty}^t e^{-\delta(t-\tau)}E\big\|g\big(\tau,x^{(n)}(\tau)\big)-g\big(\tau,x(\tau)\big)\big\|^p\,d\tau\bigg)\\
&\qquad+3^{p-1}C_pM^p\bigg(\int_{-\infty}^t e^{-2\delta(t-\tau)}E\big\|f\big(\tau,x^{(n)}(\tau)\big)-f\big(\tau,x(\tau)\big)\big\|_{L_2^0}^2\,d\tau\bigg)^{p/2}\\
&\qquad+3^{p-1}M^pE\bigg[\bigg(\sum_{t_i<t}e^{-\delta(t-t_i)}\bigg)^{p-1}\bigg(\sum_{t_i<t}e^{-\delta(t-t_i)}\big\|I_i\big(x^{(n)}(t_i)\big)-I_i\big(x(t_i)\big)\big\|^p\bigg)\bigg]\\
&\quad\le 3^{p-1}M^p\bigg(\int_{-\infty}^t e^{-\delta(t-\tau)}\bigg)^p\varepsilon\\
&\qquad+3^{p-1}C_pM^p\bigg(\int_{-\infty}^t e^{-\frac{p}{p-2}\delta(t-\tau)}\,d\tau\bigg)^{\frac{p-2}{p}}\bigg(\int_{-\infty}^t e^{-\frac{p}{2}\delta(t-\tau)}\,d\tau\bigg)\varepsilon\\
&\qquad+3^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^{p-1}}\bigg(\sum_{t_i<t}e^{-\delta(t-t_i)}\bigg)\varepsilon\\
&\quad\le 3^{p-1}M^p\bigg[\frac{1}{\delta^p}+C_p\bigg(\frac{p-2}{p\delta}\bigg)^{\frac{p-2}{2}}\frac{1}{p\delta}+\frac{2}{(1-e^{-\delta\alpha})^p}\bigg]\varepsilon.
\end{aligned}$$

For $p=2$, we have

$$E\big\|\big(\Psi_1x^{(n)}\big)(t)-(\Psi_1x)(t)\big\|^2\le 6M^2\bigg[\frac{1}{\delta^2}+\frac{1}{2\delta}+\frac{1}{(1-e^{-\delta\alpha})^2}\bigg]\varepsilon.$$

Thus Ψ_1 is continuous. Similarly, we can show that Ψ_2 is continuous and hence Ψ is continuous.

Step 4. Ψ maps bounded sets into equicontinuous sets of $PAP_T(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))\cap UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$.

Let $\tau_1,\tau_2\in(t_i,t_{i+1})$, $i\in\mathbb{Z}$, $\tau_1<\tau_2$, and $x\in B_{r^*}$. Then, by (H1)-(H6), Hölder's inequality, and the Ito integral, we have, for $p>2$,

$$\begin{aligned}
&E\big\|(\Psi_1x)(\tau_2)-(\Psi_1x)(\tau_1)\big\|^p\\
&\quad\le 6^{p-1}E\bigg\|\int_{-\infty}^{\tau_1}\big[U(\tau_2,\tau_1)-I\big]U(\tau_1,\tau)P(\tau)g\big(\tau,x(\tau)\big)\,d\tau\bigg\|^p
\end{aligned}$$

$$
\begin{aligned}
&+6^{p-1}E\left\|\int_{\tau_1}^{\tau_2}U(\tau_2,\tau)P(\tau)g\big(\tau,x(\tau)\big)\,d\tau\right\|^p\\
&+6^{p-1}E\left\|\int_{-\infty}^{\tau_1}\big[U(\tau_2,\tau_1)-I\big]U(\tau_1,\tau)P(\tau)f\big(\tau,x(\tau)\big)\,dW(\tau)\right\|^p\\
&+6^{p-1}E\left\|\int_{\tau_1}^{\tau_2}U(\tau_2,\tau)P(\tau)f\big(\tau,x(\tau)\big)\,dW(\tau)\right\|^p\\
&+3^{p-1}E\left\|\sum_{t_i<\tau_1}\big[U(\tau_2,\tau_1)-I\big]U(\tau_1,t_i)P(t_i)I_i\big(x(t_i)\big)\right\|^p\\
\le{}&6^{p-1}M^p\big\|U(\tau_2,\tau_1)-I\big\|^p\left(\int_{-\infty}^{\tau_1}e^{-\delta(\tau_1-\tau)}\,d\tau\right)^{p-1}\\
&\times\left(\int_{-\infty}^{\tau_1}e^{-\delta(\tau_1-\tau)}E\big\|g\big(\tau,x(\tau)\big)\big\|^p\,d\tau\right)\\
&+6^{p-1}M^p\left(\int_{\tau_1}^{\tau_2}e^{-\delta(\tau_2-\tau)}\,d\tau\right)^{p-1}\left(\int_{\tau_1}^{\tau_2}e^{-\delta(\tau_2-\tau)}E\big\|g\big(\tau,x(\tau)\big)\big\|^p\,d\tau\right)\\
&+6^{p-1}M^pC_pE\left[\int_{-\infty}^{\tau_1}e^{-2\delta(\tau_1-\tau)}\big\|U(\tau_2,\tau_1)-I\big\|^2\big\|f\big(\tau,x(\tau)\big)\big\|_{L_2^0}^2\,d\tau\right]^{p/2}\\
&+6^{p-1}M^pC_pE\left[\int_{\tau_1}^{\tau_2}e^{-2\delta(\tau_2-\tau)}\big\|f\big(\tau,x(\tau)\big)\big\|_{L_2^0}^2\,d\tau\right]^{p/2}\\
&+3^{p-1}M^p\big\|U(\tau_2,\tau_1)-I\big\|^p\left(\sum_{t_i<\tau_1}e^{-\delta(\tau_1-t_i)}\right)^{p-1}\\
&\times\left(\sum_{t_i<\tau_1}e^{-\delta(\tau_1-t_i)}E\big\|I_i\big(x(t_i)\big)\big\|^p\right)\\
\le{}&6^{p-1}M^p\big\|U(\tau_2,\tau_1)-I\big\|^p\left(\int_{-\infty}^{\tau_1}e^{-\delta(\tau_1-\tau)}\,d\tau\right)^{p-1}\\
&\times\left(\int_{-\infty}^{\tau_1}e^{-\delta(\tau_1-\tau)}\Theta\big(E\|x(\tau)\|^p\big)\,d\tau\right)\\
&+6^{p-1}M^p\left(\int_{\tau_1}^{\tau_2}e^{-\delta(\tau_2-\tau)}\,d\tau\right)^{p-1}\left(\int_{\tau_1}^{\tau_2}e^{-\delta(\tau_2-\tau)}\Theta\big(E\|x(\tau)\|^p\big)\,d\tau\right)\\
&+6^{p-1}M^pC_p\big\|U(\tau_2,\tau_1)-I\big\|^p\left(\int_{-\infty}^{\tau_1}e^{-\frac{p}{p-2}\delta(\tau_1-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\\
&\times\left(\int_{-\infty}^{\tau_1}e^{-\frac{p}{2}\delta(\tau_1-\tau)}\Theta\big(E\|x(\tau)\|^p\big)\,d\tau\right)\\
&+6^{p-1}M^pC_p\left(\int_{\tau_1}^{\tau_2}e^{-\frac{p}{p-2}\delta(\tau_2-s)}\,ds\right)^{\frac{p-2}{p}}\left(\int_{\tau_1}^{\tau_2}e^{-\frac{p}{2}\delta(\tau_2-\tau)}\Theta\big(E\|x(\tau)\|^p\big)\,d\tau\right)\\
&+3^{p-1}M^p\big\|U(\tau_2,\tau_1)-I\big\|^p\left(\sum_{t_i<\tau_1}e^{-\delta(\tau_1-t_i)}\right)^{p-1}\\
&\times\left(\sum_{t_i<\tau_1}e^{-\delta(\tau_1-t_i)}\tilde{\Theta}_i\big(E\|x(t_i)\|^p\big)\right)\\
\le{}&6^{p-1}M^p\big\|U(\tau_2,\tau_1)-I\big\|^p\frac{1}{\delta^p}\Theta\big(r^*\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 6^{p-1}M^p\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-\tau)}\,d\tau\right)^p \Theta\bigl(r^*\bigr) \\
&+ 6^{p-1}M^p C_p\bigl\|U(\tau_2,\tau_1)-I\bigr\|^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta}\Theta\bigl(r^*\bigr) \\
&+ 6^{p-1}M^p C_p\left(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{p-2}\delta(\tau_2-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\left(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{2}\delta(\tau_2-\tau)}\,d\tau\right)\Theta\bigl(r^*\bigr) \\
&+ 3^{p-1}M^p\bigl\|U(\tau_2,\tau_1)-I\bigr\|^p\frac{1}{(1-e^{-\delta\alpha})^p}\sup_{i\in\mathbb{Z}}\tilde{\Theta}_i\bigl(r^*\bigr).
\end{aligned}
$$

For $p = 2$, we have

$$
\begin{aligned}
&E\bigl\|(\Psi_1 x)(\tau_2)-(\Psi_1 x)(\tau_1)\bigr\|^2 \\
&\quad\le 6M^2\bigl\|U(\tau_2,\tau_1)-I\bigr\|^2\frac{1}{\delta^2}\Theta\bigl(r^*\bigr) + 6M^2\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-\tau)}\,d\tau\right)^2\Theta\bigl(r^*\bigr) \\
&\qquad + 6M^2\bigl\|U(\tau_2,\tau_1)-I\bigr\|^2\frac{2}{\delta}\Theta\bigl(r^*\bigr) + 6M^2\left(\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2-\tau)}\,d\tau\right)\Theta\bigl(r^*\bigr) \\
&\qquad + 3M^2\bigl\|U(\tau_2,\tau_1)-I\bigr\|^2\frac{1}{(1-e^{-\delta\alpha})^2}\sup_{i\in\mathbb{Z}}\tilde{\Theta}_i\bigl(r^*\bigr).
\end{aligned}
$$

The right-hand side of the above inequality is independent of $x \in B_{r^*}$ and tends to zero as $\tau_2 \to \tau_1$, since the compactness of $U(t,\tau)$ for $t-\tau > 0$ implies imply the continuity in the uniform operator topology. Thus, Ψ_1 maps B_{r^*} into an equicontinuous family of functions. Similarly, we can show that Ψ_2 maps B_{r^*} into an equicontinuous family of functions and hence Ψ maps B_{r^*} into an equicontinuous family of functions.

Step 5. $\Lambda(t) = \{(\Psi x)(t) : x \in B_{r^*}\}$ is relatively compact in $L^p(\mathbb{P},\mathbb{H})$ for each $t \in \mathbb{R}$.

For each $t \in \mathbb{R}$, and let ε be a real number satisfying $0 < \varepsilon < 1$. For $x \in B_{r^*}$, we define

$$
\begin{aligned}
\bigl(\Psi_1^\varepsilon x\bigr)(t) = U(t,t-\varepsilon)\biggl[&\int_{-\infty}^{t-\varepsilon} U(t-\varepsilon,\tau)P(\tau)g\bigl(\tau,x(\tau)\bigr)\,d\tau \\
&+ \int_{-\infty}^{t-\varepsilon} U(t-\varepsilon,\tau)P(\tau)f\bigl(\tau,x(\tau)\bigr)\,dW(\tau) \\
&+ \sum_{t_i<t-\varepsilon} U(t-\varepsilon,t_i)P(t_i)I_i\bigl(x(t_i)\bigr)\biggr].
\end{aligned}
$$

Since $U(t,\tau)$ $(t-\tau > 0)$ is compact, then the set $\Lambda_1^\varepsilon(t) = \{(\Psi_1^\varepsilon x)(t) : x \in B_{r^*}\}$ is relatively compact in $L^p(\mathbb{P},\mathbb{H})$ for each $t \in \mathbb{R}$. Moreover, for every $x \in B_{r^*}$, we have for $p > 2$,

$$
\begin{aligned}
&E\bigl\|(\Psi_1 x)(t)-\bigl(\Psi_1^\varepsilon x\bigr)(t)\bigr\|^p \\
&\quad\le 3^{p-1}E\left\|\int_{t-\varepsilon}^{t} U(t,\tau)P(\tau)g\bigl(\tau,x(\tau)\bigr)\,d\tau\right\|^p \\
&\qquad + 3^{p-1}E\left\|\int_{t-\varepsilon}^{t} U(t,\tau)P(\tau)f\bigl(\tau,x(\tau)\bigr)\,dW(\tau)\right\|^p \\
&\qquad + 3^{p-1}E\left\|\sum_{t-\varepsilon<t_i<t} U(t,t_i)P(t_i)I_i\bigl(x(t_i)\bigr)\right\|^p
\end{aligned}
$$

$$\begin{aligned}
&\leq 3^{p-1}M^p\left(\int_{t-\varepsilon}^{t} e^{-\delta(t-\tau)}\,d\tau\right)^{p-1}\left(\int_{t-\varepsilon}^{t} e^{-\delta(t-\tau)}E\big\|g\big(\tau,x(\tau)\big)\big\|^p\,d\tau\right)\\
&\quad + 3^{p-1}C_pM^pE\left(\int_{t-\varepsilon}^{t} e^{-2\delta(t-\tau)}\big\|f\big(\tau,x(\tau)\big)\big\|_{L_2^0}^2\,d\tau\right)^{p/2}\\
&\quad + 3^{p-1}M^pE\left[\left(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\right)^{p-1}\left(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\big\|I_i\big(x(t_i)\big)\big\|^p\right)\right]\\
&\leq 3^{p-1}M^p\left(\int_{t-\varepsilon}^{t} e^{-\delta(t-\tau)}\,d\tau\right)^{p-1}\left(\int_{t-\varepsilon}^{t} e^{-\delta(t-\tau)}\Theta\big(E\big\|x(\tau)\big\|^p\big)\,ds\right)\\
&\quad + 3^{p-1}C_pM^p\left(\int_{t-\varepsilon}^{t} e^{-\frac{p}{p-2}\delta(t-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\left(\int_{t-\varepsilon}^{t} e^{-\frac{p}{2}\delta(t-\tau)}\Theta\big(E\big\|x(\tau)\big\|^p\big)\,d\tau\right)\\
&\quad + 3^{p-1}M^p\left(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\right)^{p-1}\left(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\tilde{\Theta}_i\big(E\big\|x_i(t_i)\big\|^p\big)\right)\\
&\leq 3^{p-1}M^p\left(\int_{t-\varepsilon}^{t} e^{-\delta(t-\tau)}\,d\tau\right)^{p}\Theta\big(r^*\big)\\
&\quad + 3^{p-1}\left(\int_{t-\varepsilon}^{t} e^{-\frac{p}{p-2}\delta(t-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\left(\int_{t-\varepsilon}^{t} e^{-\frac{p}{2}\delta(t-\tau)}\,d\tau\right)\Theta\big(r^*\big)\\
&\quad + 3^{p-1}M^p\left(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\right)^{p}\sup_{i\in\mathbb{Z}}\tilde{\Theta}_i\big(r^*\big).
\end{aligned}$$

For $p = 2$, we have

$$\begin{aligned}
&E\big\|(\Psi_1x)(t) - \big(\Psi_1^\varepsilon x\big)(t)\big\|^2\\
&\quad\leq 3M^2\left(\int_{t-\varepsilon}^{t} e^{-\delta(t-\tau)}\,d\tau\right)^{2}\Theta\big(r^*\big)\\
&\qquad + 3M^2\left(\int_{t-\varepsilon}^{t} e^{-2\delta(t-\tau)}\,d\tau\right)\Theta\big(r^*\big)\\
&\qquad + 3M^2\left(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\right)^{2}\sup_{i\in\mathbb{Z}}\tilde{\Theta}_i\big(r^*\big).
\end{aligned}$$

Therefore, letting $\varepsilon \to 0$, it follows that there are relatively compact sets $\Lambda_1^\varepsilon(t)$ arbitrarily close to $\Lambda_1(t) = \{(\Psi_1x)(t) : x \in B_{r^*}\}$ and hence $\Lambda_1(t)$ is also relatively compact in $L^p(\mathbb{P},\mathbb{H})$ for each $t \in \mathbb{R}$. Similarly, we can show that $\Lambda_2(t) = \{(\Psi_2x)(t) : x \in B_{r^*}\}$ is also relatively compact in $L^p(\mathbb{P},\mathbb{H})$ for each $t \in \mathbb{R}$. Hence $\Lambda(t)$ is relatively compact in $L^p(\mathbb{P},\mathbb{H})$ for each $t \in \mathbb{R}$. Since $\{\Psi x : x \in B_{r^*}\} \subset PC_h^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$, then $\{\Psi x : x \in B_{r^*}\}$ is a relatively compact set by Lemma 2.4, then Ψ is a compact operator.

Step 6. We now show that there exists an open set $V \subseteq PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H})) \cap UPC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ with $x \notin \Psi x$ for $\lambda \in (0,1)$ and $x \in \partial V$.

Let $\lambda \in (0,1)$ and let $x \in L^p(\mathbb{P},\mathbb{H})$ be a possible solution of $x = \lambda\Psi(x)$ for some $0 < \lambda < 1$. Thus, for each $t \in \mathbb{R}$,

$$\begin{aligned}
x(t) &= \lambda(\Psi x)(t)\\
&= \left[\lambda\int_{-\infty}^{t} U(t,\tau)P(\tau)g\big(\tau,x(\tau)\big)\,d\tau + \lambda\int_{-\infty}^{t} U(t,\tau)P(\tau)f\big(\tau,x(\tau)\big)\,dW(\tau)\right.
\end{aligned}$$

$$+ \lambda \sum_{t_i<t} U(t,t_i)P(t_i)I_i\big(x(t_i)\big)\Bigg] + \Bigg[-\lambda \int_t^{+\infty} U_Q(t,\tau)Q(\tau)g\big(\tau,x(\tau)\big)\,d\tau$$
$$- \lambda \int_t^{+\infty} U_Q(t,\tau)Q(\tau)f\big(\tau,x(\tau)\big)\,dW(\tau) - \lambda \sum_{t<t_i} U_Q(t,t_i)Q(t_i)I_i\big(x(t_i)\big)\Bigg].$$

Then, by (H6), Hölder's inequality, and the Ito integral, we have, for $p > 2$,

$$\begin{aligned} E\big\|x(t)\big\|^p \le{}& 6^{p-1}M^p \frac{2}{\delta^p}\Theta\Big(\sup_{s\in\mathbb{R}} E\|x\|^p\Big) \\ &+ 6^{p-1}C_pM^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}} \frac{4}{p\delta}\Theta\Big(\sup_{s\in\mathbb{R}} E\|x\|^p\Big) \\ &+ 6^{p-1}M^p\frac{2}{(1-e^{-\delta\alpha})^p}\tilde{\Theta}_i\big(E\big\|x(t_i)\big\|^p\big). \end{aligned}$$

For $p = 2$, we have

$$\begin{aligned} E\big\|x(t)\big\|^2 \le{}& 12M^2\frac{1}{\delta^2}\Theta\Big(\sup_{s\in\mathbb{R}} E\|x\|^2\Big) \\ &+ 12M^2\frac{1}{2\delta}\Theta\Big(\sup_{s\in\mathbb{R}} E\|x\|^2\Big) \\ &+ 12M^2\frac{1}{(1-e^{-\delta\alpha})^2}\tilde{\Theta}_i\big(E\big\|x(t_i)\big\|^2\big). \end{aligned}$$

Taking the supremum over t, we have, for $p > 2$,

$$\begin{aligned} \sup_{t\in\mathbb{R}} E\big\|x(t)\big\|^p \le{}& 6^{p-1}M^p \frac{2}{\delta^p}\Theta\Big(\sup_{s\in\mathbb{R}} E\big\|x(s)\big\|^p\Big) \\ &+ 6^{p-1}C_pM^p\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}} \frac{4}{p\delta}\Theta\Big(\sup_{s\in\mathbb{R}} E\big\|x(s)\big\|^p\Big) \\ &+ 6^{p-1}M^p\frac{2}{(1-e^{-\delta\alpha})^p}\tilde{\Theta}_i\Big(\sup_{s\in\mathbb{R}} E\big\|x(s)\big\|^p\Big). \end{aligned}$$

For $p = 2$, we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} E\big\|x(t)\big\|^2 \le{}& 12M^2\frac{1}{\delta^2}\Theta\Big(\sup_{s\in\mathbb{R}} E\big\|x(s)\big\|^2\Big) \\ &+ 12M^2\frac{1}{2\delta}\Theta\Big(\sup_{s\in\mathbb{R}} E\big\|x(s)\big\|^2\Big) \\ &+ 12M^2\frac{1}{(1-e^{-\delta\alpha})^2}\tilde{\Theta}_i\Big(\sup_{s\in\mathbb{R}} E\big\|x(s)\big\|^2\Big). \end{aligned}$$

Therefore, we have, for $p > 2$,

$$\frac{\|x\|_\infty^p}{N_1\Theta(\|x\|_\infty^p) + N_2\sup_{i\in\mathbb{Z}}\tilde{\Theta}_i(\|x\|_\infty^p)} \le 1,$$

where $N_1 = 6^{p-1}M^p[\frac{2}{\delta^p} + C_p(\frac{p-2}{p\delta})^{\frac{p-2}{p}}\frac{4}{p\delta}]$, $N_2 = 6^{p-1}M^p\frac{2}{(1-e^{-\delta\alpha})^p}$. For the case of $p = 2$, take $N_1 = 12M^2[\frac{1}{\delta^2} + \frac{1}{2\delta}]$, $N_2 = 12M^2\frac{1}{(1-e^{-\delta\alpha})^2}$. Then, by (H7), there exists M^* such that

$\|x\|_\infty^p \neq M^*$. Set

$$V = \{x \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) \cap UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) : \|x\|_\infty^p < M^*\}.$$

As a consequence of Steps 1-6, it suffices to show that $\Psi : \overline{V} \to PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) \cap UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is a compact map. From the choice of V, there is no $x \in \partial V$ such that $x \in \lambda\Psi x$ for $\lambda \in (0,1)$. By Lemma 2.5, we deduce that Ψ has a fixed point $x \in \overline{V}$, such that $\Psi x = x$, that is,

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} U(t,\tau)P(\tau)g(\tau,x(\tau))\,d\tau + \int_{-\infty}^{t} U(t,\tau)P(\tau)f(\tau,x(\tau))\,dW(\tau) \\ &\quad + \sum_{t_i<t} U(t,t_i)P(t_i)I_i(x_i) - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)g(\tau,x(\tau))\,d\tau \\ &\quad - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)f(\tau,x(\tau))\,dW(\tau) - \sum_{t<t_i} U_Q(t,t_i)Q(t_i)I_i(x_i), \quad t \in \mathbb{R}. \end{aligned}$$

Finally, to prove that x satisfies (3.1) for all $t \geq s$, all $s \in \mathbb{R}$. For this purpose, we let

$$\begin{aligned} x(s) &= \int_{-\infty}^{s} U(s,\tau)P(\tau)g(\tau,x(\tau))\,d\tau - \int_{s}^{+\infty} U_Q(s,\tau)Q(\tau)h(\tau,x(\tau))\,d\tau \\ &\quad + \int_{-\infty}^{s} U(s,\tau)P(\tau)f(\tau,x(\tau))\,dW(\tau) - \int_{s}^{+\infty} U_Q(s,\tau)Q(\tau)f(\tau,x(\tau))\,dW(\tau) \\ &\quad + \sum_{t_i<s} U(s,t_i)P(t_i)I_i(x(t_i)) - \sum_{s<t_i} U_Q(s,t_i)Q(t_i)I_i(x(t_i)), \quad s \in \mathbb{R}. \end{aligned} \tag{3.5}$$

Multiply both sides of (3.5) by $U(t,s)$ for all $t \geq s$, then

$$\begin{aligned} U(t,s)x(s) &= \int_{-\infty}^{s} U(t,\tau)P(\tau)g(\tau,x(\tau))\,d\tau - \int_{s}^{+\infty} U_Q(t,\tau)Q(\tau)g(\tau,x(\tau))\,d\tau \\ &\quad + \int_{-\infty}^{s} U(t,\tau)P(\tau)f(\tau,x(\tau))\,dW(\tau) \\ &\quad - \int_{s}^{+\infty} U_Q(t,\tau)Q(\tau)f(\tau,x(\tau))\,dW(\tau) \\ &\quad + \sum_{t_i<s} U(t,t_i)P(t_i)I_i(x(t_i)) - \sum_{s<t_i} U_Q(t,t_i)Q(t_i)I_i(x(t_i)) \\ &= \int_{-\infty}^{t} U(t,\tau)P(\tau)g(\tau,x(\tau))\,d\tau - \int_{s}^{t} U(t,\tau)P(\tau)g(\tau,x(\tau))\,d\tau \\ &\quad - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)g(\tau,x(\tau))\,d\tau + \int_{s}^{t} U_Q(t,\tau)Q(\tau)g(\tau,x(\tau))\,d\tau \\ &\quad + \int_{-\infty}^{t} U(t,\tau)P(\tau)f(\tau,x(\tau))\,dW(\tau) - \int_{s}^{t} U(t,\tau)P(\tau)f(\tau,x(\tau))\,dW(\tau) \\ &\quad - \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)f(\tau,x(\tau))\,dW(\tau) \\ &\quad + \int_{s}^{t} U_Q(t,\tau)Q(\tau)f(\tau,x(\tau))\,dW(\tau) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{t_i<t} U(t,t_i)P(t_i)I_i\big(x(t_i)\big) - \sum_{s<t_i<t} U(t,t_i)P(t_i)I_i\big(x(t_i)\big) \\
&- \sum_{t<t_i} U_Q(t,t_i)Q(t_i)I_i\big(x(t_i)\big) + \sum_{s<t<t_i} U_Q(t,t_i)Q(t_i)I_i\big(x(t_i)\big) \\
={}& x(t) - \int_s^t U(t,\tau)g\big(\tau,x(\tau)\big)\,d\tau - \int_s^t U(t,\tau)f\big(\tau,x(\tau)\big)\,dW(\tau) \\
&- \sum_{s<t_i<t} U(t,t_i)I_i\big(x(t_i)\big).
\end{aligned}
$$

Hence x is a piecewise pseudo almost periodic mild solution to the system (1.1)-(1.2). The proof is complete. □

4 Exponential stability

In this section, we present the exponential stability of a piecewise pseudo almost periodic solution of (1.1)-(1.2). To do this, we also need the following assumptions:

(B1) There exist constants $0 < \beta < \delta$, $l_1 > 0$ and a continuous function $m : \mathbb{R} \to (0,\infty)$ with $m(t) \le l_2 e^{-\beta t}$, $l_2 > 0$, such that

$$
E\big\|g(t,x)\big\|^p + E\big\|f(t,x)\big\|^p_{L_2^0} \le l_1\|x\|^p + m(t), \quad t \in \mathbb{R}, x \in L^p(\mathbb{P},\mathbb{K}).
$$

(B2) There exists a constant $c_i > 0$, $i \in \mathbb{Z}$, such that

$$
E\big\|I_i(x)\big\|^p \le c_i E\|x\|^p, \quad x \in L^p(\mathbb{P},\mathbb{K}).
$$

Theorem 4.1 *Assume that assumptions of Theorem* 3.1 *hold and, in addition, hypotheses* (B1), (B2) *are satisfied. Then the piecewise pseudo almost periodic mild solution of* (1.1)-(1.2) *is exponentially stable, provided that*

$$
\begin{aligned}
&6^{p-1}M^p\Biggl[\biggl(\frac{2}{\delta^{p-1}(\delta-\beta)} + C_p\biggl(\frac{p-2}{p\delta}\biggr)^{\frac{p-2}{p}}\frac{2}{p\delta-2\beta}\biggr)l_1 \\
&\quad + \frac{1}{(1-e^{-\delta\alpha})^{p-1}(1-e^{-(\delta-\beta)\alpha})}\sup_{i\in\mathbb{Z}} c_i\Biggr] < 1
\end{aligned}
$$

for $p > 2$, *and*

$$
6M^p\Biggl[\biggl(\frac{2}{\delta(\delta-\beta)} + \frac{1}{2\delta-\beta}\biggr)l_1 + \frac{1}{(1-e^{-\delta\alpha})(1-e^{-(\delta-\beta)\alpha})}\sup_{i\in\mathbb{Z}} c_i\Biggr] < 1
$$

for $p = 2$.

Proof Let $x(\cdot)$ be a fixed point of Ψ in $PAP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H})) \cap UPC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$. By Theorem 3.1, any fixed point of Ψ is a mild solution of the system (1.1)-(1.2). We now can choose a positive constant β such that $0 < \beta < \delta$, and

$$
\begin{aligned}
e^{\beta t}E\big\|x(t)\big\|^p \le{}& 6^{p-1}e^{\beta t}E\biggl\|\int_{-\infty}^t U(t,\tau)P(\tau)g\big(\tau,x(\tau)\big)\,d\tau\biggr\|^p \\
&+ 6^{p-1}e^{\beta t}E\biggl\|\int_t^{+\infty} U_Q(t,\tau)Q(\tau)g\big(\tau,x(\tau)\big)\,d\tau\biggr\|^p
\end{aligned}
$$

$$
\begin{aligned}
&+ 6^{p-1}e^{\beta t}E\left\| \int_{-\infty}^{t} U(t,\tau)P(\tau)f\bigl(\tau,x(\tau)\bigr)\,dW(\tau)\right\|^p \\
&+ 6^{p-1}e^{\beta t}E\left\| \int_{t}^{+\infty} U_Q(t,\tau)Q(\tau)f\bigl(\tau,x(\tau)\bigr)\,dW(\tau)\right\|^p \\
&+ 6^{p-1}e^{\beta t}E\left\| \sum_{t_i<t} U(t,t_i)P(t_i)I_i\bigl(x(t_i)\bigr)\right\|^p \\
&+ 6^{p-1}e^{\beta t}E\left\| \sum_{t<t_i} U_Q(t,t_i)Q(t_i)I_i\bigl(x(t_i)\bigr)\right\|^p = \sum_{j=1}^{6} \nu_j.
\end{aligned}
$$

By (B1) and Hölder's inequality, we have

$$
\begin{aligned}
\nu_1 &\le 6^{p-1}M^p e^{\beta t}\left(\int_{-\infty}^{t} e^{-\delta(t-\tau)}\,d\tau\right)^{p-1}\left(\int_{-\infty}^{t} e^{-\delta(t-\tau)}E\bigl\|g\bigl(\tau,x(\tau)\bigr)\bigr\|^p\,d\tau\right) \\
&\le 6^{p-1}M^p \frac{1}{\delta^{p-1}} e^{\beta t}\left(\int_{-\infty}^{t} e^{-\delta(t-\tau)}\bigl[l_1E\bigl\|x(\tau)\bigr\|^p + m(\tau)\bigr]\,d\tau\right) \\
&\le 6^{p-1}M^p \frac{1}{\delta^{p-1}}\left(\int_{-\infty}^{t} e^{-(\delta-\beta)(t-\tau)}\bigl[l_1e^{\beta\tau}E\bigl\|x(\tau)\bigr\|^p + l_2\bigr]\,d\tau\right) \\
&\le 6^{p-1}M^p \frac{1}{\delta^{p-1}(\delta-\beta)}\Bigl[l_1\sup_{\tau\in\mathbb{R}} e^{\beta\tau}E\bigl\|x(\tau)\bigr\|^p + l_2\Bigr].
\end{aligned}
$$

Similarly, we have

$$
\nu_2 \le 6^{p-1}M^p \frac{1}{\delta^{p-1}(\delta-\beta)}\Bigl[l_1\sup_{\tau\in\mathbb{R}} e^{\beta\tau}E\bigl\|x(\tau)\bigr\|^p + l_2\Bigr].
$$

By (B1), Hölder's inequality, and the Ito integral, we have

$$
\begin{aligned}
\nu_3 &\le 6^{p-1}C_pM^p e^{\beta t}E\left(\int_{-\infty}^{t} e^{-2\delta(t-\tau)}\bigl\|f\bigl(\tau,x(\tau)\bigr)\bigr\|_{L_2^0}^2\,d\tau\right)^{p/2} \\
&\le 6^{p-1}C_pM^p e^{\beta t}\left(\int_{-\infty}^{t} e^{-\frac{p}{p-2}\delta(t-\tau)}\,d\tau\right)^{\frac{p-2}{p}}\left(\int_{-\infty}^{t} e^{-\frac{p}{2}\delta(t-\tau)}\bigl[l_1E\bigl\|x(\tau)\bigr\|^p + m(\tau)\bigr]\,d\tau\right) \\
&\le 6^{p-1}C_pM^p \left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\left(\int_{-\infty}^{t} e^{-(\frac{p\delta}{2}-\beta)(t-\tau)}\bigl[l_1e^{\beta\tau}E\bigl\|x(\tau)\bigr\|^p + l_2\bigr]\,d\tau\right) \\
&\le 6^{p-1}C_pM^p \left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta-2\beta}\Bigl[l_1\sup_{\tau\in\mathbb{R}} e^{\beta\tau}E\bigl\|x(\tau)\bigr\|^p + l_2\Bigr].
\end{aligned}
$$

Similarly, we have

$$
\nu_4 = 6^{p-1}C_pM^p \left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}}\frac{2}{p\delta-2\beta}\Bigl[l_1\sup_{\tau\in\mathbb{R}} e^{\beta\tau}E\bigl\|x(\tau)\bigr\|^p + l_2\Bigr].
$$

For $p = 2$, we have

$$
\nu_3, \nu_4 = 6M^2\frac{1}{2\delta-\beta}\Bigl[l_1\sup_{\tau\in\mathbb{R}} e^{\beta\tau}E\bigl\|x(\tau)\bigr\|^2 + l_2\Bigr].
$$

By (B2) and Hölder's inequality, we have

$$\begin{aligned}
\nu_5 &\le 6^{p-1}M^p e^{\beta t}E\Bigg[\Bigg(\sum_{t_i<t} e^{-\delta(t-t_i)}\Bigg)^{p-1}\Bigg(\sum_{t_i<t} e^{-\delta(t-t_i)}\big\|I_i\big(x(t_i)\big)\big\|^p\Bigg)\Bigg] \\
&\le 6^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^{p-1}}e^{\beta t}\Bigg(\sum_{t_i<t} e^{-\delta(t-t_i)}\big[c_iE\big\|x(t_i)\big\|^p\big]\Bigg) \\
&\le 6^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^{p}}\Bigg(\sum_{t_i<t} e^{-(\delta-\beta)(t-t_i)}\big[c_ie^{\beta t_i}E\big\|x(t_i)\big\|^p\big]\Bigg) \\
&\le 6^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^{p-1}(1-e^{-(\delta-\beta)\alpha})}\Big[\sup_{i\in\mathbb{Z}} c_ie^{\beta t_i}E\big\|x(t_i)\big\|^p\Big].
\end{aligned}$$

Similarly, we have

$$\nu_6 = 6^{p-1}M^p\frac{1}{(1-e^{-\delta\alpha})^{p-1}(1-e^{-(\delta-\beta)\alpha})}\Big[\sup_{i\in\mathbb{Z}} c_ie^{\beta t_i}E\big\|x(t_i)\big\|^p\Big].$$

Thus, from the above inequality, it follows that

$$e^{\beta t}E\big\|x(t)\big\|^p \le L^*\sup_{\tau\in\mathbb{R}} e^{\beta\tau}E\big\|x(\tau)\big\|^p + L_0.$$

Since $L^* < 1$, we have

$$\sup_{t\in\mathbb{R}} e^{\beta t}E\big\|x(t)\big\|^p \le \frac{L_0}{1-L^*},$$

where

$$\begin{aligned}
L^* &= 6^{p-1}M^p\Bigg[\Bigg(\frac{2}{\delta^{p-1}(\delta-\beta)} + C_p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{p}}\frac{2}{p\delta-2\beta}\Bigg)l_1 \\
&\quad + \frac{1}{(1-e^{-\delta\alpha})^{p-1}(1-e^{-(\delta-\beta)\alpha})}\sup_{i\in\mathbb{Z}} c_i\Bigg], \\
L_0 &= 6^{p-1}M^p\Bigg[\frac{2}{\delta^{p-1}(\delta-\beta)} + C_p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{p}}\frac{2}{p\delta-2\beta}\Bigg]l_2
\end{aligned}$$

for $p > 2$, and

$$\begin{aligned}
L^* &= 6M^p\Bigg[\Bigg(\frac{2}{\delta(\delta-\beta)} + \frac{1}{2\delta-\beta}\Bigg)l_1 + \frac{1}{(1-e^{-\delta\alpha})(1-e^{-(\delta-\beta)\alpha})}\sup_{i\in\mathbb{Z}} c_i\Bigg], \\
L_0 &= 6M^2\Bigg[\frac{2}{\delta(\delta-\beta)} + \frac{1}{2\delta-\beta}\Bigg]l_2
\end{aligned}$$

for $p = 2$. Then we get $E\|x(t)\|^p \le \frac{L_0}{1-L^*}e^{-\beta t}$, which implies that the piecewise pseudo almost periodic mild solution of (1.1)-(1.2) is exponentially stable. The proof is completed. □

5 Applications

Consider following impulsive partial stochastic differential equations of the form

$$dz(t,x) = \frac{\partial^2}{\partial x^2} z(t,x)\,dt + (-2 + \sin t + \sin \pi t) z(t,x)\,dt + a_1(t)\sin\bigl(z(t,x)\bigr)\,dt \\ + a_2(t)\sin\bigl(z(t,x)\bigr)\,dW(t), \quad t \in \mathbb{R}, t \neq t_i, i \in \mathbb{Z}, x \in [0,\pi], \tag{5.1}$$

$$\Delta z(t_i,x) = \beta_i \sin\bigl(z(t_i,x)\bigr), \quad i \in \mathbb{Z}, x \in [0,\pi], \tag{5.2}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in \mathbb{R}, \tag{5.3}$$

where $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$. In this system, $a_i \in PAP_T(\mathbb{R},\mathbb{R})$, $i = 1,2$, and $\beta_i \in PAP(\mathbb{Z},\mathbb{R})$, $t_i = i + \frac{1}{8}|\sin i + \sin\sqrt{2}i|$, $\{t_i^j\}$, $i \in \mathbb{Z}$, $j \in \mathbb{Z}$ are equipotentially almost periodic (see [28]), and $\alpha = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$, one can refer to [35] for more details.

Let $\mathbb{H} = L^2([0,\pi])$ with the norm $\|\cdot\|$ and define the operators $A : A(D) \subset \mathbb{H} \to \mathbb{H}$ by $Av = v''$ with the domain $D(A) := \{v \in \mathbb{H} : v'' \in \mathbb{H}, v(0) = v(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic compact semigroup $T(t)$ on $\mathbb{H}$ and $\|T(t)\| \le e^{-t}$ for $t \ge 0$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$ and normalized eigenfunctions given by $v_n(\xi) := (\frac{2}{\pi})^{\frac{1}{2}} \sin(n\xi)$. Moreover, $T(t)v = \sum_{n=1}^{\infty} e^{-n^2 t}\langle v, v_n\rangle v_n$ for $v \in \mathbb{H}$.

Define a family of linear operators $A(t)$ by

$$\begin{cases} D(A(t)) = D(A), \\ A(t)v(\xi) = (A - 2 + \sin t + \sin \pi t)v(\xi), \quad \xi \in [0,\pi], v \in D(A). \end{cases}$$

Then the $A(t)$ generate an evolution family $(U(t,s))_{t\ge s}$ such that

$$U(t,s)v(\xi) = T(t-s)e^{\int_s^t (-2+\sin\tau+\sin\pi\tau)\,d\tau} v(\xi).$$

Hence $\|U(t,s)\| \le e^{-(t-s)}$ for $t \ge s$. If $(n-1)^2 + 1 \le -2 + \sin t + \sin \pi t \le n^2 - 1$ for all $t \in \mathbb{R}$, and for some $n \in \mathbb{N}$. We can define $P(t) : \mathbb{H} \to \mathbb{H}$ by

$$P(t)v = \sum_{k=n}^{\infty} \langle v, v_k\rangle v_k.$$

Then

$$\begin{aligned} U(t,s)P(s)v &= \sum_{k=n}^{\infty} \langle v, v_k\rangle e^{\int_s^t (-2+\sin\tau+\sin\pi\tau)\,d\tau} T(t-s)v_k \\ &= \sum_{k=n}^{\infty} e^{-k^2(t-s)+\int_s^t (-2+\sin\tau+\sin\pi\tau)\,d\tau} \langle v, v_k\rangle v_k, \end{aligned}$$

which implies that

$$\bigl\|U(t,s)P(s)v\bigr\| \le e^{-n^2(t-s)+\int_s^t (-2+\sin\tau+\sin\pi\tau)\,d\tau} \|v\|.$$

Since $-n^2(t-s)+\int_s^t(-2+\sin\tau+\sin\pi\tau)\,d\tau\le -(t-s)$, we have

$$\|U(t,s)P(s)\|\le e^{-(t-s)},\quad s\le t.$$

On the other hand, if $Q(t)=I-P(t)$, then

$$U(t,s)Q(s)\nu=\sum_{k=1}^{n-1}e^{k^2(t-s)-\int_s^t(-2+\sin\tau+\sin\pi\tau)\,d\tau}\langle\nu,\nu_k\rangle\nu_k,$$

which implies that

$$U_Q(t,s)Q(s)\nu=\big(U(t,s)\big)^{-1}Q(s)\nu=\sum_{k=1}^{n-1}e^{-k^2(t-s)+\int_s^t(-2+\sin\tau+\sin\pi\tau)\,d\tau}\langle\nu,\nu_k\rangle\nu_k$$

and

$$\|U_Q(t,s)Q(s)\nu\|\le e^{(n-1)^2(t-s)-\int_s^t(-2+\sin\tau+\sin\pi\tau)\,d\tau}\|\nu\|.$$

Since $-(n-1)^2(t-s)+\int_s^t(-2+\sin\tau+\sin\pi\tau)\,d\tau\ge -(t-s)$, we have

$$\|U_Q(t,s)Q(s)\|\le e^{-(t-s)},\quad s\le t.$$

Furthermore, we have

$$\begin{aligned}
&\|R(\lambda,A(t+\tau))-R(\lambda,A(t))\|\\
&\quad=\|R(\lambda-(-2+\sin(t+\tau)+\sin\pi(t+\tau)),A)-R(\lambda-(-2+\sin t+\sin\pi t),A)\|\\
&\quad\le\|(-2+\sin(t+\tau)+\sin\pi(t+\tau))-b(t)\|\,\|R(\lambda-(-2+\sin(t+\tau)+\sin\pi(t+\tau)),A)\|\\
&\qquad\times\|R(\lambda-(-2+\sin t+\sin\pi t),A)\|,\quad t,\tau\in\mathbb{R}.
\end{aligned}$$

Since $-2+\sin t+\sin\pi t$ is almost periodic, $R(\lambda,A(\cdot))\in AP(L(\mathbb{R},L^p(\mathbb{P},\mathbb{H})))$. Hence $A(t)$ satisfy (H1)-(H4) with $M=1$, $\delta=1$.

Set $z(t)(x)=z(t,x)$ for $t\in\mathbb{R}$, $x\in[0,\pi]$. Taking

$$\begin{aligned}
g(t,\psi)(x)&=a_1(t)\sin(\psi(x)),\\
f(t,\psi)(x)&=a_2(t)\sin(\psi(x)),
\end{aligned}$$

and

$$I_i(\psi)(x)=\beta_i\sin(\psi(x)),\quad i\in\mathbb{Z}.$$

Then equations (5.1)-(5.3) can be written in the abstract form as the system (1.1)-(1.2). Since $a_i\in PAP_T(\mathbb{R},\mathbb{R})$, $i=1,2$, we deduce that $g,f\in PAP_T(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H}),L^p(\mathbb{P},\mathbb{H}))$, and

$$\begin{aligned}
&E\|g(t,\psi)\|^p+E\|f(t,\psi)\|^p\\
&\quad=E\bigg[\bigg(\int_0^\pi|a_1(t)\sin(\psi(x))|^2\,dx\bigg)^{1/2}\bigg]^p
\end{aligned}$$

$$
\begin{aligned}
&+ E\left[\left(\int_0^\pi \left|a_2(t)\sin\left(\psi(x)\right)\right|^2 dx\right)^{1/2}\right]^p \\
&\le \left[\left|a_1(t)\right|^p + \left|a_2(t)\right|^p\right]\|\psi\|^p \le \tilde{L}\|\psi\|^p
\end{aligned}
$$

for all $t \in \mathbb{R}$, $\psi \in L^p(\mathbb{P}, \mathbb{H})$, where $\tilde{L} = \sup_{t\in\mathbb{R}}(\max\{|a_1(t)|^p, |a_2(t)|^p\})$. Similarly, $\beta_i \in PAP(\mathbb{Z}, \mathbb{R})$ implies that $I_i \in PAP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, $i \in \mathbb{Z}$, and

$$
E\left\|I_i(\psi)\right\|^p = E\left[\left(\int_0^\pi \left|\beta_i \sin\left(\psi(x)\right)\right|^2 dx\right)^{1/2}\right]^p \le \tilde{L}_i\|\psi\|, \quad i \in \mathbb{Z},
$$

for all $\psi \in L^p(\mathbb{P}, \mathbb{H})$, where $\tilde{L}_i = |\beta_i|^p$, $i \in \mathbb{Z}$. Suppose that the assumption (H7) in Section 3 holds. Then it satisfies all the assumptions given in Theorem 3.1. Therefore, the system (5.1)-(5.3) has an p-mean piecewise pseudo almost periodic mild solution on $\mathbb{R}$.

In the above example, we can take

$$
\begin{aligned}
g(t, \psi)(x) &= \left[\sin t + \sin 2\pi\sqrt{2}t + \rho_1\sigma(t)\right]\sin\left(\psi(x)\right), \\
f(t, \psi)(x) &= \left[\sin t + \sin 2\pi\sqrt{2}t + \rho_2\sigma(t)\right]\sin\left(\psi(x)\right),
\end{aligned}
$$

and

$$
I_i(\psi) = \left[\sin i + \sin 2\pi\sqrt{2}i + \varsigma_i\sigma(i)\right]\sin\left(\psi(x)\right), \quad i \in \mathbb{Z},
$$

where $\sigma \in UPC(\mathbb{R}, \mathbb{R})$ is defined by

$$
\sigma(t) = \begin{cases} 0 & \text{for } t < 0, \\ e^{-\rho_3 t} & \text{for } t \ge 0 \end{cases}
$$

for $\rho_k > 0$, $k = 1, 2, 3$, $\varsigma_i > 0$, $i \in \mathbb{Z}$. Obviously, $[\sin t + \sin 2\pi\sqrt{2}t + \sigma(t)]\sin(\psi) \in PAP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}))$ and $[\sin i + \sin 2\pi\sqrt{2}i + \sigma(i)]\sin(\psi) \in PAP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, $i \in \mathbb{Z}$, where $[\sin t + \sin 2\pi\sqrt{2}t]\sin(\psi)$ is the almost periodic component and $\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\sigma(t)\,dt = 0$. Moreover, $E\|g(t, \psi)\|^p + E\|f(t, \psi)\|^p \le \tilde{l}\|\psi\|^p$ and $E\|I_i(\psi)\|^p \le \tilde{l}_i\|\psi\|^p$, $i \in \mathbb{Z}$, for all $t \in \mathbb{R}$, $\psi \in L^p(\mathbb{P}, \mathbb{H})$, where $\tilde{l} = (2 + \rho_1)^p + (2 + \rho_2)^p$ and $\tilde{l}_i = (2 + \varsigma_i)^p$, $i \in \mathbb{Z}$. On the other hand, we can see that

$$
E\left\|g(t, \psi)\right\|^p + E\left\|f(t, \psi)\right\|^p \le \hat{l}_1\|\psi\|^p + \hat{l}_2 e^{-p\rho_3 t}
$$

for all $t \in \mathbb{R}$, $\psi \in L^p(\mathbb{P}, \mathbb{H})$, where $\hat{l}_1 = 2^{2(p-1)}$, $\hat{l}_2 = 2^{p-1}\pi^{p/2}(\rho_1^p + \rho_2^p)$. Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 4.1, Hence, the system (5.1)-(5.3) has an p-mean piecewise exponentially stable pseudo almost periodic mild solution.

6 Conclusion

In this paper, we studied the p-mean piecewise pseudo almost periodic periodicity for a class of impulsive nonautonomous partial stochastic evolution equations in Hilbert spaces. More precisely, by using the exponential dichotomy techniques, stochastic analysis theory, and Leray-Schauder nonlinear alternative combined with the new composition theorem, we discussed the existence and exponential stability of p-mean piecewise pseudo almost periodic mild solutions for these equations. The conditions are formulated and proved under which the nonlinear terms and the jump operators satisfy the non-Lipschitz condition with the sense of the pseudo almost periodic. Finally, an example is provided to illustrate the obtained theory.

There are two direct issues which require further study. First, we will investigate the existence and exponential stability of p-mean piecewise pseudo almost periodic mild solutions for impulsive partial stochastic functional differential equations with infinite delay in Hilbert spaces. Second, we will devote our efforts to the study of the existence and exponential stability of p-mean piecewise weighted pseudo almost periodic mild solutions of impulsive partial stochastic differential equations.

Competing interests
The authors declares that they have no competing interests.

Authors' contributions
The authors completed the paper, and read and approved the final manuscript.

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