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Explicit iteration to Hadamard fractional integro-differential equations on infinite domain

Guotao Wang[1*], Ke Pei[1] and Dumitru Baleanu[2,3]

*Correspondence:
wgt2512@163.com
[1]School of Mathematics, Shanxi Normal University, Linfen, Shanxi 041004, People's Republic of China
Full list of author information is available at the end of the article

Abstract

This paper investigates the existence of the unique solution for a Hadamard fractional integral boundary value problem of a Hadamard fractional integro-differential equation with the monotone iterative technique. Two examples to illustrate our result are given.

Keywords: Hadamard derivative; Hadamard integro-differential boundary conditions; monotone iterative; infinite interval

1 Introduction

Fractional differential equations are becoming more and more popular recently in several journals and books due to their applications in a number of fields such as physics, biophysics, mechanical systems, electrical-analytical, and thermal systems [1–6]. For some recent development of this topic, see for example [7–13] and the references therein.

In 1892 [14], Hadamard presented a concept of fractional derivatives, which is different from Caputo and Riemann-Liouville type fractional derivatives and involves a logarithmic function of an arbitrary exponent in the integral kernel. It is significant that the study of Hadamard type fractional differential equations is still in its infancy and deserves further study. A detailed presentation of Hadamard fractional derivative is available in [3] and [15–23].

As was pointed out in [22], Hadamard's construction is more appropriate for problems on half axes. In this situation, we consider the following Hadamard fractional integro-differential equations with Hadamard fractional integral boundary conditions on an infinite interval:

$$\begin{cases} {}^{H}D^{\gamma}u(t)+f(t,u(t),{}^{H}I^{q}u(t))=0, & 2<\gamma<3, t\in(1,+\infty), \\ u(1)=u'(1)=0, & {}^{H}D^{\gamma-1}u(\infty)=\sum_{i=1}^{m}\lambda_i{}^{H}I^{\beta_i}u(\eta), \end{cases} \tag{1.1}$$

where ${}^{H}D^{\gamma}$ denotes Hadamard fractional derivative of order γ, $\eta\in(1,\infty)$, and ${}^{H}I^{(\cdot)}$ is the Hadamard fractional integral, $q,\beta_i>0$ $(i=1,2,\ldots,m)$, $\lambda_i\geq 0$ $(i=1,2,\ldots,m)$ are given constants and γ, η, β_i, λ_i satisfy $\Gamma(\gamma)>\sum_{i=1}^{m}\frac{\lambda_i\Gamma(\gamma)}{\Gamma(\gamma+\beta_i)}(\log\eta)^{\gamma+\beta_i-1}$.

We recall that the monotone iterative technique represents a powerful tool for seeking the solution of a nonlinear problem. For more details as regards the application of this method in fractional differential equations, see [24–38] and the references therein.

We organize the rest of our manuscript as follows: In Section 2, we show some useful preliminaries and the key lemmas that are used in subsequent part of the manuscript. Then, in Section 3, the main results and proofs are provided. Section 4, exhibits two examples to illustrate our main results.

2 Preliminaries

Below, we will present some useful definitions and related lemmas.

Define

$$E=\left\{u\in C\bigl([1,\infty),\mathbb{R}\bigr):\sup_{t\in[1,\infty)}\frac{|u(t)|}{1+(\log t)^{\gamma-1}}<\infty\right\}, \tag{2.1}$$

then E denotes a Banach space equipped with norm $\|u\|_E=\sup_{t\in[1,\infty)}\frac{|u(t)|}{1+(\log t)^{\gamma-1}}$.

Denote

$$\Omega=\Gamma(\gamma)-\sum_{i=1}^{m}\frac{\lambda_i\Gamma(\gamma)}{\Gamma(\gamma+\beta_i)}(\log\eta)^{\gamma+\beta_i-1}; \tag{2.2}$$

obviously $\Omega>0$.

Definition 2.1 [3] For a function g, the Hadamard fractional integral of order γ has the following form:

$$^{H}I^{\gamma}g(t)=\frac{1}{\Gamma(\gamma)}\int_1^t\left(\log\frac{t}{s}\right)^{\gamma-1}\frac{g(s)}{s}\,ds,\quad \gamma>0,$$

provided the integral exists.

Definition 2.2 [3] The Hadamard fractional derivative of fractional order γ for a function $g:[1,\infty)\to\mathbb{R}$ has the following form:

$$^{H}D^{\gamma}g(t)=\frac{1}{\Gamma(n-\gamma)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\log\frac{t}{s}\right)^{n-\gamma-1}\frac{g(s)}{s}\,ds,\quad n-1<\gamma<n, n=[\gamma]+1,$$

where $[\gamma]$ means the integer part of the real number γ and $\log(\cdot)=\log_e(\cdot)$.

Lemma 2.1 [3] *If* $a,\gamma,\beta>0$ *then*

$$^{H}I_a^{\gamma}\left(\log\left(\frac{t}{a}\right)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}\left(\log\frac{x}{a}\right)^{\beta+\gamma-1}.$$

Lemma 2.2 [3] *If* $a,\gamma,\beta>0$ *then*

$$^{H}D_a^{\gamma}\left(\log\left(\frac{t}{a}\right)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\left(\log\frac{x}{a}\right)^{\beta-\gamma-1}.$$

Lemma 2.3 [3] *Given $\gamma > 0$ and $x \in C[1,\infty) \cap L^1[1,\infty)$, then the solution of the Hadamard fractional differential equation ${}^H D^\gamma x(t) = 0$ is*

$$x(t) = \sum_{i=1}^{n} c_i (\log t)^{\gamma - i} \tag{2.3}$$

and

$${}^H I^{\gamma\,H} D^\gamma x(t) = x(t) + \sum_{i=1}^{n} c_i (\log t)^{\gamma - i} \tag{2.4}$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, and $n - 1 < \gamma < n$.

Lemma 2.4 *Let $h \in C[1,\infty)$ with $0 < \int_1^\infty h(s)\frac{ds}{s} < \infty$, then the Hadamard fractional integral boundary value problem*

$$\begin{cases} {}^H D^\gamma u(t) + h(t) = 0, & 2 < \gamma < 3, t \in (1, +\infty), \\ u(1) = u'(1) = 0, & {}^H D^{\gamma-1} u(\infty) = \sum_{i=1}^m \lambda_i {}^H I^{\beta_i} u(\eta), \end{cases} \tag{2.5}$$

has the unique solution

$$u(t) = \int_1^\infty G(t,s) h(s) \frac{ds}{s}, \tag{2.6}$$

where

$$G(t,s) = g(t,s) + \sum_{i=1}^{m} \frac{\lambda_i (\log t)^{\gamma-1}}{\Omega \Gamma(\gamma + \beta_i)} g_i(\eta, s), \tag{2.7}$$

and

$$g(t,s) = \frac{1}{\Gamma(\gamma)} \begin{cases} (\log t)^{\gamma-1} - (\log(\frac{t}{s}))^{\gamma-1}, & 1 \le s \le t < \infty, \\ (\log t)^{\gamma-1}, & 1 \le t \le s < \infty, \end{cases} \tag{2.8}$$

$$g_i(\eta,s) = \begin{cases} (\log \eta)^{\gamma+\beta_i-1} - (\log(\frac{\eta}{s}))^{\gamma+\beta_i-1}, & 1 \le s \le \eta < \infty, \\ (\log \eta)^{\gamma+\beta_i-1}, & 1 \le \eta \le s < \infty. \end{cases} \tag{2.9}$$

Proof We apply the Hadamard fractional integral of order γ to

$${}^H D^\gamma u(t) + h(t) = 0,$$

and we conclude that

$$u(t) = c_1 (\log t)^{\gamma-1} + c_2 (\log t)^{\gamma-2} + c_3 (\log t)^{\gamma-3} - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log \frac{t}{s}\right)^{\gamma-1} h(s) \frac{ds}{s}, \tag{2.10}$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Using the fact that $u(1) = u'(1) = 0$, we conclude that $c_2 = c_3 = 0$. Thus,

$$u(t) = c_1(\log t)^{\gamma-1} - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log \frac{t}{s}\right)^{\gamma-1} h(s)\frac{ds}{s}. \tag{2.11}$$

Lemma 2.2 implies that

$${}^{H}D^{\gamma-1}u(t) = c_1\Gamma(\gamma) - \int_1^t h(s)\frac{ds}{s}. \tag{2.12}$$

Thus, the condition

$${}^{H}D^{\gamma-1}(\infty) = \sum_{i=1}^{m} \lambda_i {}^{H}I^{\beta_i} u(\eta)$$

leads to

$$c_1 = \frac{1}{\Omega}\left(\int_1^\infty h(s)\frac{ds}{s} - \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\gamma+\beta_i)} \int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\beta_i-1} h(s)\frac{ds}{s}\right), \tag{2.13}$$

where Ω is defined by (2.2). Substituting $c_2 = c_3 = 0$ and (2.13) into (2.10), we get the unique solution of the Hadamard fractional integral boundary value problem (2.5)

$$\begin{aligned} u(t) = {}& \frac{(\log t)^{\gamma-1}}{\Omega}\left(\int_1^\infty h(s)\frac{ds}{s}\right. \\ & \left. - \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\gamma+\beta_i)} \int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\beta_i-1} h(s)\frac{ds}{s}\right) \\ & - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log\frac{t}{s}\right)^{\gamma-1} h(s)\frac{ds}{s} \\ = {}& \int_1^\infty g(t,s)h(s)\frac{ds}{s} + \sum_{i=1}^{m} \frac{\lambda_i(\log t)^{\gamma-1}}{\Omega\Gamma(\gamma+\beta_i)} \int_1^\infty g_i(\eta,s)h(s)\frac{ds}{s} \\ = {}& \int_1^\infty G(t,s)h(s)\frac{ds}{s}. \end{aligned} \tag{2.14}$$

The proof is finished. □

Lemma 2.5 *The Green's function $G(t,s)$ defined by* (2.7) *has the following properties*:

(A_1): *$G(t,s)$ is continuous and $G(t,s) \geq 0$ for $(t,s) \in [1,\infty) \times [1,\infty)$.*
(A_2): $\frac{G(t,s)}{1+(\log t)^{\gamma-1}} \leq \frac{1}{\Gamma(\gamma)} + \sum_{i=1}^{m} \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\gamma+\beta_i)}$ *for all $s,t \in [1,\infty)$.*

Proof Since (A_1) it is easy to prove, we do not present it but only prove the property (A_2).
For $\forall s,t \in [1,\infty)$,

$$\frac{G(t,s)}{1+(\log t)^{\gamma-1}} = \frac{1}{1+(\log t)^{\gamma-1}}\left[g(t,s) + \sum_{i=1}^{m} \frac{\lambda_i(\log t)^{\gamma-1} g_i(\eta,s)}{\Omega\Gamma(\gamma+\beta_i)}\right]$$

$$\begin{aligned}&\le \frac{1}{\Gamma(\gamma)} + \sum_{i=1}^{m} \frac{\lambda_i (\log t)^{\gamma-1} g_i(\eta,s)}{\Omega\Gamma(\gamma+\beta_i)(1+(\log t)^{\gamma-1})}\\&\le \frac{1}{\Gamma(\gamma)} + \sum_{i=1}^{m} \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\gamma+\beta_i)}.\end{aligned}$$

□

We present the following conditions for the sake of convenience:

(C_1): There exist two positive functions $p(t)$ and $q(t)$ such that

$$\begin{aligned}&\Lambda = \int_1^\infty \bigl[1+(\log t)^{\gamma-1}\bigr]\biggl[p(t) + \frac{q(t)(\log t)^q}{\Gamma(q)}\biggr]\frac{dt}{t} < \infty,\\&\bigl|f(t,u,v) - f(t,\overline{u},\overline{v})\bigr| \le p(t)|u-\overline{u}| + q(t)|v-\overline{v}|, \quad t\in[1,\infty), u,v,\overline{u},\overline{v}\in\mathbb{R}.\end{aligned}$$

(C_2):

$$\lambda = \int_1^\infty \bigl|f(t,0,0)\bigr|\frac{dt}{t} < \infty.$$

Lemma 2.6 *If* (C_1), (C_2) *hold, then for any* $u\in E$

$$\int_1^\infty \bigl|f(t,u(t),{}^H I^q u(t)\bigr|\frac{dt}{t} \le \Lambda\|u\|_E + \lambda. \tag{2.15}$$

Proof For any $u\in E$, taking $\overline{u}=0$, then ${}^H I^q \overline{u} = 0$. Thus, by condition (C_1) we have

$$\begin{aligned}\bigl|f\bigl(t,u(t),{}^H I^q u(t)\bigr)\bigr| &\le p(t)\bigl|u(t)\bigr| + q(t)\bigl|{}^H I^q u(t)\bigr| + \bigl|f(t,0,0)\bigr|\\&\le p(t)\bigl[1+(\log t)^{\gamma-1}\bigr]\frac{|u(t)|}{1+(\log t)^{\gamma-1}}\\&\quad + q(t)\frac{1}{\Gamma(q)}\int_1^t \biggl(\log\frac{t}{s}\biggr)^{q-1}\frac{|u(s)|}{s}\,ds + \bigl|f(t,0,0)\bigr|\\&\le p(t)\bigl[1+(\log t)^{\gamma-1}\bigr]\|u\|_E\\&\quad + q(t)\frac{1+(\log t)^{\gamma-1}}{\Gamma(q)}\\&\quad \times \int_1^t \frac{(\log\frac{t}{s})^{q-1}}{s}\frac{|u(s)|}{1+(\log s)^{\gamma-1}}\frac{1+(\log s)^{\gamma-1}}{1+(\log t)^{\gamma-1}}\,ds\\&\quad + \bigl|f(t,0,0)\bigr|\\&\le p(t)\bigl[1+(\log t)^{\gamma-1}\bigr]\|u\|_E\\&\quad + q(t)\frac{1+(\log t)^{\gamma-1}}{\Gamma(q)}\|u\|_E\int_1^t \frac{(\log t)^{q-1}}{s}\,ds + \bigl|f(t,0,0)\bigr|\\&\le p(t)\bigl[1+(\log t)^{\gamma-1}\bigr]\|u\|_E + q(t)\frac{1+(\log t)^{\gamma-1}}{\Gamma(q)}(\log t)^q\|u\|_E\\&\quad + \bigl|f(t,0,0)\bigr|,\end{aligned} \tag{2.16}$$

from which, combined with (C_1) and (C_2), we can obtain

$$\begin{aligned}\int_1^\infty \left|f\left(t,u(t),{}^HI^q u(t)\right)\right|\frac{dt}{t} &\le \int_1^\infty p(t)\left[1+(\log t)^{\gamma-1}\right]\|u\|_E\frac{dt}{t}\\ &\quad+\int_1^\infty q(t)\frac{1+(\log t)^{\gamma-1}}{\Gamma(q)}(\log t)^q\|u\|_E\frac{dt}{t}\\ &\quad+\int_1^\infty \left|f(t,0,0)\right|\frac{dt}{t}\\ &=\Lambda\|u\|_E+\lambda. \end{aligned}\tag{2.17}$$

The proof is done. □

3 Main results

Theorem 3.1 *Suppose that the conditions* (C_1) *and* (C_2) *hold. Let*

$$w=\Lambda\left(\frac{1}{\Gamma(\gamma)}+\sum_{i=1}^m\frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\gamma+\beta_i)}\right)<1.\tag{3.1}$$

Then the Hadamard fractional integral boundary value problem (1.1) *admits an unique solution* $\widetilde{u}(t)$ *in* E. *In addition, there exists a monotone iterative sequence* $u_n(t)$ *such that* $u_n(t)\to\widetilde{u}(t)$ $(n\to\infty)$ *uniformly on any finite sub-interval of* $[1,\infty)$, *where*

$$u_n(t)=\int_1^\infty G(t,s)f\left(s,u_{n-1}(s),{}^HI^q u_{n-1}(s)\right)\frac{ds}{s}.\tag{3.2}$$

Furthermore, there exists an error estimate for the approximating sequence

$$\|u_n-\widetilde{u}\|_E\le\frac{w^n}{1-w}\|u_1-u_0\|_E\quad(n=1,2,\ldots).\tag{3.3}$$

Proof Define the operator T by

$$(Tu)(t)=\int_1^\infty G(t,s)f\left(s,u(s),{}^HI^q u(s)\right)\frac{ds}{s}.\tag{3.4}$$

By Lemma 2.4, the Hadamard fractional integral boundary value problem (1.1) possesses a solution u iff u is a solution of $u=Tu$.

First, for any $t\in[1,\infty)$, by Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}\frac{|(Tu)(t)|}{1+(\log t)^{\gamma-1}} &\le \int_1^\infty \frac{G(t,s)}{1+(\log t)^{\gamma-1}}\left|f\left(s,u(s),{}^HI^q u(s)\right)\right|\frac{ds}{s}\\ &\le\left(\frac{1}{\Gamma(\gamma)}+\sum_{i=1}^m\frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\gamma+\beta_i)}\right)\left[\Lambda\|u\|_E+\lambda\right]\\ &=w\|u\|_E+k.\end{aligned}\tag{3.5}$$

This means

$$\|Tu\|_E\le w\|u\|_E+k,\quad\forall t\in[1,\infty),\tag{3.6}$$

where w is defined in (3.1) and

$$k = \lambda \left(\frac{1}{\Gamma(\gamma)} + \sum_{i=1}^{m} \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\gamma+\beta_i)} \right). \tag{3.7}$$

In addition, for any $u, \overline{u} \in E$, we have

$$\begin{aligned}
\frac{|(Tu)(t) - (T\overline{u})(t)|}{1+(\log t)^{\gamma-1}} &\leq \int_1^{\infty} \frac{G(t,s)}{1+(\log t)^{\gamma-1}} \left| f\left(s,u(s),{}^H I^q u(s)\right) - f\left(s,\overline{u}(s),{}^H I^q \overline{u}(s)\right) \right| \frac{ds}{s} \\
&\leq \int_1^{\infty} \frac{G(t,s)}{1+(\log t)^{\gamma-1}} \left[p(s)\left|u(s)-\overline{u}(s)\right| + q(s)\left|{}^H I^q u(s) - {}^H I^q \overline{u}(s)\right| \right] \frac{ds}{s} \\
&\leq \int_1^{\infty} \frac{G(t,s)}{1+(\log t)^{\gamma-1}} p(s)\left[1+(\log s)^{\gamma-1}\right] \frac{|u(s)-\overline{u}(s)|}{1+(\log s)^{\gamma-1}}\, ds \\
&\quad + \int_1^{\infty} \frac{G(t,s)}{1+(\log t)^{\gamma-1}} q(s)\left|{}^H I^q u(s) - {}^H I^q \overline{u}(s)\right| \frac{ds}{s} \\
&\leq \int_1^{\infty} \frac{G(t,s)}{1+(\log t)^{\gamma-1}} p(s)\left[1+(\log s)^{\gamma-1}\right] \|u-\overline{u}\|_E \, ds \\
&\quad + \int_1^{\infty} \frac{G(t,s)}{1+(\log t)^{\gamma-1}} q(s) \frac{[1+(\log s)^{\gamma-1}](\log s)^{\gamma}}{\Gamma(\gamma)} \|u-\overline{u}\|_E \frac{ds}{s} \\
&\leq \left(\frac{1}{\Gamma(\gamma)} + \sum_{i=1}^{m} \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\gamma+\beta_i)} \right) \\
&\quad \times \int_1^{\infty} \|u-\overline{u}\|_E \left[1+(\log s)^{\gamma-1}\right] \left[p(s) + \frac{q(s)(\log s)^{\gamma}}{\Gamma(\gamma)} \right] \frac{ds}{s} \\
&\leq \left(\frac{1}{\Gamma(\gamma)} + \sum_{i=1}^{m} \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\gamma+\beta_i)} \right) \Lambda \|u-\overline{u}\|_E \\
&= w\|u-\overline{u}\|_E.
\end{aligned} \tag{3.8}$$

Then we get

$$\|Tu - T\overline{u}\|_E \leq w\|u-\overline{u}\|_E, \quad \forall u, \overline{u} \in E. \tag{3.9}$$

Through the Banach fixed point theorem, we can ensure that T has a unique fixed point $\widetilde{u}$ in E. That is, (1.1) admits a unique solution $\widetilde{u}$ in E. In addition, for any $u_0 \in E$, $\|u_n - \widetilde{u}\|_E \to 0$ as $n \to \infty$, where $u_n = Tu_{n-1}$ ($n = 1, 2, \ldots$).

From (3.9), we have

$$\|u_n - u_{n-1}\|_E \leq w^{n-1}\|u_1 - u_0\|_E \tag{3.10}$$

and

$$\begin{aligned}
\|u_n - u_j\|_E &\leq \|u_n - u_{n-1}\|_E + \|u_{n-1} - u_{n-2}\|_E + \cdots + \|u_{j+1} - u_j\|_E \\
&\leq \frac{w^n(1-w^{n-j})}{1-w} \|u_1 - u_0\|_E.
\end{aligned} \tag{3.11}$$

Letting $n \to \infty$ on both sides of (3.11), we conclude that

$$\|u_n - \widetilde{u}\|_E \leq \frac{w^n}{1-w}\|u_1 - u_0\|_E. \tag{3.12}$$

□

4 Example

Example 4.1 In the following we discuss the Hadamard fractional integral boundary value problem

$$\begin{cases} {}^H D^{\frac{5}{2}} u(t) + \frac{e^{-3t}t}{1+(\log t)^{\frac{3}{2}}} \cos(3t^2 \\ \quad + u(t)) + \frac{3\sqrt{\pi}e^{-3t}t}{8[1+(\log t)^{\frac{3}{2}}](\log t)^{\frac{5}{2}}} \arctan({}^H I^{\frac{5}{2}} u(t)) = 0, \\ u(1) = u'(1) = 0, \qquad {}^H D^{\frac{3}{2}} u(+\infty) = \lambda_1 {}^H I^{\beta_1} u(\eta), \end{cases} \tag{4.1}$$

where $\gamma = \frac{5}{2}$, $m = 1$, $q = \frac{5}{2}$, and λ_1, β_1, η satisfy ($\lambda_1 \geq 0, \beta_1 > 0, \eta > 1$)

$$0 < \frac{\Gamma(\frac{1}{2} + \beta_1)}{3\Gamma(\frac{1}{2} + \beta_1) - 4\lambda_1 (\log \eta)^{\frac{3}{2}+\beta_1}} < \frac{e^3\sqrt{\pi}}{2} \tag{4.2}$$

(see Figure 1).

For example, we can take $\lambda_1 = \frac{1}{10}$, $\beta_1 = \frac{3}{2}$, $\eta = \frac{5}{2}$,

$$\begin{aligned} &\left|f\left(t, u, {}^H I^{\frac{5}{2}} u(t)\right) - f\left(t, \overline{u}, {}^H I^{\frac{5}{2}} \overline{u}(t)\right)\right| \\ &\quad \leq \frac{e^{-3t}t}{1+(\log t)^{\frac{3}{2}}} \left|\cos\left(3t^2 + u(t)\right) - \cos\left(3t^2 + \overline{u}(t)\right)\right| \\ &\qquad + \frac{3\sqrt{\pi}e^{-3t}t}{8[1+(\log t)^{\frac{3}{2}}](\log t)^{\frac{5}{2}}} \left|\arctan\left({}^H I^{\frac{5}{2}} u(t)\right) - \arctan\left({}^H I^{\frac{5}{2}} \overline{u}(t)\right)\right| \end{aligned}$$

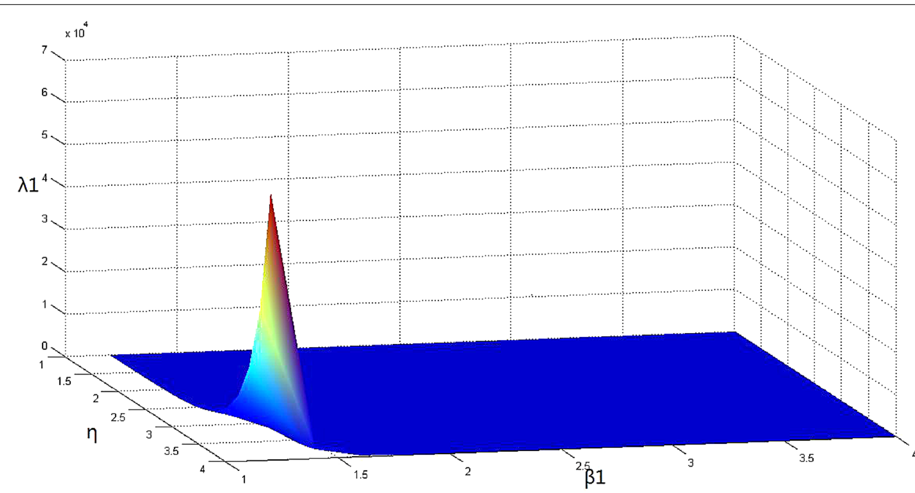

Figure 1 Set of $\lambda_1 \geq 0, \beta_1 > 0, \eta > 1$ such that $0 < \frac{\Gamma(\frac{1}{2}+\beta_1)}{3\Gamma(\frac{1}{2}+\beta_1)-4\lambda_1(\log \eta)^{\frac{3}{2}+\beta_1}} < \frac{e^3\sqrt{\pi}}{2}$.

$$\begin{aligned}&\leq \frac{e^{-3t}t}{1+(\log t)^{\frac{3}{2}}}\big|u(t)-\overline{u}(t)\big| \\ &\quad + \frac{3\sqrt{\pi}e^{-3t}t}{8[1+(\log t)^{\frac{3}{2}}](\log t)^{\frac{5}{2}}}\big|{}^{H}I^{\frac{5}{2}}u(t)-{}^{H}I^{\frac{5}{2}}\overline{u}(t)\big|.\end{aligned}$$

Since $p(t)=\frac{e^{-3t}t}{1+(\log t)^{\frac{3}{2}}}$ and $q(t)=\frac{3\sqrt{\pi}e^{-3t}t}{8[1+(\log t)^{\frac{3}{2}}](\log t)^{\frac{5}{2}}}$, we can show that

$$\begin{aligned}\Lambda &= \int_1^\infty \big[1+(\log t)^{\frac{3}{2}}\big]\biggl[\frac{e^{-3t}t}{1+(\log t)^{\frac{3}{2}}}+\frac{3\sqrt{\pi}e^{-3t}t(\log t)^{\frac{5}{2}}}{8[1+(\log t)^{\frac{3}{2}}](\log t)^{\frac{5}{2}}\Gamma(\frac{5}{2})}\biggr]\frac{dt}{t} \\ &= \frac{1}{2e^3}<\infty, \\ \lambda &= \int_1^\infty \big|f(t,0,0)\big|\frac{dt}{t}=\int_1^\infty e^{-3t}\,dt=\frac{1}{3e^3}<\infty.\end{aligned}$$

Then (C_1) and (C_2) hold. At last, by a simple computation, we have

$$\begin{aligned}\Omega &= \Gamma(\gamma)-\frac{\lambda_1\Gamma(\gamma)}{\Gamma(\gamma+\beta_1)}(\log\eta)^{\gamma+\beta_1-1}=\frac{3\sqrt{\pi}}{4}-\frac{\sqrt{\pi}\lambda_1(\log\eta)^{\frac{3}{2}+\beta_1}}{\Gamma(\frac{1}{2}+\beta_1)}>\frac{1}{2e^3}>0, \\ w &= \Lambda\biggl(\frac{1}{\Gamma(\gamma)}+\frac{\lambda_1 g_1(\eta,s)}{\Omega\Gamma(\gamma+\beta_i)}\biggr)=\frac{2}{e^3\sqrt{\pi}}\biggl(\frac{\Gamma(\frac{1}{2}+\beta_1)}{3\Gamma(\frac{1}{2}+\beta_1)-4\lambda_1(\log\eta)^{\frac{3}{2}+\beta_1}}\biggr)<1.\end{aligned}$$

As a result, the conditions of Theorem 3.1 hold. Thus, the conclusion of Theorem 3.1 implies that (4.1) possesses a unique solution.

Example 4.2 Let us discuss the following Hadamard fractional integral boundary value problem:

$$\begin{cases} {}^{H}D^{\frac{9}{4}}u(t)+f(t,u(t),{}^{H}I^{\frac{9}{2}}u(t))=0, \\ u(1)=u'(1)=0, \qquad {}^{H}D^{\frac{9}{4}}u(+\infty)=\sum_{i=1}^{3}\lambda_i{}^{H}I^{\beta_i}u(\eta), \end{cases} \tag{4.3}$$

here

$$f\big(t,u(t),{}^{H}I^{\frac{9}{2}}u(t)\big)=\frac{\sin(t^2+u(t))}{16(1+t^2)[1+(\log t)^{\frac{5}{4}}]}+\frac{{}^{H}I^{\frac{9}{2}}u(t)-\sin({}^{H}I^{\frac{9}{2}}u(t))\cos({}^{H}I^{\frac{9}{2}}u(t))}{128(1+t^2)[1+(\log t)^{\frac{5}{4}}](\log t)^{\frac{9}{2}}}.$$

Take $\gamma=\frac{9}{4}$, $m=3$, $q=\frac{9}{2}$, $\eta=\frac{9}{4}$, $\lambda_1=\frac{1}{10}$, $\beta_1=\frac{3}{4}$, $\lambda_2=\frac{\sqrt{\pi}}{2}$, $\beta_2=\frac{7}{4}$, $\lambda_3=4$, and $\beta_3=\frac{11}{4}$.

The function f satisfies the inequality

$$\begin{aligned}&\big|f\big(t,u,{}^{H}I^{\frac{9}{2}}u(t)\big)-f\big(t,\overline{u},{}^{H}I^{\frac{9}{2}}\overline{u}(t)\big)\big| \\ &\quad\leq \frac{1}{16(1+t^2)[1+(\log t)^{\frac{5}{4}}]}\big|u(t)-\overline{u}(t)\big| \\ &\qquad + \frac{1}{64(1+t^2)[1+(\log t)^{\frac{5}{4}}](\log t)^{\frac{9}{2}}}\big|{}^{H}I^{\frac{9}{2}}u(t)-{}^{H}I^{\frac{9}{2}}\overline{u}(t)\big|.\end{aligned}$$

Since $p(t) = \frac{1}{16(1+t^2)[1+(\log t)^{\frac{5}{4}}]}$ and $q(t) = \frac{1}{64(1+t^2)[1+(\log t)^{\frac{5}{4}}](\log t)^{\frac{9}{2}}}$, we can show that

$$\begin{aligned}\Lambda &= \int_1^\infty \bigl[1 + (\log t)^{\frac{5}{4}}\bigr] \\ &\quad \times \biggl[\frac{1}{16(1+t^2)[1+(\log t)^{\frac{5}{4}}]} + \frac{(\log t)^{\frac{9}{2}}}{64\Gamma(\frac{9}{2})(1+t^2)[1+(\log t)^{\frac{5}{4}}](\log t)^{\frac{9}{2}}}\biggr]\frac{dt}{t} \\ &< \frac{\pi}{32} < \infty, \\ \lambda &= \int_1^\infty \bigl|f(t,0,0)\bigr|\frac{dt}{t} < \int_1^\infty \frac{1}{4(1+t^2)}\,dt = \frac{\pi}{64} < \infty.\end{aligned}$$

Then (C_1) and (C_2) hold. At last, by a simple computation, we have

$$\begin{aligned}\Omega &= \Gamma\left(\frac{9}{4}\right) - \sum_{i=1}^{3}\frac{\lambda_i\Gamma(\frac{9}{4})}{\Gamma(\frac{9}{4}+\beta_i)}(\log\eta)^{\frac{9}{4}+\beta_i-1} \approx 0.8562 > 0, \\ w &= \Lambda\left(\frac{1}{\Gamma(\frac{9}{4})} + \sum_{i=1}^{3}\frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\frac{9}{4}+\beta_i)}\right) \approx 0.0452 < 1.\end{aligned}$$

Thus, by the application of Theorem 3.1 the Hadamard fractional integral boundary value problem (4.3) admits an unique solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics, Shanxi Normal University, Linfen, Shanxi 041004, People's Republic of China. [2]Department of Mathematics, Faculty of Art and Sciences, Çankaya University, Balgat, 06530, Turkey. [3]Institute of Space Sciences, Magurele-Bucharest, Romania.

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