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The controllability for the internally controlled 1-D wave equation via a finite difference method

Juntao Li^{1,2}, Yuhan Zheng¹ and Guojie Zheng^{1*}

*Correspondence: guojiezheng@yeah.net
¹College of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we study the controllability of the semi-discrete internally controlled 1-D wave equation by using the finite difference method. In the discrete setting of the finite difference method, we derive the observability inequality and get the exact controllability for the semi-discrete internally controlled wave equation in the one-dimensional case. Then we also analyze whether the uniform observability inequality holds for the adjoint system as $h \rightarrow 0$.

MSC: 49K20; 35J65

Keywords: wave equation; finite difference; observability inequality

1 Introduction

In this paper, we discuss the topic related to the controllability for the space semi-discretizations of the internally controlled one-dimensional wave equation. Let us first introduce certain notations and state the controlled system as studied in this paper. Let $\omega = (a, b)$ be an open and nonempty subset of $(0, 1)$, χ_ω represent the characteristic function of ω . The controlled wave equation is represented as follows:

$$\begin{cases} \partial_{tt}y(x, t) - \partial_{xx}y(x, t) = \chi_\omega u(x, t), & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = y(1, t) = 0, & t \in (0, T), \\ y(x, 0) = y_0(x), \quad \partial_t y(x, 0) = y_1(x), & x \in (0, 1), \end{cases} \quad (1.1)$$

where $T > 0$, the initial value (y_0, y_1) belongs to $H_0^1(0, 1) \times L^2(0, 1)$ and $u(\cdot)$ is a control function taken from the space $L^2(0, T; L^2(0, 1))$.

Problem (1.1) is said to be exactly controllable from the initial value $(y_0, y_1) \in H_0^1(0, 1) \times L^2(0, 1)$ in time T if there exists a control function $u(\cdot) \in L^2(0, T; L^2(0, 1))$, such that the solution of (1.1) satisfies $(y(T), \partial_t y(T)) = (0, 0)$. The problem of the controllability of wave equations has also been the object of numerous studies. Extensive related references can be found in [1–3] and the rich work cited therein.

In this work, we shall mainly focus on the issue of how the controllability property can be achieved under the numerical approximation schemes. Now, we will introduce the numerical project by using the finite difference method. Given $N \in \mathbb{N}$, we define $h = \frac{1}{N+1}$. We

consider the nodal points

$$x_0 = 0; \quad x_j = jh, \quad j = 1, \dots, N; \quad x_{N+1} = 1,$$

which divides $[0, 1]$ into $N + 1$ subintervals $I_j = [x_j, x_{j+1}]$, $j = 0, 1, \dots, N$. We suppose that the nodal points $x_{k+1}, \dots, x_{k+p} \in \omega$, and $x_1, \dots, x_k, x_{k+p+1}, \dots, x_N \in (0, 1) \setminus \omega$, for some $k, p \in \mathbb{N}$ with $k + p \leq N$.

Now, we consider the following finite difference semi-discretization of (1.1):

$$\begin{cases} y_j''(t) - \frac{1}{h^2} [y_{j+1}(t) + y_{j-1}(t) - 2y_j(t)] = (\chi_\omega u)_j(t), & t \in (0, T), \\ y_0(t) = y_{N+1}(t) = 0, & t \in (0, T), \\ y_j(0) = y_j^0, \quad y_j'(0) = y_j^1, & j = 1, \dots, N, \end{cases} \tag{1.2}$$

where

$$(\chi_\omega u)_j(t) = \begin{cases} u(x_j, t), & \text{if } x_j \in \omega, \\ 0, & \text{if } x_j \in (0, 1) \setminus \omega. \end{cases}$$

The conditions $y_0(t) = y_{N+1}(t) = 0$, $t \in (0, T)$, are the Dirichlet boundary conditions in semi-discrete case. Next, we shall rewrite equation (1.2) by vectors. Let

$$\begin{aligned} \vec{y}_h(t) &= (y_1(t), \dots, y_N(t))^T, \\ \vec{y}_h^0 &= (y_0(x_1), \dots, y_0(x_N))^T, \\ \vec{y}_h^1 &= (y_1(x_1), \dots, y_1(x_N))^T, \end{aligned}$$

and

$$\vec{u}_h(t) = (u_1(t), \dots, u_N(t))^T.$$

Define the two $N \times N$ matrices

$$A_h = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

and

$$B_h = \begin{bmatrix} O_{k \times k} & \cdots & O \\ \vdots & I_{p \times p} & \vdots \\ O & \cdots & O_{(N-k-p) \times (N-k-p)} \end{bmatrix},$$

where $I_{p \times p}$ is a $p \times p$ identity matrix, and $p, k \in \mathbb{N}$, which are mentioned above (1.2). The system (1.2) can be rewritten as follows:

$$\begin{cases} \vec{y}_h''(t) + A_h \vec{y}_h(t) = B_h \vec{u}_h(t), & t \in (0, T), \\ y_0(t) = y_{N+1}(t) = 0, & t \in (0, T), \\ \vec{y}_h(0) = \vec{y}_h^0, \quad \vec{y}_h'(0) = \vec{y}_h^1, \end{cases} \tag{1.3}$$

where ' denotes derivation with respect to time. In fact, system (1.3) is in the form of linear ordinary differential equations for an unknown vector function $\vec{y}(t)_h = (y_1(t), \dots, y_N(t))^T$, with the boundary conditions $y_0(t) = y_{N+1}(t) = 0$, and $\vec{u}_h(t)$ plays the role of control function. The adjoint system for system (1.3) can be represented as

$$\begin{cases} \vec{\phi}_h''(t) + A_h \vec{\phi}_h(t) = 0, & t \in (0, T), \\ \phi_0(t) = \phi_{N+1}(t) = 0, & t \in (0, T), \\ \vec{\phi}_h(0) = \vec{\phi}_0^h, \quad \vec{\phi}_h'(0) = \vec{\phi}_1^h, \end{cases} \tag{1.4}$$

where the initial date $\vec{\phi}_0^h = (\phi_1^0, \dots, \phi_N^0)^T$, and $\vec{\phi}_1^h = (\phi_1^1, \dots, \phi_N^1)^T$. It is easy to check that $\phi_0^0 = \phi_{N+1}^0 = 0$ and $\phi_0^1 = \phi_{N+1}^1 = 0$ are the compatibility conditions. Throughout the paper, we suppose that the compatibility conditions hold for any initial value.

Taking the boundary conditions $\phi_0(t) = \phi_{N+1}(t) = 0, \forall t \in [0, T]$, for the solution of (1.4) we define the energy of the semi-discrete system (1.4) as

$$E_h(t) = \frac{h}{2} \sum_{i=0}^N \left(|\phi_i'(t)|^2 + \left| \frac{\phi_{i+1}(t) - \phi_i(t)}{h} \right|^2 \right), \quad \forall t \in [0, T].$$

Since B_h is a symmetric matrix, the observability inequality of (1.4) can be formulated as: To find a constant $C(T, h)$ such that (see [4–6])

$$E_h(0) \leq C(T, h) \int_0^T \|B_h \vec{\phi}_h(t)\|_{\mathbb{R}^N}^2 dt, \tag{1.5}$$

where

$$E_h(0) = \frac{h}{2} \sum_{i=0}^N \left(|\phi_i^1|^2 + \left| \frac{\phi_{i+1}^0 - \phi_i^0}{h} \right|^2 \right).$$

Remark 1.1 The energy $E_h(t)$ is conserved for the solution of (1.4). Namely, for any $h > 0$ and solution $\vec{\phi}_h(t)$ of (1.4), we have (see [5])

$$E_h(t) = E_h(0), \quad \text{for any } t \in [0, T].$$

In this paper, we will study whether the inequality (1.5) for adjoint system (1.4) holds. We are also interested in whether the constant $C(T, h)$ is bounded as $h \rightarrow 0$. The main results of the paper are presented as follows.

Theorem 1.1 *For any $T > 0$, the observability estimate (1.5) for the adjoint system (1.4) holds.*

Remark 1.2 Theorem 1.1 shows that the semi-discrete system (1.2) or (1.3) is controllable for any time $T > 0$.

Theorem 1.2 For any $T > 0$, we have

$$\sup_{\text{solution of (1.4)}} \frac{E_h(0)}{\int_0^T \|B_h \vec{\phi}_h(t)\|_{\mathbb{R}^N}^2 dt} \rightarrow +\infty, \quad \text{as } h \rightarrow 0. \tag{1.6}$$

To the best of our knowledge, Infante and Zuazua made the first study of this topic in [5]. They studied a controllability result for the semi-discrete 1-D wave equation with boundary control. However, the uniform controllability for the semi-discrete systems in [5] cannot be derived as the discretization parameter $h \rightarrow 0$. The main differences between [5] and our paper are as follows. In [5], the authors focused on a one-dimensional boundary controlled wave equation, and we mainly study the internally controlled 1-D wave equation. In this case, the controller is more complicated than the case with controller on boundary. Regarding other works on this subject, we mention [4, 7, 8] and [6].

The paper is organized as follows: Section 2 briefly describes some preliminary results on the finite difference scheme. The proofs of Theorem 1.1 and Theorem 1.2 are provided in Section 3.

2 The finite difference scheme

In this section, we will discuss the numerical project by the finite difference method. We consider the numerical problem for (1.1) in the state space \mathbb{R}^N with the usual Euclidean norm and inner product denoted by $\|\cdot\|_{\mathbb{R}^N}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^N}$, respectively. To this end, we first introduce some properties for the eigenvalues and eigenvectors of the matrix A_h . The spectrum for A_h can be explicitly computed in this case (see [9]). The eigenvalues $\lambda_i(h)$ ($i = 1, \dots, N$) satisfy

$$\lambda_i(h) = \frac{4}{h^2} \sin^2\left(\frac{i\pi h}{2}\right), \tag{2.1}$$

and the corresponding unit eigenvectors in \mathbb{R}^N are

$$\vec{w}_i^h = (w_{i,1}^h, \dots, w_{i,N}^h)^T, \quad \text{and} \quad w_{ij}^h = \sqrt{\frac{2}{N+1}} \sin(i\pi jh), \quad j = 1, \dots, N. \tag{2.2}$$

Clearly, the family of eigenvectors

$$\{\vec{w}_1^h, \vec{w}_2^h, \dots, \vec{w}_N^h\} \text{ forms an orthonormal basis of } \mathbb{R}^N. \tag{2.3}$$

Indeed, we can easily get the following properties for these eigenvectors.

Lemma 2.1

- (i) For any eigenvector $w = (w_1, w_2, \dots, w_N)^T$ with eigenvalue $\lambda(h)$ of matrix A_h , the following identity holds:

$$\sum_{j=0}^N \left| \frac{w_j - w_{j+1}}{h} \right|^2 = \lambda(h) \sum_{j=1}^N |w_j|^2, \tag{2.4}$$

where $w_0 = w_{N+1} = 0$.

(ii) If $w_k = (w_{k,1}, w_{k,2}, \dots, w_{k,N})^T$ and $w_l = (w_{l,1}, w_{l,2}, \dots, w_{l,N})^T$ are eigenvectors associated to eigenvalue λ_k, λ_l , and $\lambda_k \neq \lambda_l$, then we have

$$\sum_{j=0}^N (w_{k,j} - w_{k,j+1})(w_{l,j} - w_{l,j+1}) = 0, \tag{2.5}$$

where $w_{k,0} = w_{k,N+1} = 0$, and $w_{l,0} = w_{l,N+1} = 0$.

This lemma is quoted from [5].

Lemma 2.2 *Assume N is large enough so that ω contains at least two consecutive nodal points. Then*

$$B_h \vec{w}_i^h \neq 0, \quad \text{for all } i = 1, 2, \dots, N.$$

Proof Since there are more than two consecutive nodal points in $\omega = (a, b)$, we let $l(h)$ denote the first natural number such that $l(h)h \in (a, b)$, and $m(h)$ denote the last natural number such that $m(h)h \in (a, b)$. Then we have

$$B_h \vec{w}_i^h = (0, \dots, 0, w_{i,l(h)}^h, \dots, w_{i,m(h)}^h, 0, \dots, 0)^T. \tag{2.6}$$

Here,

$$w_{i,l(h)}^h = \sqrt{\frac{2}{N+1}} \sin(i\pi l(h)h) = \sqrt{\frac{2}{N+1}} \sin\left(\frac{il(h)\pi}{N+1}\right),$$

and

$$w_{i,l(h)+1}^h = \sqrt{\frac{2}{N+1}} \sin(i\pi (l(h) + 1)h) = \sqrt{\frac{2}{N+1}} \sin\left(\frac{i(l(h) + 1)\pi}{N+1}\right)$$

could not be zero at the same time. Thus, we complete the proof of the lemma. □

Epecially, we can get the following property for the eigenvectors for the matrix A_h , which will play a key role in the proof of the main results.

Proposition 2.1 *Assume N is large enough so that ω contains at least two consecutive nodal points. Then there exists a positive constant L which is independent on h such that*

$$\|B_h \vec{w}_i^h\|_{\mathbb{R}^N} > L$$

holds for the unit eigenvector \vec{w}_i^h ($i = 1, \dots, N$) of the matrix A_h .

Proof First of all, we claim that there exists a positive number M which is independent on r such that

$$\int_a^b \sin^2(rt) dt > M, \quad \text{for } r = 1, 2, \dots$$

To this end, we calculate the following integrations:

$$\begin{aligned} \int_a^b \sin^2(r\pi t) dt &= \frac{b-a}{2} - \frac{\sin(2r\pi b) - \sin(2r\pi a)}{4r\pi} \\ &\rightarrow \frac{b-a}{2}, \quad \text{as } r \rightarrow \infty. \end{aligned} \tag{2.7}$$

On the one hand, there exists a positive number N_1 depending only on $b - a$, such that

$$\int_a^b \sin^2(r\pi t) dt > \frac{b-a}{4}, \quad \text{as } r > N_1.$$

On the other hand, there exists a positive number M_1 depending only on N_1 , such that

$$\int_a^b \sin^2(r\pi t) dt > M_1, \quad \text{as } 0 < r \leq N_1.$$

Taking $M = \min\{\frac{b-a}{4}, M_1\}$, we see that the claim is correct.

After directly calculating, we obtain

$$\begin{aligned} \|B_h \vec{w}_i^h\|_{\mathbb{R}^N}^2 &= \sum_{jh \in (a,b)} \frac{2}{N+1} [\sin(i\pi jh)]^2 \\ &= \sum_{jh \in (a,b)} \frac{2}{N+1} \frac{1 - \cos(2i\pi jh)}{2} \\ &= \sum_{jh \in (a,b)} \frac{1}{N+1} - \frac{1}{N+1} \sum_{jh \in (a,b)} \cos(2i\pi jh). \end{aligned} \tag{2.8}$$

Let $l(h)$ denote the first natural number such that $l(h)h \in (a, b)$, and $m(h)$ denote the last natural number such that $m(h)h \in (a, b)$. Obviously, we have

$$l(h) = \left\lceil \frac{a}{h} \right\rceil + 1$$

and

$$m(h) = \begin{cases} \lfloor \frac{b}{h} \rfloor, & \text{if } \frac{b}{h} \notin \mathbb{N}, \\ \frac{b}{h} - 1, & \text{if } \frac{b}{h} \in \mathbb{N}. \end{cases}$$

Note that

$$\frac{1}{N+1} \sum_{jh \in (a,b)} \cos(2i\pi jh) = \frac{1}{2(N+1)} \frac{\sin([2m(h)+1]i\pi h) - \sin([2l(h)-1]i\pi h)}{\sin(i\pi h)}. \tag{2.9}$$

Hence, there exists a natural number I_1 depending only on $b - a$, such that $\frac{1}{i} < \frac{b-a}{4}$, as $i > I_1$. From the theory of classic analysis, we have

$$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1, \quad \text{as } x \in \left(0, \frac{\pi}{2}\right],$$

i.e.,

$$\sin x \geq \frac{2}{\pi}x, \quad \text{as } x \in \left(0, \frac{\pi}{2}\right].$$

Combining the above inequality with (2.9), we see that

$$\left| \frac{1}{N+1} \sum_{jh \in (a,b)} \cos(2i\pi jh) \right| \leq \frac{1}{2(N+1)} \frac{2}{i\pi h} = \frac{1}{i\pi} < \frac{b-a}{4}, \tag{2.10}$$

If $0 < i \leq I_1$, then

$$\sum_{jh \in (a,b)} \frac{2}{N+1} [\sin(i\pi jh)]^2 \rightarrow 2 \int_a^b \sin^2(i\pi t) dt,$$

for $i = 1, 2, \dots, I_1$, as $N \rightarrow \infty$, and

$$\sum_{jh \in (a,b)} \frac{1}{N+1} = \frac{m(h) - l(h)}{N+1} \rightarrow b - a, \quad \text{as } N \rightarrow \infty.$$

Thus, there exists a positive number $N_2 > I_1$ depending only on a, b and $b - a$, such that

$$\sum_{jh \in (a,b)} \frac{1}{N+1} > \frac{b-a}{2} \tag{2.11}$$

and

$$\sum_{jh \in (a,b)} \frac{2}{N+1} [\sin(i\pi jh)]^2 > M, \tag{2.12}$$

for $i = 1, 2, \dots, I_1$, when $N > N_2$.

Case I: $N > N_2, N \geq i > I_1$. From (2.8), (2.10), and (2.11), we can easily see that

$$\|B_h \vec{w}_i^h\|_{\mathbb{R}^N}^2 = \sum_{jh \in (a,b)} \frac{1}{N+1} - \frac{1}{N+1} \sum_{jh \in (a,b)} \cos(2i\pi jh) > \frac{b-a}{4}.$$

Case II: $N > N_2, 0 < i \leq I_1$. According to (2.12), we can derive

$$\|B_h \vec{w}_i^h\|_{\mathbb{R}^N}^2 = \sum_{jh \in (a,b)} \frac{2}{N+1} [\sin(i\pi jh)]^2 > M.$$

Case III: $N \leq N_2$. According to Lemma 2.2, there exists a positive constant L_1 , which is independent on h , such that

$$\|B_h \vec{w}_i^h\|_{\mathbb{R}^N} > L_1,$$

for any unit eigenvector \vec{w}_i^h ($i = 1, \dots, N$).

In summary, taking $L = \min\{\frac{b-a}{4}, \frac{M}{2}, L_1\}$, we can complete the proof of this conclusion. \square

Remark 2.1 This theorem gives a fundamental property for the unit eigenvectors \vec{w}_i^h ($i = 1, \dots, N$) of the discrete Laplacian operator. It shows that the energy for these unit eigenvectors have a uniform lower boundary, which is positive and not dependent on h , in a nonempty and open subset $\omega \subset (0, 1)$.

Now, we need to introduce certain notations. Let X_0^h denote the space \mathbb{R}^N equipped with the norm $\|\cdot\|_0$

$$\|u\|_0 = h \sum_{j=1}^N |u_j|^2, \quad \text{for any } u = (u_1, u_2, \dots, u_N)^T \in \mathbb{R}^N.$$

Let X_1^h denote the space \mathbb{R}^N equipped with the norm $\|\cdot\|_1$,

$$\|u\|_1 = h \sum_{j=0}^N \left| \frac{u_j - u_{j+1}}{h} \right|^2, \quad \text{for any } u = (u_1, u_2, \dots, u_N)^T \in \mathbb{R}^N,$$

where $u_0 = u_{N+1} = 0$. According to the definitions of the discrete norms, the energy can be represented as $E_h(t) = \frac{1}{2}(\|\vec{\phi}'_h(t)\|_0 + \|\vec{\phi}_h(t)\|_1)$, where $\vec{\phi}_h(t)$ is the solution of equation (1.4).

Lemma 2.3 For any vector $u = (u_1, u_2, \dots, u_N)^T \in \mathbb{R}^N$, we have the following inequality:

$$\lambda_1(h)\|u\|_0 \leq \|u\|_1.$$

This lemma can easily be deduced from Lemma 2.1.

Remark 2.2

- (i) According to Lemma 2.3, it is easy to find that the spaces X_0^h and X_1^h are both Banach spaces. In fact, X_0^h and X_1^h can be regarded as the discrete version of the space $L^2(0, 1)$ and $H_0^1(0, 1)$, respectively. Thus, Lemma 2.3 can be regarded as the discrete version of Poincaré’s inequality.
- (ii) Since $\mathbb{R}^N \times \mathbb{R}^N$ is a finite dimensional space, thus all norms of this space are equivalent. In particular, there exist positive numbers C_1, C_2 , such that

$$C_1 \|(z_1, z_2)\|_{X_0^h \times X_1^h} \leq \|(z_1, z_2)\|_{\mathbb{R}^N \times \mathbb{R}^N} \leq C_2 \|(z_1, z_2)\|_{X_0^h \times X_1^h} \tag{2.13}$$

hold for any $(z_1, z_2) \in \mathbb{R}^N \times \mathbb{R}^N$.

3 The proof of Theorem 1.1 and Theorem 1.2

3.1 The proof of Theorem 1.1

Proof First of all, we will prove that there exists a positive constant $C(T, h)$ such that the inequality

$$\|(\vec{\phi}_0^h, \vec{\phi}_1^h)\|_{\mathbb{R}^N \times \mathbb{R}^N}^2 \leq C(T, h) \int_0^T \|B_h \vec{\phi}_h(t)\|_{\mathbb{R}^N}^2 dt \tag{3.1}$$

holds, where $\vec{\phi}_h(t)$ is the solution of (1.4) with initial data $(\vec{\phi}_0^h, \vec{\phi}_1^h)$.

Let $T > 0$. We first define a function $F : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ as

$$F(\vec{\phi}_0^h, \vec{\phi}_1^h) = \int_0^T \|B_h \vec{\phi}_h(t)\|_{\mathbb{R}^N}^2 dt,$$

where $\vec{\phi}_h(t)$ is the solution of (1.4) with initial data $(\vec{\phi}_0^h, \vec{\phi}_1^h)$. Obviously, F is continuous. Now, we will prove that

$$\min\{F(\vec{\phi}_0^h, \vec{\phi}_1^h); \|(\vec{\phi}_0^h, \vec{\phi}_1^h)\|_{\mathbb{R}^N \times \mathbb{R}^N} = 1\} \geq L(h, T) \tag{3.2}$$

holds for certain positive constant $L(h, T)$ only depending on h and T .

Suppose that there exists an unit vector $(\vec{\varphi}_0^h, \vec{\varphi}_1^h)$ in $\mathbb{R}^N \times \mathbb{R}^N$ such that $F(\vec{\varphi}_0^h, \vec{\varphi}_1^h) = 0$. Since $\|(\vec{\varphi}_0^h, \vec{\varphi}_1^h)\|_{\mathbb{R}^N \times \mathbb{R}^N} = 1$, $\vec{\varphi}_0^h$ and $\vec{\varphi}_1^h$ could not be zero at the same time. Without loss of generality, we assume that $\vec{\varphi}_0^h \neq 0$. According to (2.3), we have

$$\vec{\varphi}_0^h = \sum_{j=1}^N \varphi_0^j \vec{w}_j^h$$

and

$$\vec{\varphi}_1^h = \sum_{j=1}^N \varphi_1^j \vec{w}_j^h,$$

where $\sum_{j=1}^N |\varphi_0^j|^2 = \|\vec{\varphi}_0^h\|_{\mathbb{R}^N}^2 \neq 0$. Solving (1.4), we can deduce that

$$\vec{\phi}_h(t) = \sum_{j=1}^N \beta_j(t) \vec{w}_j^h, \tag{3.3}$$

where $\beta_j(t) = \varphi_0^j \cos(\sqrt{\lambda_j(h)}t) + \frac{\varphi_1^j}{\sqrt{\lambda_j(h)}} \sin(\sqrt{\lambda_j(h)}t)$.

From the definition of the function F and the assumption that $F(\vec{\varphi}_0^h, \vec{\varphi}_1^h) = 0$, we have

$$0 = F(\vec{\varphi}_0^h, \vec{\varphi}_1^h) = \int_0^T \|B_h \vec{\phi}_h(t)\|_{\mathbb{R}^N}^2 dt = \int_0^T \sum_{j=1}^N |\beta_j(t) B_h \vec{w}_j^h|^2 dt.$$

Thus,

$$\sum_{j=1}^N \beta_j(t) B_h \vec{w}_j^h = 0, \quad \text{for any } t \in [0, T]. \tag{3.4}$$

It follows from Lemma 2.1 or Proposition 2.1 that $B_h \vec{w}_j^h \neq 0$ for any $j = 1, 2, \dots, N$. It is obvious that the rank of subspace spanned by $\{B_h \vec{w}_1^h, \dots, B_h \vec{w}_N^h\}$ is less than N . Therefore, we can assume that $B_h \vec{w}_1^h, \dots, B_h \vec{w}_\alpha^h$, with $1 \leq \alpha < N$, are linear independent in \mathbb{R}^N , and

$$\text{span}\{B_h \vec{w}_1^h, \dots, B_h \vec{w}_\alpha^h\} = \text{span}\{B_h \vec{w}_1^h, \dots, B_h \vec{w}_N^h\}.$$

Hence,

$$B_h \vec{w}_q^h = \sum_{j=1}^{\alpha} b_{qj} B_h \vec{w}_j^h, \quad \text{for any } q = \alpha + 1, \dots, N.$$

For any $q = \alpha + 1, \dots, N$, there exists at least one scalar $b_{qj(q)}$ ($1 \leq j(q) \leq \alpha$) such that $b_{qj(q)} \neq 0$. This, together with (3.4), indicates that

$$\begin{aligned} 0 &= \sum_{j=1}^{\alpha} \beta_j(t) B_h \vec{w}_j^h + \sum_{q=\alpha+1}^N \beta_q(t) \left(\sum_{j=1}^{\alpha} b_{qj} B_h \vec{w}_j^h \right) \\ &= \sum_{j=1}^{\alpha} \left(\beta_j(t) + \sum_{q=\alpha+1}^N \beta_q(t) b_{qj} \right) B_h \vec{w}_j^h, \quad \text{for any } t \in [0, T]. \end{aligned} \tag{3.5}$$

According to the linear independence of $\{B_h \vec{w}_j^h\}_{j=1}^{\alpha}$, we can deduce that

$$\beta_j(t) + \sum_{q=\alpha+1}^N \beta_q(t) b_{qj} = 0, \quad \text{for any } j = 1, 2, \dots, \alpha, \text{ and for any } t \in [0, T]. \tag{3.6}$$

Taking $t = 0$, we get

$$\varphi_0^j + \sum_{q=\alpha+1}^N \varphi_0^q b_{qj} = 0, \quad \text{for any } j = 1, 2, \dots, \alpha.$$

Differentiating (3.6) twice and taking $t = 0$, we have

$$\lambda_j(h) \varphi_0^j + \sum_{q=\alpha+1}^N \lambda_q(h) \varphi_0^q b_{qj} = 0, \quad \text{for any } j = 1, 2, \dots, \alpha.$$

By induction, we obtain

$$\lambda_j^m(h) \varphi_0^j + \sum_{q=\alpha+1}^N \lambda_q^m(h) \varphi_0^q b_{qj} = 0, \quad \text{for any } j = 1, 2, \dots, \alpha, \text{ and } m \in \mathbb{N}^+. \tag{3.7}$$

It follows from (2.1) that $\{\lambda_j(h)\}_{j=1}^N$ are different from each other. Thus, we can deduce that

$$\varphi_0^j = 0, \quad \text{for any } j = 1, 2, \dots, \alpha, \tag{3.8}$$

and

$$\varphi_0^q b_{qj} = 0, \quad \text{for any } q = \alpha + 1, \dots, N, \text{ and } j = 1, 2, \dots, \alpha.$$

Taking $j = j(q)$, we have

$$\varphi_0^q = 0, \quad \text{for any } q = \alpha + 1, \dots, N.$$

This, together with (3.8), leads to a contradiction to the assumption that $\vec{\varphi}_0^h \neq 0$. Thus, (3.2) holds. Note that (3.3) implies $F(\mu v_1, \mu v_2) = \mu^2 F(\mu v_1, \mu v_2)$ for every $(v_1, v_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and $\mu \in \mathbb{R}$. Thus, it is obvious that inequality (3.2) leads to (3.1).

From (3.1) with (2.13), it is easy to obtain the observability inequality (1.5) of the semi-discrete system (1.4). This completes the proof of this theorem. □

3.2 The proof of Theorem 1.2

Proof Given the initial data $\vec{\phi}_0^h = \vec{w}_N^h$, and $\vec{\phi}_1^h = 0$, the solution of equation (1.4) can be represented as

$$\vec{\phi}_h(t) = \cos(\sqrt{\lambda_j(h)}t)\vec{w}_N^h.$$

By Lemma 2.1, one shows that

$$E_h(0) = \frac{h}{2} \sum_{i=0}^N \left| \frac{w_{N,i+1} - w_{N,i}}{h} \right|^2 = \frac{h}{2} \lambda_N(h), \tag{3.9}$$

where $w_{N,0} = w_{N,N+1} = 0$. Then

$$\int_0^T \|B_h \vec{\phi}_h(t)\|_{\mathbb{R}^N}^2 dt = \int_0^T |\cos(\sqrt{\lambda_N(h)}t) B_h \vec{w}_N^h|^2 dt \tag{3.10}$$

$$= \|B_h \vec{w}_N^h\|_{\mathbb{R}^N}^2 \int_0^T |\cos(\sqrt{\lambda_N(h)}t)|^2 dt \leq T. \tag{3.11}$$

It follows from (3.9) and (3.10) that

$$\frac{E_h(0)}{\int_0^T \|B_h \vec{\phi}_h(t)\|_{\mathbb{R}^N}^2 dt} \geq \frac{\frac{h}{2} \lambda_N(h)}{T} = \frac{h \lambda_N(h)}{2T}. \tag{3.12}$$

By (2.1), we derive that

$$\frac{h \lambda_N(h)}{2T} = \frac{2 \sin^2(\frac{\pi N h}{2})}{Th} \rightarrow \infty, \text{ as } h \rightarrow 0. \tag{3.13}$$

This completes the proof of this theorem. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

JL provided the questions. YZ and GZ gave the proof for the main result together. All authors read and approved the final manuscript.

Author details

¹College of Mathematics and Information Science, Henan Normal University, Xixiang, 453007, P.R. China. ²College of Computer and Information Engineering, Henan University of Economics and Law, Zhengzhou, 450002, P.R. China.

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