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Solvability of a class of product-type systems of difference equations

Stevo Stević*

*Correspondence: sstevic@ptt.rs
Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, Beograd, 11000, Serbia
Operator Theory and Applications Research Group, Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia

Abstract

A solvable class of product-type systems of difference equations with two dependent variables on the complex domain is presented. The main results complement some recent ones in the literature, while their proofs contain some refined methodological details. We provide closed form formulas for general solutions to the system or give procedures for how to get them.

MSC: 39A20; 39A45

Keywords: system of difference equations; product-type system; solvable system; solutions in closed form

1 Introduction

Nonlinear difference equations and systems have been studied a lot in the last few decades (see, *e.g.*, [1–29]). Two of the topics of recent interest are symmetric and closely related systems (see, for example, [3, 6–12, 15, 16, 19, 20, 22, 23, 26–29]), whose investigation was considerably influenced by some papers by Papaschinopoulos and Schinas (see, for example, [8–10]), and the solvable difference equations and systems (see, for example, [3, 14, 18–22, 24–29] and the references therein). For some classical methods for solving the equations and systems see, for example, [1, 30–33]. It has been shown recently that many nonlinear equations and systems can be solved by transforming them to linear ones (see, for example, [3, 14, 18, 21, 24, 25] and the related references therein).

Some of the equations and systems that we have studied recently, such as the ones in [17] and [23] (see also [13]), are obtained by adding constants to the right-hand sides of some product-type equations/systems or by taking the maximum of some constants and the right-hand sides of the equations/systems. This means that they are related to the product-type ones, which are usually some kind of boundary cases. The case of positive initial values and multipliers is simple, since in that case the equations/systems can easily be treated by one of the simplest transformation methods. The case of complex initial values is more complex due to the fact that complex functions need not be single valued. Hence, our methods in [3, 18, 21] and other related papers cannot be applied. This has motivated us to start investigating product-type systems on the complex domain. In a series of papers, see [19, 20, 22, 26–29], we have obtained some results in the area (during the study of the equation in [21] we came across a product-type equation). In our first papers on the topic (see [20, 22, 26]) the systems have not had coefficients different from one. However, not long after that we have introduced two coefficients and also got solvable systems (this was done for the first time in [19], and somewhat later in [27–29]). We have also observed the fact that there are only a few solvable product-type systems of difference equations with two dependent variables. Hence, our aim is to describe all such product-type systems and present formulas for the general solutions for each of them.

This paper complements our previous results on the solvability of product-type systems of difference equations with two dependent variables, by studying the following one:

$$z_n = \alpha z_{n-1}^a w_{n-2}^b, \qquad w_n = \beta w_{n-2}^c z_{n-1}^d, \quad n \in \mathbb{N}_0, \tag{1}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ and $z_{-1}, w_{-2}, w_{-1} \in \mathbb{C}$. To do this we will modify ideas and methods from our previous papers, for example, the ones in [19, 27–29].

If the initial values belong to the following set:

$$\{(z_{-1}, w_{-2}, w_{-1}) \in \mathbb{C}^3 : z_{-1} = 0 \text{ or } w_{-2} = 0 \text{ or } w_{-1} = 0\},$$

and if some of the exponents a, b, c, d are negative, then such solutions are not defined. Hence, this set of initial values will not be taken into consideration. Besides, if $\alpha = 0$ or $\beta = 0$, we get $z_n = 0$ and $w_n = 0$ for every $n \in \mathbb{N}_0$, respectively, which are quite simple cases, or also obtain solutions which are not well defined, so, we will also assume that $\alpha \neq 0 \neq \beta$. We use the convention $\sum_{i=k}^{l} a_i = 0$, when $l < k$, throughout the paper.

2 Main results

Our main results are presented here. The following three cases will be considered separately:

(i) $c = 0$, $ac = bd$;

(ii) $c \neq 0$, $ac = bd$;

(iii) $ac \neq bd$.

Clearly, in case (i) from $c = 0$ and $ac = bd$ it immediately follows that $bd = 0$, but we have chosen to write $ac = bd$ at this point, to point out that the whole analysis essentially depends on the values of the quantities c and $ac - bd$, that is, if they are equal to zero or not.

First, we will consider case (i), then case (iii) and at the end case (ii), for the presentational reasons.

Theorem 1 *Assume that* $a, b, d \in \mathbb{Z}$, $c = 0$, $bd = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ *and* $z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$. *Then the following statements hold.*

(a) *If* $a \neq 1$, *then the general solution to system* (1) *is given by the following formulas*:

$$z_n = \alpha^{\frac{1-a^{n+1}}{1-a}} \beta^{b\frac{1-a^{n-1}}{1-a}} z_{-1}^{a^{n+1}} w_{-2}^{ba^n} w_{-1}^{ba^{n-1}}, \quad n \geq 2, \tag{2}$$

and

$$w_n = \alpha^{d\frac{1-a^n}{1-a}} \beta z_{-1}^{da^n}, \quad n \geq 2. \tag{3}$$

(b) *If* $a = 1$, *then the general solution to system* (1) *is given by the following formulas*:

$$z_n = \alpha^{n+1}\beta^{b(n-1)}z_{-1}w_{-2}^b w_{-1}^b, \quad n \geq 2, \tag{4}$$

and

$$w_n = \alpha^{dn}\beta z_{-1}^d, \quad n \geq 2. \tag{5}$$

Proof Since $c = 0$, we have

$$z_n = \alpha z_{n-1}^a w_{n-2}^b, \qquad w_n = \beta z_{n-1}^d, \quad n \in \mathbb{N}_0. \tag{6}$$

From (6) and $bd = 0$ is obtained

$$z_n = \alpha\beta^b z_{n-1}^a, \quad n \geq 2. \tag{7}$$

From (7) we easily get

$$z_n = \left(\alpha\beta^b\right)^{\sum_{j=0}^{n-2} a^j} z_1^{a^{n-1}}, \quad n \geq 2,$$

which, along with

$$z_1 = \alpha^{1+a} z_{-1}^{a^2} w_{-2}^{ab} w_{-1}^b, \tag{8}$$

yields

$$z_n = \alpha^{\sum_{j=0}^{n} a^j}\beta^{b\sum_{j=0}^{n-2} a^j} z_{-1}^{a^{n+1}} w_{-2}^{ba^n} w_{-1}^{ba^{n-1}}, \quad n \geq 2. \tag{9}$$

By using equation (9) along with the formula for the sum of the geometric progression we see that equation (2) holds when $a \neq 1$, while equation (4) is directly obtained for $a = 1$.

From (9), the equality $w_n = \beta z_{n-1}^d$, and the condition $bd = 0$, we obtain

$$\begin{aligned} w_n &= \alpha^{d\sum_{j=0}^{n-1} a^j}\beta^{1+bd\sum_{j=0}^{n-3} a^j} z_{-1}^{da^n} w_{-2}^{bda^{n-1}} w_{-1}^{bda^{n-2}} \\ &= \alpha^{d\sum_{j=0}^{n-1} a^j}\beta z_{-1}^{da^n}, \quad n \geq 2. \end{aligned} \tag{10}$$

By using equation (10) along with the formula for the sum of the geometric progression we see that equation (3) holds when $a \neq 1$, while equation (5) is directly obtained for $a = 1$, completing the proof of the theorem. □

Theorem 2 *Assume that* $a, b, c, d \in \mathbb{Z}$, $ac \neq bd$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ *and* $z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$. *Then system* (1) *is solvable in closed form.*

Proof Since $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$, using (1) it easily follows that $z_n \neq 0$ for $n \geq -1$ and $w_n \neq 0$ for $n \geq -2$. Thus, from (1) we have

$$w_{n-2}^b = \frac{z_n}{\alpha z_{n-1}^a}, \quad n \in \mathbb{N}_0, \tag{11}$$

and

$$w_n^b = \beta^b w_{n-2}^{bc} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0. \tag{12}$$

From (11) and (12) we get

$$z_{n+2} = \alpha^{1-c}\beta^b z_{n+1}^a z_n^c z_{n-1}^{bd-ac}, \quad n \in \mathbb{N}_0. \tag{13}$$

Let $\mu = \alpha^{1-c}\beta^b$,

$$a_1 = a, \qquad b_1 = c, \qquad c_1 = bd - ac, \qquad y_1 = 1. \tag{14}$$

Then

$$z_{n+2} = \mu^{y_1} z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1}, \quad n \in \mathbb{N}_0. \tag{15}$$

From (15) is obtained

$$\begin{aligned} z_{n+2} &= \mu^{y_1}\left(\mu z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1}\right)^{a_1} z_n^{b_1} z_{n-1}^{c_1} \\ &= \mu^{y_1+a_1} z_n^{a_1a_1+b_1} z_{n-1}^{b_1a_1+c_1} z_{n-2}^{c_1a_1} \\ &= \mu^{y_2} z_n^{a_2} z_{n-1}^{b_2} z_{n-2}^{c_2}, \end{aligned} \tag{16}$$

for $n \in \mathbb{N}$, where

$$a_2 := a_1a_1 + b_1, \qquad b_2 := b_1a_1 + c_1, \qquad c_2 := c_1a_1, \qquad y_2 := y_1 + a_1. \tag{17}$$

Suppose that

$$z_{n+2} = \mu^{y_k} z_{n+2-k}^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k}, \tag{18}$$

for a $k \in \mathbb{N}$ and every $n \geq k-1$, and

$$a_k = a_1a_{k-1} + b_{k-1}, \qquad b_k = b_1a_{k-1} + c_{k-1}, \qquad c_k = c_1a_{k-1}, \tag{19}$$

$$y_k := y_{k-1} + a_{k-1}. \tag{20}$$

From (15) with $n \to n-k$ and (18), it follows that

$$\begin{aligned} z_{n+2} &= \mu^{y_k}\left(\mu z_{n+1-k}^{a_1} z_{n-k}^{b_1} z_{n-k-1}^{c_1}\right)^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} \\ &= \mu^{y_k+a_k} z_{n+1-k}^{a_1a_k+b_k} z_{n-k}^{b_1a_k+c_k} z_{n-k-1}^{c_1a_k} \\ &= \mu^{y_{k+1}} z_{n+1-k}^{a_{k+1}} z_{n-k}^{b_{k+1}} z_{n-k-1}^{c_{k+1}}, \end{aligned} \tag{21}$$

for $n \geq k$, where

$$a_{k+1} := a_1a_k + b_k, \qquad b_{k+1} := b_1a_k + c_k, \qquad c_{k+1} := c_1a_k, \tag{22}$$

$$y_{k+1} := y_k + a_k. \tag{23}$$

Equalities (16), (17), (21), (22), (23), and the induction show that (18), (19), and (20), hold for every $k, n \in \mathbb{N}$ such that $2 \le k \le n+1$. Note that (18) holds also for $k = 1$.

For $k = n+1$, (18) becomes

$$\begin{aligned} z_{n+2} &= \mu^{y_{n+1}} z_1^{a_{n+1}} z_0^{b_{n+1}} z_{-1}^{c_{n+1}} \\ &= \left(\alpha^{1-c}\beta^b\right)^{y_{n+1}} \left(\alpha^{1+a} z_{-1}^{a^2} w_{-2}^{ab} w_{-1}^{b}\right)^{a_{n+1}} \left(\alpha z_{-1}^{a} w_{-2}^{b}\right)^{b_{n+1}} z_{-1}^{c_{n+1}} \\ &= \alpha^{(1-c)y_{n+1}+(1+a)a_{n+1}+b_{n+1}} \beta^{by_{n+1}} z_{-1}^{a^2 a_{n+1}+ab_{n+1}+c_{n+1}} \\ &\quad \times w_{-2}^{aba_{n+1}+bb_{n+1}} w_{-1}^{ba_{n+1}} \\ &= \alpha^{y_{n+3}-cy_{n+1}} \beta^{by_{n+1}} z_{-1}^{a_{n+3}-ca_{n+1}} w_{-2}^{ba_{n+2}} w_{-1}^{ba_{n+1}}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{24}$$

where we have also used the fact $z_0 = \alpha z_{-1}^{a} w_{-2}^{b}$, (8), (19), and (20).

Further note that (19) implies that $(a_k)_{k\ge 4}$ is a solution to the equation

$$a_k = a_1 a_{k-1} + b_1 a_{k-2} + c_1 a_{k-3}. \tag{25}$$

Since $c_1 \neq 0$, from (25), we get

$$a_{k-3} = \frac{a_k - a_1 a_{k-1} - b_1 a_{k-2}}{c_1}, \tag{26}$$

from which it follows that we can calculate a_l for every non-positive l, that is, for $k \le 3$. A direct calculation shows that

$$a_0 = 1, \qquad a_{-1} = a_{-2} = 0. \tag{27}$$

Moreover, it is shown that $(a_k)_{k\ge -2}$, $(b_k)_{k\ge -2}$, $(c_k)_{k\ge -2}$ are solutions to (25) such that

$$\begin{aligned} &a_{-2} = 0, \qquad a_{-1} = 0, \qquad a_0 = 1; \\ &b_{-2} = 0, \qquad b_{-1} = 1, \qquad b_0 = 0; \\ &c_{-2} = 1, \qquad c_{-1} = 0, \qquad c_0 = 0; \end{aligned}$$

respectively, whereas $(y_k)_{k\ge -2}$ satisfies (20) and

$$y_{-2} = y_{-1} = y_0 = 0, \qquad y_1 = 1. \tag{28}$$

From (20) and since $a_0 = 1$, we get

$$y_k = \sum_{j=0}^{k-1} a_j. \tag{29}$$

The solvability of (25) shows that for $(a_k)_{k\ge -2}$ can be found a closed form formula. Therefore, using equation (29) and known formulas for the following sums:

$$s_m^{(j)}(\zeta) = \sum_{k=1}^{m} k^j \zeta^k, \quad m \in \mathbb{N}_0, \tag{30}$$

where $j = \overline{0,2}$ (see, *e.g.*, [31, 33]), the closed form formula for $(y_k)_{k\in\mathbb{N}}$ is easily obtained. This along with (24) shows the solvability of (13).

From (1), we also have

$$z_{n-1}^d = \frac{w_n}{\beta w_{n-2}^c}, \quad n \in \mathbb{N}_0, \tag{31}$$

and

$$z_n^d = \alpha^d z_{n-1}^{ad} w_{n-2}^{bd}, \quad n \in \mathbb{N}_0. \tag{32}$$

Equalities (31) and (32) yield

$$w_{n+1} = \alpha^d \beta^{1-a} w_n^a w_{n-1}^c w_{n-2}^{bd-ac}, \quad n \in \mathbb{N}_0. \tag{33}$$

We have

$$w_0 = \beta w_{-2}^c z_{-1}^d. \tag{34}$$

Similarly as above we get

$$w_{n+1} = \eta^{y_k} w_{n+1-k}^{a_k} w_{n-k}^{b_k} w_{n-k-1}^{c_k}, \quad n \geq k-1, \tag{35}$$

where $\eta = \alpha^d \beta^{1-a}$, $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ are defined by (14) and (19), whereas $(y_k)_{k\in\mathbb{N}}$ satisfies (20) and (28), so it is given by (29).

From (35) with $k = n+1$ and (34) is obtained

$$\begin{aligned} w_{n+1} &= \eta^{y_{n+1}} w_0^{a_{n+1}} w_{-1}^{b_{n+1}} w_{-2}^{c_{n+1}} \\ &= \left(\alpha^d \beta^{1-a}\right)^{y_{n+1}} \left(\beta w_{-2}^c z_{-1}^d\right)^{a_{n+1}} w_{-1}^{b_{n+1}} w_{-2}^{c_{n+1}} \\ &= \alpha^{dy_{n+1}} \beta^{(1-a)y_{n+1}+a_{n+1}} z_{-1}^{da_{n+1}} w_{-2}^{ca_{n+1}+c_{n+1}} w_{-1}^{b_{n+1}} \\ &= \alpha^{dy_{n+1}} \beta^{y_{n+2}-ay_{n+1}} z_{-1}^{da_{n+1}} w_{-2}^{a_{n+3}-aa_{n+2}} w_{-1}^{a_{n+2}-aa_{n+1}}, \end{aligned} \tag{36}$$

for $n \in \mathbb{N}_0$.

The solvability of (25) along with (27) shows that for $(a_k)_{k\geq -2}$ we can find a closed form formula, from which along with (29) the formulas for y_k can also be obtained, as described above. These facts along with (36) imply the solvability of equation (33). It is not difficult to show that formulas (24) and (36) really represent solutions to system (1). Thus, system (1) is also solvable in this case, as claimed. □

Corollary 1 *Consider system* (1) *with* $a, b, c, d \in \mathbb{Z}$, $ac \neq bd$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, *and* $z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$. *Then the general solution to system* (1) *is given by* (24) *and* (36), *where* $(a_k)_{k\in\mathbb{N}}$ *satisfies equation* (25) *with initial conditions* (27), *while* $(y_k)_{k\in\mathbb{N}}$ *is given by* (29) *and can be found by using formulas for the sums in* (30).

Theorem 3 *Assume that* $a, b, c, d \in \mathbb{Z}$, $c \neq 0$, $ac = bd$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ *and* $z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$. *Then the following statements hold.*

(a) *If* $a^2 \neq -4c$ *and* $a + c \neq 1$, *then the general solution to system* (1) *is given by the following formulas*:

$$z_n = \alpha^{\frac{a(\lambda_2-1)\lambda_1^{n+1}-a(\lambda_1-1)\lambda_2^{n+1}+(\lambda_1-\lambda_2)(1-c)}{(\lambda_1-1)(\lambda_2-1)(\lambda_1-\lambda_2)}}\beta^{b\frac{(\lambda_2-1)\lambda_1^{n}-(\lambda_1-1)\lambda_2^{n}+\lambda_1-\lambda_2}{(\lambda_1-1)(\lambda_2-1)(\lambda_1-\lambda_2)}} \times z_{-1}^{a\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}} w_{-2}^{b\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}} w_{-1}^{b\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}} \tag{37}$$

and

$$w_n = \alpha^{d\frac{(\lambda_2-1)\lambda_1^{n+1}-(\lambda_1-1)\lambda_2^{n+1}+\lambda_1-\lambda_2}{(\lambda_1-1)(\lambda_2-1)(\lambda_1-\lambda_2)}}\beta^{\frac{c(\lambda_2-1)\lambda_1^{n}-c(\lambda_1-1)\lambda_2^{n}+(\lambda_1-\lambda_2)(1-a)}{(\lambda_1-1)(\lambda_2-1)(\lambda_1-\lambda_2)}} \times z_{-1}^{d\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}} w_{-2}^{c\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}} w_{-1}^{c\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}}, \tag{38}$$

where

$$\lambda_{1,2} = \frac{a \pm \sqrt{a^2+4c}}{2}. \tag{39}$$

(b) *If* $a^2 \neq -4c$ *and* $a + c = 1$, *then the general solution to system* (1) *is given by the following formulas*:

$$z_n = \alpha^{\frac{a\lambda_1^{n+1}+((c-1)n-2)\lambda_1+(1-c)n+1+c}{(1-\lambda_1)^2}}\beta^{b\frac{n-1-n\lambda_1+\lambda_1^{n}}{(1-\lambda_1)^2}} z_{-1}^{a\frac{\lambda_1^{n+1}-1}{\lambda_1-1}} w_{-2}^{b\frac{\lambda_1^{n+1}-1}{\lambda_1-1}} w_{-1}^{b\frac{\lambda_1^{n}-1}{\lambda_1-1}} \tag{40}$$

and

$$w_n = \alpha^{d\frac{n-(n+1)\lambda_1+\lambda_1^{n+1}}{(1-\lambda_1)^2}}\beta^{\frac{c\lambda_1^{n}+((a-1)n+a-2)\lambda_1+(1-a)n+1}{(1-\lambda_1)^2}} \times z_{-1}^{d\frac{\lambda_1^{n+1}-1}{\lambda_1-1}} w_{-2}^{c\frac{\lambda_1^{n+1}-1}{\lambda_1-1}} w_{-1}^{c\frac{\lambda_1^{n}-1}{\lambda_1-1}}, \tag{41}$$

where

$$\lambda_1 = -c. \tag{42}$$

(c) *If* $a^2 = -4c$ *and* $a + c \neq 1$, *then the general solution to system* (1) *is given by the following formulas*:

$$z_n = \alpha^{\frac{an\lambda_1^{n+1}-a(n+1)\lambda_1^{n}+1-c}{(1-\lambda_1)^2}}\beta^{b\frac{1-n\lambda_1^{n-1}+(n-1)\lambda_1^{n}}{(1-\lambda_1)^2}} \times z_{-1}^{a(n+1)\lambda_1^{n}} w_{-2}^{b(n+1)\lambda_1^{n}} w_{-1}^{bn\lambda_1^{n-1}} \tag{43}$$

and

$$w_n = \alpha^{d\frac{1-(n+1)\lambda_1^{n}+n\lambda_1^{n+1}}{(1-\lambda_1)^2}}\beta^{\frac{c(n-1)\lambda_1^{n}-cn\lambda_1^{n-1}+1-a}{(1-\lambda_1)^2}} \times z_{-1}^{d(n+1)\lambda_1^{n}} w_{-2}^{c(n+1)\lambda_1^{n}} w_{-1}^{cn\lambda_1^{n-1}}, \tag{44}$$

where

$$\lambda_1 = \frac{a}{2}. \tag{45}$$

(d) *If* $a^2 = -4c$ *and* $a + c = 1$, *then the general solution to system* (1) *is given by the following formulas*:

$$z_n = \alpha^{\frac{(1-c)n^2+(c+3)n+2}{2}} \beta^{b\frac{(n-1)n}{2}} z_{-1}^{a(n+1)} w_{-2}^{b(n+1)} w_{-1}^{bn} \tag{46}$$

and

$$w_n = \alpha^{d\frac{n(n+1)}{2}} \beta^{\frac{(n+1)((1-a)n+2)}{2}} z_{-1}^{d(n+1)} w_{-2}^{c(n+1)} w_{-1}^{cn}. \tag{47}$$

Proof Similar to the proof of Theorem 2 we have $z_n \neq 0$ for $n \geq -1$ and $w_n \neq 0$ for $n \geq -2$, and for every such solution (13) holds, from which, along with the condition $ac = bd$, it follows that

$$z_{n+2} = \alpha^{1-c} \beta^b z_{n+1}^a z_n^c, \tag{48}$$

for $n \in \mathbb{N}_0$.

Let $\nu = \alpha^{1-c}\beta^b$,

$$\hat{a}_1 = a, \qquad \hat{b}_1 = c, \qquad \hat{y}_1 = 1. \tag{49}$$

Then

$$z_{n+2} = \nu^{\hat{y}_1} z_{n+1}^{\hat{a}_1} z_n^{\hat{b}_1}, \tag{50}$$

for $n \in \mathbb{N}_0$.

From (50) is obtained

$$\begin{aligned} z_{n+2} &= \nu^{\hat{y}_1} \left(\nu z_n^{\hat{a}_1} z_{n-1}^{\hat{b}_1}\right)^{\hat{a}_1} z_n^{\hat{b}_1} \\ &= \nu^{\hat{y}_1+\hat{a}_1} z_n^{\hat{a}_1\hat{a}_1+\hat{b}_1} z_{n-1}^{\hat{b}_1\hat{a}_1} \\ &= \nu^{\hat{y}_2} z_n^{\hat{a}_2} z_{n-1}^{\hat{b}_2}, \end{aligned} \tag{51}$$

for $n \in \mathbb{N}$, where

$$\hat{a}_2 := \hat{a}_1\hat{a}_1 + \hat{b}_1, \qquad \hat{b}_2 := \hat{b}_1\hat{a}_1, \qquad \hat{y}_2 := \hat{y}_1 + \hat{a}_1. \tag{52}$$

Suppose that

$$z_{n+2} = \nu^{\hat{y}_k} z_{n+2-k}^{\hat{a}_k} z_{n+1-k}^{\hat{b}_k}, \tag{53}$$

holds for some $k \in \mathbb{N}$ and for every $n \geq k-1$, and that

$$\hat{a}_k = \hat{a}_1\hat{a}_{k-1} + \hat{b}_{k-1}, \qquad \hat{b}_k = \hat{b}_1\hat{a}_{k-1}, \qquad \hat{y}_k := \hat{y}_{k-1} + \hat{a}_{k-1}. \tag{54}$$

Then from (50), with $n \to n-k$ and (53), we get

$$\begin{aligned} z_{n+2} &= \nu^{\hat{y}_k}\left(\nu z_{n+1-k}^{\hat{a}_1} z_{n-k}^{\hat{b}_1}\right)^{\hat{a}_k} z_{n+1-k}^{\hat{b}_k} \\ &= \nu^{\hat{y}_k+\hat{a}_k} z_{n+1-k}^{\hat{a}_1\hat{a}_k+\hat{b}_k} z_{n-k}^{\hat{b}_1\hat{a}_k} \\ &= \nu^{\hat{y}_{k+1}} z_{n+1-k}^{\hat{a}_{k+1}} z_{n-k}^{\hat{b}_{k+1}}, \end{aligned} \tag{55}$$

for $n \geq k$, where

$$\hat{a}_{k+1} := \hat{a}_1\hat{a}_k + \hat{b}_k, \qquad \hat{b}_{k+1} := \hat{b}_1\hat{a}_k, \qquad \hat{y}_{k+1} := \hat{y}_k + \hat{a}_k. \tag{56}$$

Equalities (51), (52), (55), (56), and the induction show that (53) and (54), hold for every $k, n \in \mathbb{N}$ such that $2 \leq k \leq n+1$. Note that the equality in (53) also holds for $k = 1$ (see (50)).

For $k = n+1$, (53) becomes

$$\begin{aligned} z_{n+2} &= \nu^{\hat{y}_{n+1}} z_1^{\hat{a}_{n+1}} z_0^{\hat{b}_{n+1}} \\ &= \left(\alpha^{1-c}\beta^b\right)^{\hat{y}_{n+1}} \left(\alpha^{1+a} z_{-1}^{a^2} w_{-2}^{ab} w_{-1}^{b}\right)^{\hat{a}_{n+1}} \left(\alpha z_{-1}^{a} w_{-2}^{b}\right)^{\hat{b}_{n+1}} \\ &= \alpha^{(1-c)\hat{y}_{n+1}+(1+a)\hat{a}_{n+1}+\hat{b}_{n+1}} \beta^{b\hat{y}_{n+1}} z_{-1}^{a^2\hat{a}_{n+1}+a\hat{b}_{n+1}} \\ &\quad \times w_{-2}^{ab\hat{a}_{n+1}+b\hat{b}_{n+1}} w_{-1}^{b\hat{a}_{n+1}} \\ &= \alpha^{\hat{y}_{n+3}-c\hat{y}_{n+1}} \beta^{b\hat{y}_{n+1}} z_{-1}^{a\hat{a}_{n+2}} w_{-2}^{b\hat{a}_{n+2}} w_{-1}^{b\hat{a}_{n+1}}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{57}$$

where we have used the fact $z_0 = \alpha z_{-1}^a w_{-2}^b$, (8), and (54).

Now note that $(\hat{a}_k)_{k\geq 3}$ is a solution to the equation

$$\hat{a}_k = \hat{a}_1\hat{a}_{k-1} + \hat{b}_1\hat{a}_{k-2}. \tag{58}$$

As in the proof of Theorem 2, it is shown that $(\hat{a}_k)_{k\geq -2}$ and $(\hat{b}_k)_{k\geq -2}$ are solutions to equation (58), satisfying the initial conditions

$$\begin{aligned} &\hat{a}_{-1} = 0, \qquad \hat{a}_0 = 1; \\ &\hat{b}_{-1} = 1, \qquad \hat{b}_0 = 0, \end{aligned} \tag{59}$$

respectively, whereas $(\hat{y}_k)_{k\geq -1}$ satisfies the third equation in (54) and

$$\hat{y}_{-1} = \hat{y}_0 = 0, \qquad \hat{y}_1 = 1. \tag{60}$$

This and $\hat{a}_0 = 1$ imply

$$\hat{y}_k = \sum_{j=0}^{k-1} \hat{a}_j. \tag{61}$$

Since (33) also holds, using the condition $ac = bd$, we get

$$w_{n+1} = \alpha^d \beta^{1-a} w_n^a w_{n-1}^c, \quad n \in \mathbb{N}_0. \tag{62}$$

Similar to the above is obtained

$$w_{n+1} = \eta^{\hat{y}_k} w_{n+1-k}^{\hat{a}_k} w_{n-k}^{\hat{b}_k}, \tag{63}$$

for $k, n \in \mathbb{N}$, $1 \le k \le n+1$, where $\eta = \alpha^d \beta^{1-a}$, sequences $(\hat{a}_k)_{k\in\mathbb{N}}$, and $(\hat{b}_k)_{k\in\mathbb{N}}$ satisfy (49) and (54), whereas $(\hat{y}_k)_{k\in\mathbb{N}}$ satisfies the third equation in (54) and (60), so, formula (61) holds.

From (63) with $k = n+1$ and (34) it follows that

$$\begin{aligned} w_{n+1} &= \eta^{\hat{y}_{n+1}} w_0^{\hat{a}_{n+1}} w_{-1}^{\hat{b}_{n+1}} \\ &= \left(\alpha^d \beta^{1-a}\right)^{\hat{y}_{n+1}} \left(\beta w_{-2}^c z_{-1}^d\right)^{\hat{a}_{n+1}} w_{-1}^{\hat{b}_{n+1}} \\ &= \alpha^{d\hat{y}_{n+1}} \beta^{\hat{y}_{n+2} - a\hat{y}_{n+1}} z_{-1}^{d\hat{a}_{n+1}} w_{-2}^{c\hat{a}_{n+1}} w_{-1}^{c\hat{a}_n}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{64}$$

The characteristic polynomial associated to the linear difference equation (58) is the following:

$$P_2(\lambda) = \lambda^2 - a\lambda - c,$$

from which it follows that the corresponding characteristic roots are given by the formulas in (39).

Since $a_{-1} = 0$ and $a_0 = 1$, then, if $a^2 \ne -4c$ and $a + c \ne 1$, we easily get

$$a_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} \tag{65}$$

and consequently

$$\hat{y}_n = \sum_{j=0}^{n-1} \frac{\lambda_1^{j+1} - \lambda_2^{j+1}}{\lambda_1 - \lambda_2} = \frac{(\lambda_2 - 1)\lambda_1^{n+1} - (\lambda_1 - 1)\lambda_2^{n+1} + \lambda_1 - \lambda_2}{(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - \lambda_2)}. \tag{66}$$

If $a^2 \ne -4c$ and $a + c = 1$, then one of the characteristic roots, say λ_2, is equal to one, and we have

$$a_n = \frac{\lambda_1^{n+1} - 1}{\lambda_1 - 1} \tag{67}$$

and

$$\hat{y}_n = \sum_{j=0}^{n-1} \frac{\lambda_1^{j+1} - 1}{\lambda_1 - 1} = \frac{n - (n+1)\lambda_1 + \lambda_1^{n+1}}{(1 - \lambda_1)^2}. \tag{68}$$

If $a^2 = -4c$ and $a + c \ne 1$, then we have

$$a_n = (n+1)\lambda_1^n, \quad n \in \mathbb{N}_0, \tag{69}$$

and

$$\hat{y}_n = \sum_{j=0}^{n-1}(j+1)\lambda_1^j = \frac{1-(n+1)\lambda_1^n + n\lambda_1^{n+1}}{(1-\lambda_1)^2}. \tag{70}$$

If $a^2 = -4c$ and $a + c = 1$, then we have

$$a_n = n+1, \quad n \in \mathbb{N}_0, \tag{71}$$

and

$$\hat{y}_n = \sum_{j=0}^{n-1}(j+1) = \frac{n(n+1)}{2}. \tag{72}$$

Using (65)-(72) into (57) and (64) and by some calculations equations (37), (38), (40), (41), (43), (44), (46), and (47) are obtained. By some standard, but time-consuming calculations, it is shown that these formulas really represent solutions to system (1) in each if these four cases. □

Remark 1 Note that if $a^2 = -4c$ and $a + c = 1$, then $(a-2)^2 = 0$, from which it follows that $a = 2$ and consequently $c = -1$. Hence, equations (46) and (47) can also be written in the following, more concrete, forms:

$$z_n = \alpha^{n^2+n+1}\beta^{b\frac{(n-1)n}{2}} z_{-1}^{2(n+1)} w_{-2}^{b(n+1)} w_{-1}^{bn}$$

and

$$w_n = \alpha^{d\frac{n(n+1)}{2}}\beta^{-\frac{(n+1)(n-2)}{2}} z_{-1}^{d(n+1)} w_{-2}^{-(n+1)} w_{-1}^{-n}.$$

Competing interests
The author declares that he has no competing interests.

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