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A note on existence of infinitely many periodic solutions for second-order nonlinear difference systems

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Abstract

The paper deals with the existence of infinitely many periodic solutions for second-order nonlinear difference systems. A variational approach is applied using the saddle point theorem.

Keywords: second-order difference systems; infinitely many periodic solutions; variational methods; saddle point theorem

1 Introduction and statement of main results

The interest in the second-order difference systems

$$\begin{cases} \triangle^2 u(t-1) + \nabla F(t,u(t)) = 0, & t \in \mathbb{Z}[1,T], \\ u(0) = u(T), \end{cases} \tag{1}$$

has been aroused, where $\triangle u(t) = u(t+1) - u(t)$, $\triangle^2 u(t) = \triangle(\triangle u(t))$, $\nabla F(t,x) = \frac{\partial F(t,x)}{\partial x}$, T is a positive integer, and $\mathbb{Z}$ and R denote the set of integers and the set of real numbers, respectively, and $\mathbb{Z}[a,b] = \{a, a+1, \dots, b\}$, for $a, b \in \mathbb{Z}$ and $a \leq b$. Assume that $F(t,x)$ is T-periodic in t for all $x \in R^N$ and $F(t,x) \in C^1(\mathbb{Z} \times R^N, R)$.

In 2003, Yu and Guo [1–3] established a variational structure and variational methods to study discrete Hamiltonian systems and obtain the solvability condition of a periodic solution for discrete systems, based on operator theory. Since then more and more authors have contributed to study second-order discrete Hamiltonian systems, with an effective tool named the critical point theory, and one obtained many interesting results [4–11]. In [8], with operator theory, Xue and Tang constructed a variational setting unlike the one in [1] to study the second-order superquadratic discrete Hamiltonian systems (1) and obtained the existence of periodic solutions. This result generalized the one in [4]. In [7], Xue and Tang obtained the existence of one periodic solution of systems (1) under the hypothesis there exist $M_1 > 0$, $M_2 > 0$ and $0 \leq \alpha < 1$ such that

$$\left|\nabla F(t,x)\right| \leq M_1 |x|^\alpha + M_2,$$

for all $(t,x) \in \mathbb{Z}[1,T] \times R^N$, and the condition

$$|x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) \to +\infty \quad \text{as } |x| \to \infty.$$

Subsequently, in [9], Yan, Wu and Tang extended these results in [7] to the case that $F(t,x)$ is T_i-periodic in x_i and obtained the existence of multiple periodic solutions, where x_i is the ith component of $x = (x_1, x_2, \ldots, x_N)$, $i \in [1,N]$. Especially in [12], Che and Xue obtained the existence of infinitely many periodic solutions for systems in the case that:

(F1) there exist $f, g : \mathbb{Z}[0,T] \to R^+$ and $\alpha \in [0,1)$ such that

$$\left|\nabla F(t,x)\right| \le f(t)|x|^{\alpha} + g(t) \quad \text{for all } (t,x) \in \mathbb{Z}[0,T] \times R^N,$$

and a suitable oscillating behaviour at infinity,

(F2) $\liminf_{r\to\infty} \sup_{x\in R^N,|x|=r} |x|^{-2\alpha} \sum_{t=0}^{T} F(t,x) = -\infty$,

(H3) $\limsup_{r\to\infty} \inf_{x\in R^N,|x|=r} \sum_{t=0}^{T} F(t,x) = +\infty$.

Consequently, it is natural to ask if infinitely many solutions still exist when $\alpha = 1$. With the fact that $\alpha = 1$, (F1) and (F2), respectively, change to the linearly bounded gradient condition:

(F1′) there exist $f, g : \mathbb{Z}[0,T] \to R^+$ such that

$$\left|\nabla F(t,x)\right| \le f(t)|x| + g(t) \quad \text{for all } (t,x) \in \mathbb{Z}[0,T] \times R^N,$$

and the condition

(F2′) $\liminf_{r\to\infty} \sup_{x\in R^N,|x|=r} |x|^{-2} \sum_{t=0}^{T} F(t,x) = -\infty$.

However, similarly to what was pointed in [13], (F2′) does not hold if $\sum_{t=0}^{T} f(t)$ is bounded. Therefore, it is necessary to improve condition (F2′). Inspired by [7, 12, 13], in this paper, we will use minmax methods to further study systems (1) under the following assumptions:

(H1) there exist $f, g : \mathbb{Z}[0,T] \to R^+$ with $\sum_{t=0}^{T} f(t) < \frac{\lambda_1}{2}$ such that

$$\left|\nabla F(t,x)\right| \le f(t)|x| + g(t) \quad \text{for all } (t,x) \in \mathbb{Z}[0,T] \times R^N,$$

(H2) $\liminf_{r\to+\infty} \sup_{x\in R^N,|x|=r} |x|^{-2} \sum_{t=0}^{T} F(t,x) < -\frac{4\sum_{t=0}^{T} f^2(t)}{\lambda_1}$,

where $\lambda_k = 2 - 2\cos\frac{2k\pi}{T}$ satisfy the eigenvalue problem

$$-\triangle^2 u(t-1) = \lambda_k u(t), \quad k \in \mathbb{Z}\left[0, \left[\frac{T}{2}\right]\right],$$

and we note

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{[\frac{T}{2}]} \le 4.$$

The main result on the existence of infinitely many periodic solutions of systems (1) is obtained. The details are described.

Theorem 1.1 *Under the hypotheses of* (H1), (H2), *and* (H3):

(a) *discrete systems* (1) *have a sequence* $\{u_n\}$ *of solutions such that* $\{u_n\}$ *is a critical point of the functional* φ *and* $\varphi(u_n) \to +\infty$ *as* $n \to \infty$;

(b) *discrete systems* (1) *have a sequence* $\{u_n^*\}$ *of solutions such that* $\{u_n^*\}$ *is a local minimizer of* φ *and* $\varphi(u_n^*) \to -\infty$ *as* $n \to \infty$,

where the variational functional φ *is*

$$\varphi(u) = \frac{1}{2}\sum_{t=0}^{T}\left|\triangle u(t)\right|^2 - \sum_{t=0}^{T} F\bigl(t, u(t)\bigr),$$

on Hilbert space H_T *defined by*

$$H_T = \bigl\{u : \mathbb{Z} \to R^N | u(t) = u(t+T), t \in \mathbb{Z}\bigr\},$$

with the inner product and the norm, respectively, written as

$$\langle u, v\rangle = \sum_{t=0}^{T}\bigl(u(t), v(t)\bigr), \quad \forall u, v \in H_T$$

and

$$\|u\| = \left(\sum_{t=0}^{T}\left|u(t)\right|^2\right)^{\frac{1}{2}}.$$

Remark 1.2 *As pointed out in* [12], *the nonlinearity potential F does not need the symmetry condition in the paper. Moreover, Theorem* 1.1 *is a complement to and development of Theorem* 1.1 *in* [12] *corresponding to* $\alpha = 1$.

2 Proof of main result

For all $u \in H_T$, $\|u\|_\infty = \sup_{t\in\mathbb{Z}[0,T]} |u(t)|$ is defined. It is obvious that

$$\|u\|_\infty \le \|u\| = \left(\sum_{t=0}^{T}\left|u(t)\right|^2\right)^{\frac{1}{2}}. \tag{2}$$

By the definition of H_T, H_T is finite dimensional. By (H1), one gets $\varphi \in C^1(H_T, R)$ and

$$\bigl\langle\varphi'(u), v\bigr\rangle = \sum_{t=0}^{T}\bigl(\triangle u(t), \triangle v(t)\bigr) - \sum_{t=0}^{T}\bigl(\nabla F\bigl(t, u(t)\bigr), v(t)\bigr),$$

for all $u, v \in H_T$.

Subsequently, an important lemma in [8] is shown for the reader's convenience. The lemma is constructed in a variational setting, with the operator theory, unlike the one in [1]. Details can be found in [8].

Lemma 1 ([8]) *Let* N_k *be a subspace of* H_T *written as*

$$N_k := \bigl\{u \in H_T | -\Delta^2 u(t-1) = \lambda_k u(t)\bigr\},$$

where $\lambda_k = 2 - 2\cos\frac{2k\pi}{T}$, $k \in \mathbb{Z}[0, [\frac{T}{2}]]$. *Then one has the results as follows*:

(i) $N_k \perp N_j$, $k \neq j$, $k,j \in \mathbb{Z}[0,[\frac{T}{2}]]$.
(ii) $H_T = \bigoplus_{k=0}^{[T/2]} N_k$.

Let $V = N_0$ *and* $W = N_0^{\perp} = \bigoplus_{k=1}^{[T/2]} N_k$. *Via Lemma* 2 *in* [8], *we have*

$$H_T = N_0 \oplus N_0^{\perp} = V \oplus W$$

and

$$\sum_{t=0}^{T} \left|\triangle u(t)\right|^2 \geq \lambda_1 \|u\|^2, \quad \forall u \in W = N_0^{\perp}. \tag{3}$$

From the definition of N_0, *one gets* $u(t) = u(0) = C \in R^N$, *for all* $u \in N_0$ *and* $t \in \mathbb{Z}[0,T]$. *By Lemma* 1, *one rewrites* u *as* $u = \bar{u} + \tilde{u} \in V \oplus W = N_0 \oplus N_0^{\perp}$, *where*

$$\bar{u} = \frac{1}{T} \sum_{t=0}^{T} u(t).$$

From the fact that

$$\begin{aligned}
\|u\| &= \left(\sum_{t=0}^{T} \left|u(t)\right|^2\right)^{\frac{1}{2}} = \left(\sum_{t=0}^{T} \left|\bar{u} + \tilde{u}(t)\right|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{t=0}^{T} \langle \bar{u} + \tilde{u}, \bar{u} + \tilde{u} \rangle\right)^{\frac{1}{2}} \\
&= \left(\sum_{t=0}^{T} \left(|\bar{u}|^2 + \left|\tilde{u}(t)\right|^2\right)\right)^{\frac{1}{2}} = \left(T|\bar{u}|^2 + \|\tilde{u}\|^2\right)^{\frac{1}{2}},
\end{aligned}$$

one has

$$\|u\| \leq \sqrt{T+1}\left(|\bar{u}|^2 + \|\tilde{u}\|^2\right)^{\frac{1}{2}} \quad \textit{and} \quad \|u\| \geq \left(|\bar{u}|^2 + \|\tilde{u}\|^2\right)^{\frac{1}{2}}.$$

Thus one obtains that $\|u\| \to \infty$ *if and only if* $(|\bar{u}|^2 + \|\tilde{u}\|^2)^{\frac{1}{2}} \to \infty$.

Proof of Theorem 1.1 The proof of Theorem 1.1 relies on a minimax theorem (Corollary 4.3) in [14]. We complete the proof with a series of statements below.

Step 1, we claim that φ is coercive in the subspace W.

Due to (H1), there exists a constant C_1 satisfying the following inequality:

$$\begin{aligned}
\left|F(t,x)\right| &\leq \left|\int_0^1 \left(\nabla F(t,sx), x\right) ds\right| + \left|F(t,0)\right| \\
&\leq \int_0^1 \left|\nabla F(t,sx)\right||x| \, ds + C_1 \leq \frac{f(t)}{2}|x|^2 + g(t)|x| + C_1
\end{aligned} \tag{4}$$

for all $t \in \mathbb{Z}[0,T]$ and $x \in R^N$. Hence, using the Hölder inequality, (2), (3), and (4), for all $u \in W$, one derives

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\sum_{t=0}^{T}\left|\triangle u(t)\right|^2 - \sum_{t=0}^{T}F(t,u) \\
&\geq \frac{1}{2}\lambda_1\|u\|^2 - \sum_{t=0}^{T}\left(\frac{1}{2}f(t)\left|u(t)\right|^2 + g(t)\left|u(t)\right| + C_1\right) \\
&\geq \frac{1}{2}\lambda_1\|u\|^2 - \|u\|_\infty^2\sum_{t=0}^{T}\frac{1}{2}f(t) - \|u\|_\infty\sum_{t=0}^{T}g(t) - C_1T \\
&\geq \frac{1}{2}\lambda_1\|u\|^2 - \|u\|^2\frac{1}{2}\sum_{t=0}^{T}f(t) - \|u\|\sum_{t=0}^{T}g(t) - C_1T \\
&= \left(\frac{1}{2}\lambda_1 - \frac{1}{2}\sum_{t=0}^{T}f(t)\right)\|u\|^2 - \|u\|\sum_{t=0}^{T}g(t) - C_1T.
\end{aligned}$$

Combining this with the fact $\sum_{t=0}^{T}f(t) < \frac{1}{2}\lambda_1$, one deduces $\lim_{\|u\|\to+\infty}\varphi(u) = +\infty$.

Step 2, we claim that there are positive sequences $\{a_n\}, \{b_m\}$ satisfying

(c) $\lim_{n\to\infty}a_n = +\infty$ and $\lim_{n\to\infty}\sup_{u\in V,\|u\|=a_n}\varphi(u) = -\infty$,

(d) $\lim_{m\to\infty}b_m = +\infty$ and $\lim_{m\to\infty}\inf_{u\in H_{b_m}}\varphi(u) = +\infty$,

where $H_{b_m} = \{u \in V, \|u\| = b_m\} \oplus W$.

The detailed proof of (c) can be founded in [12]. On the other hand, by (H2), one can take a constant

$$a > \frac{8}{\lambda_1}.$$

Thus one gets

$$\liminf_{r\to\infty}\sup_{x\in R^N,|x|=r}|x|^{-2}\sum_{t=0}^{T}F(t,x) < -\frac{a}{2}\sum_{t=0}^{T}f^2(t). \tag{5}$$

For any $u \in H_{b_m}$, rewritten $u = \tilde{u} + \bar{u}$, where $\tilde{u} \in W$ and $\bar{u} \in V$, by the Hölder inequality, (H1), and (H2), one has

$$\begin{aligned}
&\left|\sum_{t=0}^{T}\left(F\left(t,u(t)\right) - F(t,\bar{u})\right)\right| \\
&\quad\leq \sum_{t=0}^{T}\int_0^1\left|\left(\nabla F\left(t,\bar{u}+s\tilde{u}(t)\right),\tilde{u}(t)\right)\right|ds \\
&\quad\leq \sum_{t=0}^{T}\int_0^1\left(f(t)\left|\bar{u}+s\tilde{u}(t)\right| + g(t)\right)\cdot\left|\tilde{u}(t)\right|ds \\
&\quad\leq \sum_{t=0}^{T}f(t)\left(|\bar{u}| + \frac{1}{2}\left|\tilde{u}(t)\right|\right)\left|\tilde{u}(t)\right| + \sum_{t=0}^{T}g(t)\left|\tilde{u}(t)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq |\bar{u}|\left(\sum_{t=0}^{T} f^2(t)\right)^{\frac{1}{2}}\left(\sum_{t=0}^{T}\left|\tilde{u}(t)\right|^2\right)^{\frac{1}{2}} + \frac{1}{2}\|\tilde{u}\|_\infty^2 \sum_{t=0}^{T} f(t) + \|\tilde{u}\|_\infty \sum_{t=0}^{T} g(t) \\
&\leq \frac{1}{2a}\|\tilde{u}\|^2 + \frac{a}{2}\sum_{t=0}^{T} f^2(t)|\bar{u}|^2 + \frac{1}{2}\sum_{t=0}^{T} f(t)\|\tilde{u}\|^2 + \|\tilde{u}\|\sum_{t=0}^{T} g(t) \\
&\leq \left(\frac{1}{2a} + \frac{1}{2}\sum_{t=0}^{T} f(t)\right)\|\tilde{u}\|^2 + \frac{a}{2}\sum_{t=0}^{T} f^2(t)|\bar{u}|^2 + \|\tilde{u}\|\sum_{t=0}^{T} g(t). \qquad (6)
\end{aligned}$$

Hence, for all $u \in H_{b_m}$, it follows from inequalities (3) and (6) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\sum_{t=0}^{T}\left|\triangle u(t)\right|^2 - \sum_{t=0}^{T} F\big(t, u(t)\big) \\
&= \frac{1}{2}\sum_{t=0}^{T}\left|\triangle \tilde{u}(t)\right|^2 - \sum_{t=0}^{T}\big(F\big(t, u(t)\big) - F(t, \bar{u})\big) - \sum_{t=0}^{T} F(t, \bar{u}) \\
&\geq \frac{1}{2}\lambda_1\|\tilde{u}\|^2 - \left(\frac{1}{2a} + \frac{1}{2}\sum_{t=0}^{T} f(t)\right)\|\tilde{u}\|^2 - \frac{a}{2}\sum_{t=0}^{T} f^2(t)|\bar{u}|^2 \\
&\quad - \|\tilde{u}\|\sum_{t=0}^{T} g(t) - \sum_{t=0}^{T} F(t, \bar{u}) \\
&= \left(\frac{\lambda_1}{2} - \frac{1}{2a} - \frac{1}{2}\sum_{t=0}^{T} f(t)\right)\|\tilde{u}\|^2 - \|\tilde{u}\|\sum_{t=0}^{T} g(t) \\
&\quad - |\bar{u}|^2\left(\frac{\sum_{t=0}^{T} F(t, \bar{u})}{|\bar{u}|^2} + \frac{a}{2}\sum_{t=0}^{T} f^2(t)\right). \qquad (7)
\end{aligned}$$

By $\sum_{t=0}^{T} f(t) < \frac{\lambda_1}{2}$ and $a > \frac{8}{\lambda_1}$,

$$\frac{\lambda_1}{2} - \frac{1}{2a} - \frac{1}{2}\sum_{t=0}^{T} f(t) > 0$$

is verified. By (5), (7), and the fact $\|u\| \to \infty$ if and only if $(|\bar{u}|^2 + \|\tilde{u}\|^2)^{\frac{1}{2}} \to \infty$, the conclusion (d) is achieved.

Now we have a family of maps Γ_n expressed as

$$\Gamma_n = \left\{\gamma \in C(B_{a_n}, H_T) | \gamma|_{\partial B_{a_n}} = \mathrm{Id}|_{\partial B_{a_n}}\right\}$$

and minimax values c_n formulated as

$$c_n = \inf_{\gamma \in \Gamma_n} \max_{u \in B_{a_n}} \varphi\big(\gamma(u)\big)$$

for each n, where B_{a_n} is a ball in V and a_n is the radius of B_{a_n}. One gets

$$\gamma(B_{a_n}) \cap W \neq \emptyset$$

for any $\gamma \in \Gamma_n$ from Theorem 4.6 in [14].

Step 3, we claim that, for sufficiently large n, there exist sequences $\{\gamma_k\} \subset \Gamma_n$ and $\{\nu_k\}$ in H_T, respectively, satisfying

$$\max_{u\in B_{a_n}} \varphi\bigl(\gamma_k(u)\bigr) \to c_n,$$
$$\varphi(\nu_k) \to c_n, \qquad \varphi'(\nu_k) \to 0, \qquad \operatorname{dist}\bigl(\nu_k, \gamma_k(B_{a_n})\bigr) \to 0 \quad \text{as } k \to \infty. \tag{8}$$

By Step 1, we know $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$, $u \in W$. Therefore there exists a constant C_2 satisfying

$$\max_{u\in B_{a_n}} \varphi\bigl(\gamma(u)\bigr) \geq \inf_{u\in W} \varphi(u) \geq C_2.$$

Furthermore, one has

$$c_n \geq \inf_{u\in W} \varphi(u) \geq C_2,$$

for sufficiently large n. By the fact $\gamma(B_{a_n}) \cap W \neq \emptyset$ and the conclusion of Step 2, one obtains

$$c_n > \max_{u\in \partial B_{a_n}} \varphi(u)$$

for sufficiently large n. Therefore, for a fixed n, this claim is proved from Theorem 4.3 and Corollary 4.3 in [14].

Step 4, we draw the conclusion that the sequence $\{\nu_k\}$ is bounded in H_T.

For sufficiently large k, by (8), one has

$$c_n \leq \max_{u\in B_{a_n}} \varphi\bigl(\gamma_k(u)\bigr) \leq c_n + 1.$$

We choose $w_k \in \gamma_k(B_{a_n})$ satisfying

$$\|\nu_k - w_k\| \leq 1. \tag{9}$$

From the conclusion (d) of Step 2, for a fixed n, a sufficiently large m exists, rendering the formula

$$b_m > a_n \quad \text{and} \quad \inf_{u\in H_{b_m}} \varphi(u) > c_n + 1.$$

These inequalities imply that $\gamma_k(B_{a_n}) \cap H_{b_m} = \emptyset$ for each k. We now write $w_k = \bar{w}_k + \tilde{w}_k$, where $\bar{w}_k \in V$ and $\tilde{w}_k \in W$. Then one has

$$\|\bar{w}_k\| < b_m \tag{10}$$

for each k. Moreover, by (2), (3), (4), and (10), one gets

$$\begin{aligned} 1 + c_n \geq \varphi(w_k) &= \frac{1}{2}\sum_{t=0}^{T}\bigl|\triangle w_k(t)\bigr|^2 - \sum_{t=0}^{T} F\bigl(t, w_k(t)\bigr) \\ &\geq \frac{1}{2}\lambda_1\|\tilde{w}_k\|^2 - \sum_{t=0}^{T}\left(\frac{1}{2}f(t)\bigl|w_k(t)\bigr|^2 + g(t)\bigl|w_k(t)\bigr| + C_1\right) \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2}\lambda_1\|\tilde{w}_k\|^2 - \sum_{t=0}^{T} f(t)\bigl[|\bar{w}_k|^2 + \bigl|\tilde{w}_k(t)\bigr|^2\bigr] - \sum_{t=0}^{T} g(t)\bigl(|\bar{w}_k| + \bigl|\tilde{w}_k(t)\bigr|\bigr) - C_1 T \\
&\geq \frac{1}{2}\lambda_1\|\tilde{w}_k\|^2 - \|\tilde{w}_k\|_\infty^2 \sum_{t=0}^{T} f(t) - \|\bar{w}_k\|^2 \sum_{t=0}^{T} f(t) \\
&\quad - \|\tilde{w}_k\|_\infty \sum_{t=0}^{T} g(t) - \|\bar{w}_k\| \sum_{t=0}^{T} g(t) - C_1 T \\
&\geq \frac{1}{2}\lambda_1\|\tilde{w}_k\|^2 - \|\tilde{w}_k\|^2 \sum_{t=0}^{T} f(t) - b_m^2 \sum_{t=0}^{T} f(t) \\
&\quad - \|\tilde{w}_k\| \sum_{t=0}^{T} g(t) - b_m \sum_{t=0}^{T} g(t) - C_1 T \\
&= \left(\frac{\lambda_1}{2} - \sum_{t=0}^{T} f(t)\right)\|\tilde{w}_k\|^2 - \|\tilde{w}_k\| \sum_{t=0}^{T} g(t) - b_m^2 \sum_{t=0}^{T} f(t) - b_m \sum_{t=0}^{T} g(t) - C_1 T. \qquad (11)
\end{aligned}
$$

We can combine equation (11) and the fact that $\sum_{t=0}^{T} f(t) < \frac{\lambda_1}{2}$, $\|\tilde{w}_k\|$ is bounded. Thus, by combining (10) and the fact that $\|w_k\| = (T|\bar{w}_k|^2 + \|\tilde{w}_k\|^2)^{\frac{1}{2}}$, $\{w_k\}$ is bounded. Then $\{v_k\}$ is bounded in H_T via (9). The conclusion is proved.

Step 5, we claim that c_n is a critical value of φ.

Since $\{v_k\}$ is bounded and H_T is finite dimensional space, $\{v_k\}$ contains a convergent subsequence that is still denoted as $\{v_k\}$ for convenience, meeting

$$\lim_{k\to\infty} v_k = u_n.$$

Then, by (8), one has

$$\varphi(u_n) = c_n \quad \text{and} \quad \varphi'(u_n) = 0.$$

Thus φ has a critical point u_n.

We prove part (a) of Theorem 1.1. One chooses sufficiently large n satisfying $a_n > b_m$, then one has $\gamma(B_{a_n}) \cap H_{b_m} \neq \emptyset$ for any $\gamma \in \Gamma_n$. It follows that

$$\max_{u\in B_{a_n}} \varphi\bigl(\gamma(u)\bigr) \geq \inf_{u\in H_{b_m}} \varphi(u).$$

With this and the conclusion (d) of Step 2,

$$\lim_{n\to\infty} c_n = +\infty$$

is implied. Part (a) of Theorem 1.1 is proved.

A follow-up is to prove part (b) in Theorem 1.1. For a given m, let P_m be a subset of H_T, where

$$P_m = \bigl\{u = \bar{u} + \tilde{u} \in H_T | \bar{u} \in V, \|\bar{u}\| \leq b_m, \tilde{u} \in W\bigr\}.$$

For all $u \in P_m$, similar to (11), one obtains

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\sum_{t=0}^{T}\left|\triangle u(t)\right|^2 - \sum_{t=0}^{T}F\bigl(t,u(t)\bigr) \\ &\geq \frac{1}{2}\lambda_1\|\tilde{u}\|^2 - \sum_{t=0}^{T}\left(\frac{1}{2}f(t)\left|u(t)\right|^2 + g(t)\left|u(t)\right| + C_1\right) \\ &\geq \left(\frac{\lambda_1}{2} - \sum_{t=0}^{T}f(t)\right)\|\tilde{u}\|^2 - \|\tilde{u}\|\sum_{t=0}^{T}g(t) - b_m^2\sum_{t=0}^{T}f(t) - b_m\sum_{t=0}^{T}g(t) - C_1T. \end{aligned} \tag{12}$$

Due to (12), φ is bounded below on P_m. Take

$$\mu_m = \inf_{u\in P_m}\varphi(u)$$

and a sequence $\{u_k\} \subset P_m$, satisfying

$$\varphi(u_k) \to \mu_m \quad \text{as } k \to \infty.$$

Similar to the proof of the boundedness of $\{w_k\}$ in Step 4, $\{u_k\}$ is bounded in H_T via (12). Then $\{u_k\}$ contains a convergent subsequence that is still denoted $\{u_k\}$ for convenience, satisfying

$$u_k \rightharpoonup u_m^* \quad \text{weakly in } H_T, \quad \text{as } k \to \infty.$$

Noting that P_m is convex and closed in H_T, one has $u_m^* \in P_m$. Moreover, in view of the weakly lower semi-continuity of φ, one has

$$\mu_m = \lim_{k\to\infty}\varphi(u_k) \geq \varphi\bigl(u_m^*\bigr)$$

and

$$\mu_m = \varphi\bigl(u_m^*\bigr).$$

Next, we draw the conclusion that u_m^* is an interior point of P_m. Thus u_m^* is a critical point of φ.

Taking

$$u_m^* = \bar{u}_m^* + \tilde{u}_m^*,$$

where $\bar{u}_m^* \in V$, $\tilde{u}_m^* \in W$. If $a_n < b_m$, one has $\partial B_{a_n} \subset P_m$, which implies that

$$\varphi\bigl(u_m^*\bigr) = \inf_{u\in P_m}\varphi(u) \leq \sup_{u\in\partial B_{a_n}}\varphi(u).$$

From the inequality above and the result (d) of Step 2, one gets

$$\varphi\bigl(u_m^*\bigr) \to -\infty \quad \text{as } m \to \infty.$$

By the conclusion of Step 3, one has $\bar{u}_m^* \neq b_m$ for large m. From this one deduces that u_m^* is an interior point of P_m and u_m^* is a critical point of φ. Then, the proof of Theorem 1.1 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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