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Generating integrable lattice hierarchies by some matrix and operator Lie algebras

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Abstract

Two types of matrix Lie algebras are presented. We make use of the first loop algebra to obtain a new (1 + 1)-dimensional integrable discrete hierarchy, which generalizes a result given by Gordoa *et al.*, whose reduction is a discrete modified KdV system. Then we produce another new (2 + 1)-dimensional integrable discrete hierarchy with three fields under a (2 + 1)-dimensional non-isospectral linear problem. We again generalize the (1 + 1)- and (2 + 1)-dimensional discrete hierarchies to obtain a positive and negative integrable discrete hierarchy. In addition, we obtain a discrete integrable coupling system of the (1 + 1)-dimensional discrete hierarchy presented in the paper by enlarging such the loop algebras. Next, we apply the second matrix loop algebra to introduce an isospectral problem and deduce a new integrable discrete hierarchy, whose quasi-Hamiltonian structure is derived from the trace identity proposed by Tu Guizhang, which can be reduced to some modified Toda lattice equations. A type of Darboux transformation of a reduced discrete system of the latter integrable discrete hierarchy is obtained as well. We introduce two types of operator-Lie algebras according to a given spectral problem by a matrix Lie algebra and apply the *r*-matrix theory to obtain a few lattice integrable systems, including two (2 + 1)-dimensional lattice systems.

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1 Introduction

It has been an important work to search for new lattice integrable systems, since such lattice systems not only have rich mathematical structures, *e.g.*, Lax pairs, Bäcklund transformations, Hamiltonian structures, soliton solutions, and so on, but also they have many applications in mathematical physics, statistical physics, and quantum physics. Therefore, one tries to seek for various integrable discrete systems via various methods including mathematical and physical methods, such as the Ablowitz-Ladik lattice, the Toda lattice, the Lotka-Volterra lattice, the differential-difference KdV equation, the Suris lattices, and so on [1–10]. Fan and Yang [11] introduced an isospectral problem and derived a lattice hierarchy which reduced to the Ablowitz-Ladik and the Volterra hierarchies, respectively. As far as the (2 + 1)-dimensional integrable discrete systems and their some properties are concerned, there are few works. For example, the (2 + 1)-dimensional Toda lattice was presented and it was verified that it has a Lax pair, a Hamiltonian structure, and soliton solutions [12]. Two (2 + 1)-dimensional integrable discrete hierarchies with three fields were constructed in terms of discrete zero curvature equations in [13]. Again in the case of a (2 + 1)-dimensional non-isospectral linear problem, a new (2 + 1)-dimensional integrable lattice hierarchy, which is a generalization of the discrete second Painlevé hierarchy, was investigated in [14]. By introducing fourth-order Lax matrices, two (2 + 1)-dimensional integrable lattice hierarchies, which reduce to the two Mlaszak-Marciniak integrable lattice hierarchies, were generated [15]. Tu [16] once applied the Lie-algebra method to deduce the Toda lattice hierarchy and its Hamiltonian structure combined with the variational method. By following the way proposed by Zhang *et al.* [17] one obtained some integrable discrete hierarchies. One advantage for applying the Lie-algebra method to deduce integrable discrete hierarchies lies in adopting the well-known the Tu scheme [18], which conveniently introduces linear spectral problems and manipulates similar steps as the case of generating continuous integrable systems. Based on the scheme, Zhang and Tam [19] obtained two integrable discrete integrable coupled systems of the Toda lattice, including the linear and nonlinear discrete integrable couplings. All the works mentioned above were performed under matrix Lie algebras. In the paper, we would like to employ the first matrix loop algebra to generate (1 + 1)- and (2 + 1)-dimensional integrable discrete hierarchies, which generalize some results obtained in [14], furthermore, we also obtain a positive and negative integrable discrete hierarchy which implements the well-known results presented in [10, 13, 16, 17, 19–22]. We again discuss a discrete integrable coupling of the (1 + 1)-dimensional integrable discrete hierarchy which possesses an arbitrary parameter derived by using an enlarging matrix loop algebra. Finally, we apply the second matrix loop algebra to generate a new integrable discrete hierarchy which can be reduced to a generalized Toda lattice equation, whose quasi-Hamiltonian structure is obtained. Furthermore, a Darboux transformation of a reduced differential-difference equation system of the latter discrete hierarchy is obtained. We introduce a discrete-operator associated algebra whose elements are just like the form

$$L = u_{\alpha+n}E^{\alpha+n} + u_{\alpha+n-1}E^{\alpha+n-1} + \cdots + u_\alpha E^\alpha, \quad -n < \alpha \le -1.$$

Blaszak and Marciniak [23] discovered two types of operator Lie algebras based on the above general associative algebra:

$$\begin{aligned} k = 0: &\quad L = E^{\alpha+n} + u_{\alpha+n-1}E^{\alpha+n-1} + \cdots + u_\alpha E^\alpha, \qquad u_{\alpha+n} = 1, \\ k = 1: &\quad \bar{L} = \bar{u}_{\alpha+n}E^{\alpha+n} + \bar{u}_{\alpha+n-1}E^{\alpha+n-1} + \cdots + E^\alpha, \qquad \bar{u}_\alpha = 1. \end{aligned}$$

According to the operator Lie algebras, we shall introduce different isospectral problems according to deforms of the spectral problem (54) to deduce various lattice integrable systems, including the Toda lattice system, further we derive their Lax pairs by using the r-matrix theory. In the following, we first recall the simplest matrix Lie algebra,

$$A_1 = \operatorname{span}\{h_1, h_2, e, f\},$$

where

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

equipped with the commutative relations $h_1h_1 = h_1, h_2h_2 = h_2, h_1h_2 = h_2h_1 = ee = ff = 0, h_1e = e, eh_1 = 0, h_1f = 0, fh_1 = f, h_2f = f, fh_2 = 0, h_2e = 0, eh_2 = e, ef = h_1, fe = h_2$, from which we have $[h_1, e] = e, [h_1, f] = -f, [h_2, e] = -e, [h_2, f] = f, [e, f] = h \equiv h_1 - h_2, [h, e] = 2e, [h, f] = -2f$. The first loop algebra corresponding to the Lie algebra A_1 can be defined as

$$\tilde{A}_1 = \operatorname{span}\{h_1(n), h_2(n), e(n), f(n)\},$$

where $h_1(n) = h_1\lambda^n, h_2(n) = h_2\lambda^n, e(n) = e\lambda^n, f(n) = f\lambda^n, n \in \mathbf{Z}$.

The second loop algebra is given by

$$\bar{A}_1 = \operatorname{span}\{h_1(n), h_2(n), e(n), f(n)\},$$

where $h_1(n) = h_1\lambda^{2n}, h_2(n) = h_2\lambda^{2n}, e(n) = e\lambda^{2n+1}, f(n) = f\lambda^{2n+1}$.

The purpose for recalling the above two-loop algebras aims at introducing spectral Lax pairs, then with the help of various compatibility conditions, that is, various zero curvature equations, to generate different discrete integrable hierarchies. It is remarkable that the compatibility of some spectral Lax pairs can be transformed into Lax equations. Discussions of the tensorial form of the Lax pair equations were discovered in a compact and geometrically transparent form in the presence of Cartan's torsion tensor, therefore, three dimensional spacetimes admitting Lax tensors were analyzed in [24]. Besides, Balean *et al.* in [25] investigated the connection between Killing tensors and Lax operators, and two examples, *i.e.*, the Toda lattice system and the Rindler system, were analyzed in detail. Further developments on discrete equations focus on fractional difference equations and their different properties emerged. Wu *et al.* [26] showed that the Caputo-like delta derivative is adopted as the difference operator and the master-slave synchronization for the fractional difference equation was studied with a nonlinear coupling method. A lattice fractional diffusion equation was proposed in Ref. [27], and the numerical simulation of the diffusion procession was discussed for various difference orders. In addition, Wu *et al.* [28] proposed the fractional logistic map and fractional Lorenz maps of Riemann-Liouville type and the feedback control method was extended to discrete fractional equations. In Ref. [29], by the use of the Riemann-Liouville differences on time scales, the Riesz difference was defined in a consideration for discrete fractional modeling. Specially, the Adomian decomposition method was adopted to solve the fractional partial difference equations numerically. All the results presented in [24–29] could motivate us going on investigating the generating discrete equations and discussing their properties applied to physical and mathematical sciences.

2 Two integrable discrete hierarchies with three fields in 1 + 1 and 2 + 1 dimensions

Tu in [16] proposed a method for generating discrete integrable hierarchies by the use of loop algebras whose specific steps are as follows.

First introduce the spectral problem

$$\psi_{n+1} = U_n\psi_n, \qquad \psi_{n,t} = V_n\psi_n.$$

Then solve the stationary discrete zero curvature equation

$$(E\Gamma)U_n - U_n\Gamma = 0,$$

where

$$\Gamma = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{m\geq 0} \begin{pmatrix} a_m(n,t) & b_m(n,t) \\ c_m(n,t) & -a_m(n,t) \end{pmatrix} \lambda^{-m},$$

to obtain some recurrence relations among a_m, b_m, c_m.

Third, solve the discrete zero curvature equation

$$\frac{dU_n}{dt} = \left(EV_n^{(m)}\right)U_n - U_n V^{(m)},$$

where

$$V_n^{(m)} = \left(\lambda^m \Gamma\right)_+ + \Delta_m(u,\lambda) = \sum_{i=0}^{m} \Gamma_i \lambda^{m-i} + \Delta_m(u,\lambda).$$

Finally, apply the trace identity

$$\frac{\delta}{\delta u} \operatorname{tr}\left(W \frac{\partial U_n}{\partial \lambda}\right) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \operatorname{tr}(WU_{u_i}), \quad i = 1, 2, \ldots, p,$$

to deduce the Hamiltonian structure of the discrete integrable hierarchies obtained by the discrete zero curvature equations. The above procedure for generating discrete integrable systems is called the Tu scheme.

In the following, we shall apply the Tu scheme and the first loop algebra $\tilde{A}_1$ to generate (1 + 1)- and (2 + 1)-dimensional integrable discrete hierarchies, then generalize them to a unified model which is a positive and negative integrable discrete system.

2.1 A (1 + 1)-dimensional integrable discrete hierarchy

Consider an isospectral problem

$$\psi_{n+1} = U_n \psi_n, \qquad \psi_{n,t} = V_n \psi_n, \tag{1}$$

where $U_n = s_n h_1(1) + h_2(-1) + q_n e(0) + r_n f(0)$, $V_n = A_n(h_1(0) - h_2(0)) + B_n e(0) + C_n f(0)$, where

$$A_n = \sum_{j\geq 0} a_j \lambda^{-2j}, \qquad B_n = \sum_{j\geq 0} b_j \lambda^{-2j+1}, \qquad C_n = \sum_{j\geq 0} c_j \lambda^{-2j+1}. \tag{2}$$

Denoting $\Delta = E - 1, Ef(n) = f(n+1), E^{-1}f(n) = f(n-1)$, and solving the stationary discrete zero curvature equation

$$(\Delta V_n)U_n = [U_n, V_n] \tag{3}$$

yields

$$\begin{cases} \lambda s_n \Delta A_n + r_n \Delta B_n = q_n C_n - r_n B_n, \\ q_n \Delta A_n + \lambda^{-1} \Delta B_n = \lambda s_n B_n - 2q_n A_n - \lambda^{-1} B_n, \\ \lambda s_n \Delta C_n - r_n \Delta A_n = 2r_n A_n + \lambda^{-1} C_n - \lambda s_n C_n, \\ q_n \Delta C_n - \lambda^{-1} \Delta A_n = r_n B_n - q_n C_n. \end{cases} \tag{4}$$

Substituting (2) into (4) gives rise to

$$\begin{cases} s_n \Delta a_j + r_n \Delta b_j = q_n c_j - r_n b_j, \\ q_n \Delta a_j + \Delta b_j = s_n b_{j+1} - 2q_n a_j - b_j, \\ s_n \Delta c_{j+1} - r_n \Delta a_j = 2r_n a_j + c_j - s_n c_{j+1}, \\ q_n \Delta c_{j+1} - \Delta a_j = r_n b_{j+1} - q_n c_{j+1}. \end{cases} \tag{5}$$

Taking the initial values $b_0 = c_0 = 0, a_0 = 1$, then we get from (5)

$$\begin{aligned} &b_1 = \frac{2q_n}{s_n}, \qquad c_1 = \frac{2r_{n-1}}{s_{n-1}}, \qquad a_1 = \frac{-2q_n r_{n-1}}{s_n s_{n-1}}, \\ &b_2 = \frac{2q_{n+1}}{s_n s_{n+1}} - \frac{2q_n r_n q_{n+1}}{s_n^2 s_{n+1}} - \frac{2q_n^2 r_{n-1}}{s_n^2 s_{n-1}}, \\ &c_2 = \frac{-2q_n r_{n-1}^2}{s_n s_{n-1}^2} - \frac{2q_{n-1} r_{n-1} r_{n-2}}{s_{n-1}^2 s_{n-2}} + \frac{2r_{n-2}}{s_{n-1} s_{n-2}}, \\ &a_2 = -2(E+1)\frac{q_n r_{n-2}}{s_n s_{n-1} s_{n-2}} + 2(E+1)\frac{q_n q_{n-1} r_{n-1} r_{n-2}}{s_n s_{n-1}^2 s_{n-2}} + \frac{2q_n^2 r_{n-1}^2}{s_n^2 s_{n-1}^2}, \\ &\ldots . \end{aligned}$$

Remark 1 Equation (3) is similar to the stationary zero curvature equation of continuous spectral problems

$$V_x = [U, V].$$

Therefore, by the Tu scheme, we decompose equation (3) into the following form:

$$-\left(\Delta V_n^{(m)}\right)_+ U_n + \left[U_n, \left(V_n^{(m)}\right)_+\right] = \left(\Delta V_n^{(m)}\right)_- U_n - \left[U_n, \left(V_n^{(m)}\right)_-\right], \tag{6}$$

where

$$\left(V_n^{(m)}\right)_+ = \sum_{j=0}^{m} V_n \lambda^{2m} = \lambda^{2m} V_n - \left(V_n^{(m)}\right)_-. \tag{7}$$

The degree of the elements of the left-hand side of equation (6) is higher than –1, while the right-hand side is smaller than 0. Thus, the degree of both sides of equation (6) is –1, 0. Therefore, we obtain

$$-\left(\Delta V_n^{(m)}\right)_+ U_n + \left[U_n, \left(V_n^{(m)}\right)_+\right] = (\Delta a_m) h_2(-1) - s_n b_{m+1} e(0) + s_n E c_{m+1} f(0).$$

Assuming $V^{(m)}_{(n)} = (V^{(m)}_n)_+ - a_m h_1(0) + a_m h_2(0)$, a direct calculation yields

$$-\big(\Delta V^{(m)}_{(n)}\big)U_n + \big[U_n, V^{(m)}_{(n)}\big] = s_n \Delta a_m h_1(1) - Eb_m e(0) + c_m f(0).$$

The compatibility condition of the following Lax pair:

$$\psi_{n+1} = U_n \psi_n, \qquad \psi_{n,t_m} = V^{(m)}_{(n)} \psi_n$$

admits an integrable discrete hierarchy

$$\begin{pmatrix} s_n \\ q_n \\ r_n \end{pmatrix}_{t_m} = \begin{pmatrix} -s_n \Delta a_m \\ Eb_m \\ -c_m \end{pmatrix}. \tag{8}$$

Taking $m = 2$, equation (8) reduces to an integrable discrete system with three fields

$$\begin{cases} s_{n,t} = 2s_n(E^2-1)\frac{q_n r_{n-2}}{s_n s_{n-1} s_{n-2}} - 2s_n(E^2-1)\frac{q_n q_{n-1} r_{n-1} r_{n-2}}{s_n s_{n-1}^2 s_{n-2}} - 2s_n \Delta \frac{q_n^2 r_{n-1}^2}{s_n^2 s_{n-1}^2}, \\ q_{n,t} = \frac{2q_{n+2}}{s_{n+1} s_{n+2}} - \frac{2q_{n+1} r_{n+1} q_{n+2}}{s_{n+1}^2 s_{n+2}} - \frac{2q_{n+1}^2 r_n}{s_n s_{n+1}^2}, \\ r_{n,t} = \frac{2q_n r_{n-1}^2}{s_n s_{n-1}^2} + \frac{2q_{n-1} r_{n-1} r_{n-2}}{s_{n-1}^2 s_{n-2}} - \frac{2r_{n-2}}{s_{n-1} s_{n-2}}, \end{cases} \tag{9}$$

which generalizes the positive part of a result in [14] except for constants.

Assuming $m = 1$, equation (8) reduces to the much simpler integrable discrete system

$$\begin{cases} s_{n,t} = \frac{2q_{n+1} r_n}{s_{n+1}} - \frac{2q_n r_{n-1}}{s_{n-1}}, \\ q_{n,t} = \frac{2q_{n+1}}{s_{n+1}}, \\ r_{n,t} = \frac{-2r_{n-1}}{s_{n-1}}. \end{cases} \tag{10}$$

It is easy to see that there exists an explicit relation among the three fields in (10) as follows:

$$s_n = q_n r_n + f(n),$$

where $f(n)$ is an arbitrary function with respect to variable n.

Let $s_n = 1$, equation (9) becomes

$$\begin{cases} q_{n,t} = 2q_{n+2} - 2q_{n+1} r_{n+1} q_{n+2} - 2q_{n+1}^2 r_n, \\ r_{n,t} = 2q_n r_{n-1}^2 + 2q_{n-1} r_{n-1} r_{n-2} - 2r_{n-2}, \end{cases} \tag{11}$$

and

$$(E+1) q_n r_{n-2} - (E+1) q_n q_{n-1} r_{n-1} r_{n-2} - q_n^2 r_{n-1}^2 = c, \tag{12}$$

where c is a constant independent of n, t. Equation (11) is a modified integrable discrete KdV system with the constraint (12). In fact, substituting (12) into (11) yields a reduced integrable discrete mKdV system

$$\begin{cases} q_{n,t} = 2q_{n+2} - 2q_{n+1}r_{n+1}q_{n+2} - 2q_{n+1}^2 r_n, \\ r_{n,t} = 2\frac{q_{n+1}}{q_n} r_{n-1} - 2q_{n+1}r_n r_{n-1} - 2\frac{c}{q_n}. \end{cases}$$

In the following, we discuss the quasi-Hamiltonian form of equation (8). A direct computation gives

$$U_n^{-1} = \frac{1}{s_n - q_n r_n} \begin{pmatrix} \lambda^{-1} & -q_n \\ -r_n & \lambda s_n \end{pmatrix} \equiv \frac{1}{\rho_n} \begin{pmatrix} \lambda^{-1} & -q_n \\ -r_n & \lambda s_n \end{pmatrix},$$
$$W \equiv V_n U_n^{-1} = \frac{1}{\rho_n} \begin{pmatrix} \lambda^{-1}A_n - B_n r_n & -q_n A_n + \lambda s_n B_n \\ \lambda^{-1}C_n + r_n A_n & -q_n C_n - \lambda s_n A_n \end{pmatrix}.$$

Denoting $\langle a, b\rangle = \operatorname{tr}(ab)$, we find that

$$\left\langle W, \frac{\partial U_n}{\partial \lambda} \right\rangle = \frac{\lambda^{-1}A_n - r_n B_n}{\rho_n} s_n + \frac{q_n C_n + \lambda s_n A_n}{\rho_n \lambda^2},$$
$$\left\langle W, \frac{\partial U_n}{\partial s_n} \right\rangle = \frac{A_n - r_n B_n \lambda}{\rho_n}, \qquad \left\langle W, \frac{\partial U_n}{\partial q_n} \right\rangle = \frac{\lambda^{-1}C_n + r_n A_n}{\rho_n},$$
$$\left\langle W, \frac{\partial U_n}{\partial r_n} \right\rangle = \frac{\lambda s_n B_n - q_n A_n}{\rho_n}.$$

Substituting these results into the trace identity [16] yields

$$\frac{\delta}{\delta Q_n}\left(\frac{\lambda^{-1}A_n - r_n B_n}{\rho_n} s_n + \frac{q_n C_n + \lambda s_n A_n}{\rho_n \lambda^2} \right) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{pmatrix} \frac{A_n - r_n B_n \lambda}{\rho_n} \\ \frac{\lambda^{-1}C_n + r_n A_n}{\rho_n} \\ \frac{\lambda s_n B_n - q_n A_n}{\rho_n} \end{pmatrix}, \tag{13}$$

where $\frac{\delta}{\delta Q_n} = (\frac{\delta}{\delta s_n}, \frac{\delta}{\delta q_n}, \frac{\delta}{\delta r_n})^T$.

Inserting (2) into (13), one infers that

$$\frac{\delta}{\delta Q_n}\left(\frac{q_n c_m - r_n s_n b_{m+1} + 2s_n a_m}{\rho_n} \right) = (\gamma - 2m) \begin{pmatrix} \frac{a_m - r_n b_{m+1}}{\rho_n} \\ \frac{c_m + r_n a_m}{\rho_n} \\ \frac{s_n b_{m+1} - q_n a_m}{\rho_n} \end{pmatrix} \equiv (\gamma - 2m) P_m.$$

It is easy to verify from the initial values in (5) that $\gamma = 0$. Thus, we have

$$P_m = \frac{\delta H_{m+1}}{\delta Q_n}, \qquad H_{m+1} = \frac{r_n s_n b_{m+1} - q_n c_m - 2s_n a_m}{2m\rho_n}. \tag{14}$$

Therefore, equation (8) can be written in Hamiltonian form

$$\begin{pmatrix} s_n \\ q_n \\ r_n \end{pmatrix}_{t_m} = \begin{pmatrix} -s_n \Delta a_m \\ s_n b_{m+1} - q_n(E+1)a_m \\ -c_m \end{pmatrix} = JP_m = J\frac{\delta H_{m+1}}{\delta Q_n}, \tag{15}$$

where

$$J = \begin{pmatrix} -s_n r_n q_n \Delta - s_n \rho_n \Delta & 0 & -s_n r_n \Delta \\ -q_n^2 r_n E - q_n \rho_n E & 0 & s_n - q_n r_n (E+1) \\ r_n(\rho_n + q_n r_n) & -\rho_n & r_n^2 \end{pmatrix}$$

is not obviously a Hamiltonian operator.

Remark 2 Equation (15) is only a form of Hamiltonian structure. Perhaps it becomes a Hamiltonian structure by introducing various modified terms Δ_n in generating the integrable hierarchy (8); of course, if changing the modified terms Δ_n, the discrete hierarchy is also changed. As to this question, we shall discuss it as presented in [10] in the forthcoming time.

2.2 A (2 + 1)-dimensional integrable discrete hierarchy

Consider the following (2 + 1)-dimensional discrete non-isospectral linear problem [13–15]:

$$\begin{cases} E\psi_n(\lambda) = U_n \psi_n(\lambda), \\ \frac{d\psi_n(\lambda)}{dt} = \omega(\lambda)\frac{d\psi(\lambda)}{dy} + V_n^{(m)}\psi_n(\lambda), \end{cases} \tag{16}$$

where the spectral parameter $\lambda = \lambda(y,t)$ satisfies a non-isospectral condition

$$\lambda_t = \omega(\lambda)\lambda_y + \beta(\lambda), \tag{17}$$

here $\omega(\lambda)$ and $\beta(\lambda)$ are two functions to be determined. The compatibility condition of (16) along with (17) reads

$$\frac{\partial U_n}{\partial t} = \omega(\lambda)\frac{\partial U_n}{\partial y} - \beta(\lambda)\frac{\partial U_n}{\partial \lambda} + \left(\Delta V_n^{(m)}\right)U_n - \left[U_n, V_n^{(m)}\right]. \tag{18}$$

Assume

$$\begin{cases} \omega(\lambda) = \lambda^{2m}, \qquad \beta(\lambda) = \sum_{j=0}^m \beta_{2j-1}\lambda^{2j-1}, \\ A_n = \sum_{j=0}^m a_j(n,y,t)\lambda^{2m-2j}, \qquad B_n = \sum_{j=0}^m b_j(n,y,t)\lambda^{2m-2j+1}, \\ C_n = \sum_{j=0}^m c_j(n,y,t)\lambda^{2m-2j+1}. \end{cases} \tag{19}$$

The discrete stationary equation of (18) admits the following:

$$\begin{cases} s_{n,y} + s_n\Delta a_0 + r_n\Delta b_1 - q_n c_1 - r_n b_1 = 0, \\ q_{n,y} + q_n\Delta a_0 + \Delta b_0 + s_n b_1 - 2q_n a_0 - b_0 = 0, \\ r_{n,y} + s_n\Delta c_1 - r_n\Delta a_0 + 2r_n a_0 + c_0 - s_n c_1 = 0, \\ -\beta_{2m-1} + q_n\Delta c_1 - \Delta a_0 + r_n b_1 - q_n c_1 = 0, \end{cases} \tag{20}$$

$$\begin{cases} -s_n\beta_{2m-2j+3} + s_n\Delta a_j + r_n\Delta b_j - q_n c_j - r_n b_j = 0, \\ q_n\Delta a_j + \Delta b_j + s_n b_{j+1} - 2q_n a_j - b_j = 0, \\ s_n\Delta c_{j+1} - r_n\Delta a_j + 2r_n a_j - c_j - s_n c_{j+1} = 0, \\ -\beta_{2m-2j-1} + q_n\Delta c_{j+1} - \Delta a_j + r_n b_{j+1} - q_n c_{j+1} = 0, \quad j = 1,2,\dots,m-1, \end{cases} \tag{21}$$

$$\begin{cases} s_n\Delta a_m + r_n\Delta b_m - q_n c_m - r_n b_m = 0, \\ q_n\Delta a_m + \Delta b_m - 2q_n a_m - b_m = 0, \\ -r_n\Delta a_m + 2r_n a_m + c_m = 0, \\ \Delta a_m = 0. \end{cases} \tag{22}$$

Assume

$$\begin{aligned} &\bigl(V_n^{(m)}\bigr)_+ = \sum_{j=0}^{m} \lambda^{2m} V_n = \lambda^{2m} V_n - \bigl(V_n^{(m)}\bigr)_-, \\ &-\bigl(\Delta V_n^{(m)}\bigr)_+ U_n + \bigl[U_n, \bigl(V_n^{(m)}\bigr)_+\bigr] \\ &\quad = -\omega(\lambda)U_{n,y} + \beta(\lambda)U_{n,\lambda} = (q_n\Delta a_m + \Delta b_m - 2q_n a_m - b_m)e(0) \\ &\qquad + (2r_n a_m - r_n\Delta a_m + c_m)f(0) - (\Delta a_m)h_2(-1). \end{aligned} \tag{23}$$

Suppose

$$V_{(n)}^{(m)} = \bigl(V_n^{(m)}\bigr)_+ + \Delta_n = \bigl(V_n^{(m)}\bigr)_+ - a_m h_1(0) + a_m h_2(0). \tag{24}$$

Substituting (23), (24) into equation (18) replacing $V_n^{(m)}$ by $V_{(n)}^{(m)}$ gives

$$\begin{cases} s_{n,t_m} = -s_n\Delta a_m, \\ q_{n,t_m} = Eb_m - 2q_n a_m, \\ r_{n,t_m} = c_m + 4r_n a_m, \end{cases} \tag{25}$$

which is a $(2+1)$-dimensional integrable discrete hierarchy. In the following, we consider some of its reductions. Taking $b_0 = c_0 = 0, a_0 = 1$ in (20), we can deduce from (21) and (22) that

$$\begin{aligned} &b_1 = \frac{1}{s_n}(2q_n - q_{n,y}), \qquad c_1 = -\frac{2r_{n-1} + r_{n-1,y}}{s_{n-1}}, \\ &a_1 = n\beta_{2m-1} - \frac{2q_n r_{n-1}}{s_n s_{n-1}} - 2\Delta^{-1}\left(\frac{r_n q_{n+1,y}}{s_n s_{n+1}} + \frac{q_n r_{n-1,y}}{s_n s_{n-1}}\right), \end{aligned}$$

Let $m = 2$, equation (25) reduces to a new $(2+1)$-dimensional integrable discrete coupled system

$$\begin{aligned} s_{n,t_2} &= -s_n\Delta a_2 = -\beta_3 s_n + (2n+1)\beta_3\frac{r_n q_{n+1}}{s_{n+1}} + \frac{2q_{n+1}r_{n+1}r_n q_{n+2} + q_{n+1}r_{n+1}r_n q_{n+2,y}}{s_{n+1}^2 s_{n+2}} \\ &\quad - \frac{6q_{n+1}^2 r_n^2 - q_{n+1}^2 r_n q_{n,y}}{s_n s_{n+1}^2} - \frac{2q_{n+1}q_{n+2}r_n - q_n r_n q_{n+2,y}}{s_{n+1}^2 s_{n+2}} \end{aligned}$$

$$
\begin{aligned}
&+(2n-3)\beta_3\frac{q_nr_{n-1}}{s_{n-1}}+\frac{2r_{n-1}^2q_n^2+q_nr_{n-1}^2q_{n,y}}{s_ns_{n-1}^2}\\
&-\frac{6q_nq_{n-1}r_{n-1}r_{n-2}-q_nr_{n-1}q_{n-1}r_{n-2,y}}{s_{n-1}^2s_{n-2}}+\frac{2q_nr_{n-2}+q_nr_{n-2,y}}{s_{n-1}s_{n-2}}\\
&-\frac{4q_nr_{n-1}}{s_{n-1}}E^{-1}\Delta^{-1}\left(\frac{r_nq_{n+1,y}}{s_ns_{n+1}}+\frac{q_nr_{n-1,y}}{s_ns_{n-1}}\right),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
q_{n,t_2}={}&Eb_2-2q_na_2=(2n+1)\beta_3\frac{q_{n+1}}{r_{n+1}}-2\beta_3nq_n+\frac{2q_{n+1}r_{n+1}q_{n+2}+q_{n+1}r_{n+1}q_{n+2,y}}{s_{n+1}^2s_{n+2}}\\
&-\frac{6q_{n+1}^2r_n-q_{n+1}r_{n,y}}{s_ns_{n+1}^2}-\frac{2q_{n+1}q_{n+2}-q_{n+1}q_{n+2,y}}{s_{n+1}^2s_{n+2}}+2\beta_3q_n(E+1)\frac{q_nr_{n-1}}{s_ns_{n-1}}\\
&-\frac{4q_n^3r_{n-1}^2}{s_n^2s_{n-1}^2}+4q_n(E+1)\frac{q_nq_{n-1}r_{n-1}r_{n-2}}{s_ns_{n-1}^2s_{n-2}}\\
&-\frac{4q_{n+1}}{s_{n+1}}E\Delta^{-1}\left(\frac{r_nq_{n+1,y}}{s_ns_{n+1}}+\frac{q_nr_{n-1,y}}{s_ns_{n-1}}\right)-2q_nR(a_2),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
r_{n,t_2}={}&c_2+4r_na_2=(3-2n)\beta_3\frac{r_{n-1}}{s_{n-1}}-\frac{2r_{n-1}^2q_n+r_{n-1}^2q_{n,y}}{s_ns_{n-1}^2}+\frac{6q_{n-1}r_{n-1}r_{n-2}-q_{n-1}r_{n-1}r_{n-2,y}}{s_{n-1}^2s_{n-2}}\\
&-\frac{2r_{n-2}+r_{n-2,y}}{s_{n-1}s_{n-2}}+4n\beta_3r_n-4r_n(E+1)(3-2n)\frac{q_nr_{n-1}}{s_ns_{n-1}}\\
&+\frac{8r_nq_n^2r_{n-1}^2}{s_n^2s_{n-1}^2}-8r_n(E+1)\frac{q_nq_{n-1}r_{n-1}r_{n-2}}{s_ns_{n-1}^2s_{n-2}}\\
&+\frac{16q_nr_nr_{n-1}}{s_ns_{n-1}}E^{-1}\Delta^{-1}\left(\frac{r_nq_{n+1,y}}{s_ns_{n+1}}+\frac{q_nr_{n-1,y}}{s_ns_{n-1}}\right)+4r_nR(a_2).
\end{aligned}
\tag{28}
$$

When taking $\Delta a_2=0$, we may take $a_2=\alpha,\beta_3=0,s_n=1$, equations (26)-(28) reduce to

$$
\begin{cases}
q_{n,t}=Eb_2-2\alpha q_n,\\
r_{n,t}=c_2+4\alpha r_n,
\end{cases}
\tag{29}
$$

which can be written as

$$
q_nr_{n,t}-r_nq_{n,t}=6\alpha q_nr_n.
$$

If $q_n\neq 0$, we have

$$
r_n=q_ng(n,y)e^{6\alpha t},
\tag{30}
$$

where $g(n,y)\neq 0$ is an arbitrary function independent of time t. Hence, equation (29) can be reduced to a (2 + 1)-dimensional integrable discrete equation

$$
g(n,y)q_{n,t}+2\alpha g(n,y)q_n=\bar{c}_2,
$$

here $\bar{c}_2=\frac{c_2}{e^{6\alpha t}}$.

2.3 A positive and negative integrable discrete hierarchy

Based on [14], we introduce a (2 + 1)-dimensional non-isospectral linear problem

$$\begin{cases} \psi_{n+1} = U_n\psi_n, \qquad U_n = s_nh_1(1) + q_ne(0) + r_nf(0) + h_2(-1) + p_nh_2(1), \\ \frac{d\psi_n}{dt} = \omega(\lambda)\frac{d\psi_n}{dy} + V_n^{(m)}\psi_n, \qquad \lambda_t = \omega(\lambda)\lambda_y + \beta(\lambda), \end{cases} \tag{31}$$

where

$$\begin{aligned} &V_n^{(m)} = A_n^{(m)}h_1(0) + D_n^{(m)}h_2(0) + B_n^{(m)}e(0) + C_n^{(m)}f(0), \\ &A_n^{(m)} = \sum_{j=0}^{m} a_j\lambda^{2m-2j} + \sum_{j=0}^{m-1} \bar{a}_j\lambda^{-(2m-2j)}, \qquad B_n^{(m)} = \sum_{j=1}^{m} b_j\lambda^{2(m-j)+1} + \sum_{j=1}^{m} \bar{b}_j\lambda^{-(2m-2j+1)}, \\ &C_n^{(m)} = \sum_{j=1}^{m} c_j\lambda^{2(m-j)+1} + \sum_{j=1}^{m} \bar{c}_j\lambda^{-(2m-2j+1)}, \qquad D_n^{(m)} = \sum_{j=0}^{m} d_j\lambda^{2m-2j} + \sum_{j=0}^{m-1} \bar{d}_j\lambda^{-(2m-2j)}, \\ &\omega(\lambda) = \lambda^{2m} + \lambda^{-2m}, \qquad \beta(\lambda) = \sum_{j=2}^{m}\bigl(\alpha_{2j-1}\lambda^{2j-1} + \alpha_{3-2j}\lambda^{3-2j}\bigr). \end{aligned}$$

The compatibility condition of (31) has the same form as equation (18). Substituting the U_n and $V_n^{(m)}$ in (31) into equation (18), combining the operation relations of the loop algebra $\tilde{A}_1$ leads to

$$\begin{cases} s_n(E-1)a_0 = -s_{n,y}, \qquad q_nEa_0 + p_nb_1 - q_nd_0 - s_nb_1 = -q_{n,y}, \\ s_nEc_1 + r_nEd_0 - r_na_0 - p_nc_1 = -r_{n,y}, \\ q_nEc_1 + Ed_0 + p_n\Delta d_1 - r_nb_1 - d_0 = 0, \\ s_n(E-1)a_j + r_nEb_j - q_nc_j = s_n\alpha_{2m-2j+1}, \\ q_nEc_j + p_n\Delta d_j + Ed_{j-1} - r_nb_j = 2\alpha_{2m-2j+1}, \\ s_nEc_{j+1} + r_nEd_j - r_na_j - p_nc_{j+1} - c_j = 0, \\ q_nEa_j + Eb_j + p_nEb_{j+1} - s_nb_{j+1} - q_nd_j = 0, \quad j = 1, 2, \ldots, m-1, \\ s_nEc_m + r_nEd_{m-1} - p_nc_m - r_na_{m-1} - c_{m-1} = 0, \\ p_nEb_m - s_nb_m + q_nEa_{m-1} - q_nd_{m-1} = 0, \\ \Delta d_m + p_n\Delta\bar{d}_{m-1} + q_nE\bar{c}_m - r_n\bar{b}_m = \alpha_{-1}, \end{cases} \tag{32}$$

$$\begin{cases} s_n(E-1)\bar{a}_0 + r_nE\bar{b}_1 - q_n\bar{c}_1 = -s_{n,y}, \\ r_nE\bar{d}_0 - r_n\bar{a}_0 - \bar{c}_1 = -r_{n,y}, \\ q_nE\bar{c}_1 + \Delta\bar{d}_1 + p_n\Delta\bar{d}_0 - r_n\bar{b}_1 = -p_{n,y}, \\ \Delta\bar{d}_0 = -p_{n,y}, \\ s_n(E-1)\bar{a}_j + r_nE\bar{b}_{j+1} - q_n\bar{c}_{j+1} = s_n\alpha_{1-2m+2j}, \\ s_nE\bar{c}_j + r_nE\bar{d}_j - r_n\bar{a}_j - \bar{c}_{j+1} - p_n\bar{c}_j = 0, \\ q_nE\bar{a}_j + E\bar{b}_{j+1} + p_nE\bar{b}_j - s_n\bar{b}_j - q_n\bar{d}_j = 0, \\ q_nE\bar{c}_j + E\bar{d}_j - r_n\bar{b}_j + p_nE\bar{d}_{j-1} - \bar{d}_j - p_n\bar{d}_{j-1} = \alpha_{-2(m-j)-1}, \quad j = 1, 2, \ldots, m-1, \\ r_nE\bar{b}_m - q_n\bar{c}_m = s_n\alpha_{-1}, \\ E\bar{b}_m - s_n\bar{b}_{m-1} + p_n\bar{b}_{m-1} - q_n\bar{d}_{m-1} = 0. \end{cases} \tag{33}$$

The corresponding $(2+1)$-dimensional positive and negative integrable discrete hierarchy is obtained as follows:

$$\begin{cases} s_{n,t_m} = s_n(E-1)a_m + r_nEb_m - q_nc_m, \\ r_{n,t_m} = r_nEd_m - r_na_m - c_m + (s_nE-p_n)\bar{c}_m, \\ q_{n,t_m} = q_nEa_m + Eb_m - q_nd_m + (p_nE-s_n)\bar{b}_m, \\ p_{n,t_m} = q_nEc_m + p_n(E-1)d_m - r_nb_m, \quad m \geq 2. \end{cases} \tag{34}$$

$$q_nEc_m - r_nb_m = 0. \tag{35}$$

Given some initial values in terms of (32) and (33), we could obtain some explicit $(2+1)$-dimensional positive and negative integrable discrete hierarchies as long as $a_m, b_m, c_m, d_m, \bar{c}_m$, and $\bar{b}_m$ are obtained. Here we only discuss the case where $p_n = 0$. It is easy to see that (34) reduces to

$$\begin{cases} s_{n,t_m} = s_n(E-1)a_m + r_nEb_m - q_nc_m, \\ r_{n,t_m} = r_nEd_m - r_na_m - c_m + s_nE\bar{c}_m, \\ q_{n,t_m} = q_nEa_m + Eb_m - q_nd_m - s_n\bar{b}_m. \end{cases} \tag{36}$$

Equation (35) is an obvious generalization of equations (2.18) and (2.19) presented in [14]. Specially, when taking $m = 2$, (35) becomes the following:

$$\begin{cases} s_{n,t_2} = s_n(E-1)a_2 + r_nEb_2 - q_nc_2, \\ r_{n,t_2} = r_nd_2 - r_na_2 - c_2 + s_nE\bar{c}_2, \\ q_{n,t_2} = q_nEa_2 + Eb_2 - q_nd_2 - s_n\bar{b}_2, \end{cases} \tag{37}$$

and (35) turns to

$$r_nb_2 = q_nEc_2. \tag{38}$$

From (32) and (33), we can compute that

$$\begin{aligned} &a_0 = -\Delta^{-1}\frac{s_{n,y}}{s_n}, \qquad d_0 = \Delta^{-1}\left(\frac{(q_nr_n)_y - q_nr_ns_{n,y}}{\rho_n}\right), \\ &b_1 = \frac{q_{n,y}}{s_n} - \frac{q_n}{s_n}\Delta^{-1}\frac{s_{n+1,y}}{s_{n+1}} - \frac{q_n}{s_n}d_0, \\ &c_1 = -\frac{r_{n-1,y}}{s_{n-1}} - \frac{r_{n-1}}{s_{n-1}}E^{-1}\Delta^{-1}\frac{s_{n,y}}{s_n} - \frac{r_{n-1}}{s_{n-1}}d_0, \\ &(E-1)a_1 \equiv \delta_n = -\frac{r_nq_{n+1,y}}{s_ns_{n+1}} - \frac{q_nr_{n-1,y}}{s_ns_{n-1}} + \frac{r_nq_{n+1}}{s_ns_{n+1}}E^2\Delta^{-1}\frac{s_{n,y}}{s_n} + \frac{q_{n+1}r_n}{s_ns_{n+1}}Ed_0 \\ &\qquad\qquad - \frac{q_nr_{n-1}}{s_ns_{n-1}}E^{-1}\Delta^{-1}\frac{s_{n,y}}{s_n} - \frac{q_nr_{n-1}}{s_ns_{n-1}}d_0 + \alpha_{2m-1}, \\ &a_1 = \Delta^{-1}\delta_n, \qquad \bar{b}_1 = q_{n-1} - q_{n-1,y} + q_{n-1}\Delta^{-1}\frac{(\rho_n)_y}{\rho_n}, \\ &\bar{c}_1 = r_n + r_{n,y} + r_n\Delta^{-1}\frac{\rho_{n,y}}{\rho_n}, \qquad d_1 = \Delta^{-1}\frac{2\alpha_1s_n + \alpha_3q_nr_n}{\rho_n}, \end{aligned}$$

$$\begin{aligned}\Delta\bar{a}_1 = \frac{1}{\rho_n}\bigg\{&-r_ns_nq_{n-1}+r_ns_nq_{n+1,y}-r_ns_nq_{n-1}\Delta^{-1}\frac{\rho_{n,y}}{\rho_n}+s_nq_nr_{n+1}+s_nq_nr_{n+1,y}\\&+s_nq_nr_{n+1}E\Delta^{-1}\frac{\rho_{n,y}}{\rho_n}-q_n^2r_nr_{n+1,y}-q_nr_n^2q_{n-1,y}-\alpha_{-3}q_nr_n-q_nr_n\Delta(q_nr_n)\\&-q_nr_n\Delta\left(q_{n-1}r_n\Delta^{-1}\frac{\rho_{n,y}}{\rho_n}\right)+\alpha_{-1}s_n\bigg\}.\end{aligned}$$

Substituting the related results obtained above into (37), we can get one $(2+1)$-dimensional positive and one negative discrete hierarchy with three fields; here we do not write it down again.

When taking $\partial_y = 0$, the $(2+1)$-dimensional integrable discrete system (37) reduces to a $(1+1)$-dimensional discrete system as follows:

$$\begin{aligned}s_{n,t_2} = {}&\alpha_1s_n+\alpha_3(n+1)\frac{r_nq_{n+1}}{s_{n+1}}-\frac{r_nq_{n+1}}{s_{n+1}}E\Delta^{-1}\left(\frac{2\alpha_1s_n+\alpha_3q_nr_n}{\rho_n}\right)\\&+s_n\Delta\left[\frac{q_nr_{n-1}}{s_ns_{n-1}}\Delta^{-1}\left(\frac{2\alpha_1s_n+\alpha_3q_nr_n}{\rho_n}\right)\right]\\&-\alpha_3n\frac{q_nr_{n-1}}{s_{n-1}}+\frac{q_nr_{n-1}}{s_{n-1}}\Delta^{-1}\left(\frac{2\alpha_1s_n+\alpha_3q_nr_n}{\rho_n}\right),\end{aligned}\tag{39}$$

$$\begin{aligned}q_{n,t_2} = {}&\alpha_1(n+1)q_n+\frac{q_nr_nq_{n+1}}{s_ns_{n+1}}E\Delta^{-1}\left(\frac{2\alpha_1s_n+\alpha_3q_nr_n}{\rho_n}\right)-\alpha_1nq_n-\alpha_{-3}q_nr_nq_{n-1}\\&-q_nR(d_2)-s_ns_{n-1}q_{n-2}-\alpha_3(n-1)s_nq_{n-1}+s_nr_nq_{n-1}^2+s_nq_{n-1}\bar{a}_1,\end{aligned}\tag{40}$$

$$\begin{aligned}r_{n,t_2} = {}&\alpha_{-1}(n+1)r_n+\alpha_{-3}q_nr_nr_{n+1}+r_nE\big(R(d_2)\big)-\alpha_1nr_n\\&+\left(\frac{r_{n-1}}{s_{n-1}}-\frac{q_nr_nr_{n-1}}{s_ns_{n-1}}\right)\Delta^{-1}\left(\frac{2\alpha_1s_n+\alpha_3q_nr_n}{\rho_n}\right)-\alpha_3n\frac{r_{n-1}}{s_{n-1}}+s_ns_{n+1}r_{n+2}\\&-s_nr_{n+1}r_{n+2}q_{n+2}+\alpha_3(n+2)s_nr_{n+1}-s_nE(r_n\bar{a}_1),\end{aligned}\tag{41}$$

where

$$\begin{aligned}R(d_2) &= \Delta^{-1}\big\{r_ns_{n-1}q_{n-2}+r_nr_{n-1}q_{n-1}^2-(r_nq_{n-1}-q_nEr_n)\bar{a}_1\\&\qquad-q_ns_{n+1}s_{n+2}+q_nr_{n+1}r_{n+2}q_{n+2}\big\},\\ \bar{a}_1 &= \Delta^{-1}\left\{\frac{1}{\rho_n}\big[s_nq_nr_{n+1}-q_nr_n\Delta(q_nr_n)-\alpha_{-1}s_n-\alpha_{-3}q_nr_n\big]\right\}.\end{aligned}$$

When taking $s_n = 1$, equations (39)-(41) can reduce to a new modified integrable discrete system. Specially, if we take various values of the parameters $\alpha_1, \alpha_3, \alpha_{-1}$, and α_{-3}, we can get different three-field discrete systems. For example, assume $\alpha_1 = \alpha_3 = \alpha_{-1} = \alpha_{-3} = 0$, equations (39)-(41) reduce to

$$\begin{aligned}s_{n,t_2} &= -\frac{q_{n+1}r_n}{s_n}+\frac{q_nr_{n-1}}{s_{n-1}}+s_n\Delta\left(\frac{q_nr_{n-1}}{s_ns_{n-1}}\right),\\ r_{n,t_2} &= -\frac{q_nr_nr_{n-1}}{s_ns_{n-1}}+\frac{r_{n-1}}{s_{n-1}}+s_ns_{n+1}r_{n+2}-s_nr_{n+1}r_{n+2}q_{n+2}\\&\quad+r_nE\Delta^{-1}\bigg\{r_ns_{n-1}q_{n-2}+r_nr_{n-1}q_{n-1}^2+(q_nEr_n-r_nq_{n-1})\Delta^{-1}\frac{q_ns_nr_{n+1}-q_nr_n\Delta(q_nr_n)}{\rho_n}\end{aligned}$$

$$
\begin{aligned}
&\quad - q_n s_{n+1} r_{n+2} + q_n r_{n+1} r_{n+2} q_{n+2} \Big\} - s_n r_{n+2} E \Delta^{-1} \frac{s_n q_n r_{n+1} - q_n r_n \Delta(q_n r_n)}{\rho_n}, \\
q_{n,t_2} &= \frac{q_n r_n q_{n+1}}{s_n s_{n+1}} - \frac{q_{n+1}}{s_{n+1}} - s_n s_{n-1} q_{n-2} + s_n r_n q_{n-1}^2 \\
&\quad - q_n \Delta^{-1} \Big\{ r_n s_{n-1} q_{n-2} + r_n r_{n-1} q_{n-1}^2 - q_n s_{n+1} r_{n+2} + q_n r_{n+1} r_{n+2} q_{n+2} \\
&\quad + (q_n E r_n - r_n q_{n-1}) \Delta^{-1} \left(\frac{s_n q_n r_{n+1} - q_n r_n \Delta(q_n r_n)}{\rho_n} \right) \Big\} \\
&\quad + q_{n-1} s_n \Delta^{-1} \left(\frac{s_n q_n r_{n+1} - q_n r_n \Delta(q_n r_n)}{\rho_n} \right).
\end{aligned}
$$

Remark 3 If we could made use of the constrained condition (39) when deducing the above integrable discrete systems, the local integrable discrete equations could be obtained, here we do not go into that investigation again.

2.4 A discrete integrable coupling system

Obviously, the integrable discrete system (34) is an expanding integrable hierarchy, however, it is not a discrete integrable coupling. Because nonlinear integrable couplings could lead to new integrable systems different from the original ones, it has been an interesting work for us to seek new integrable couplings, specially discrete integrable couplings. In this section we could have discussed the discrete integrable couplings of the (2 + 1)-dimensional positive and negative integrable discrete hierarchy obtained in the paper; however, for the sake of simpler computations, we only want to investigate discrete nonlinear integrable couplings of the positive part of equation (35). It is remarkable that equation (35) is different from equation (8) - why is that so? Actually, we can verify that if eliminating the constrained condition (36) according to the Tu scheme, equation (35) is just equivalent to equation (8). Therefore, in the following, we apply the Tu scheme to deduce some discrete integrable couplings of the integrable discrete system (8). For this purpose, we must enlarge the Lie algebra A_1 as done in [19]: Take

$$Q = \operatorname{span}\{H_1, H_2, E, F, T_1, T_2, T_3, T_4\},$$

where

$$
H_1 = \begin{pmatrix} h_1 & 0 \\ 0 & h_1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} h_2 & 0 \\ 0 & h_2 \end{pmatrix}, \qquad E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \qquad F = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix},
$$
$$
T_1 = \begin{pmatrix} 0 & h_1 \\ 0 & h_1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & h_2 \\ 0 & h_2 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & e \\ 0 & e \end{pmatrix}, \qquad T_4 = \begin{pmatrix} 0 & f \\ 0 & f \end{pmatrix}.
$$

We denote

$$Q = \operatorname{span}\{H_1, H_2, E, F, T_1, T_2, T_3, T_4\}, \qquad Q = Q_1 \oplus Q_2,$$

here $Q_1 = \operatorname{span}\{H_1, H_2, E, F\}$, $Q_2 = \operatorname{span}\{T_1, T_2, T_3, T_4\}$. It is easy to verify that

$$[Q_2, Q_2] \subset Q_2, \qquad [Q_1, Q_2] \subset Q_2, \tag{42}$$

which implies the Lie group corresponding to the Lie algebra Q is a symmetric space [20]. Usually, in the case of a symmetric space, the obtained integrable couplings according to the Tu scheme are nonlinear. First of all, we investigate an analog of equation (8) in terms of the Tu scheme which contains an arbitrary parameter. Then we further discuss its discrete integrable coupling system. Based on the above idea, we deduce discrete integrable couplings of equation (46).

Assume

$$A=\sum_{j\geq 0} a_j\lambda^{-j},\qquad B=\sum_{j\geq 0} b_j\lambda^{-j},\qquad C=\sum_{j\geq 0} c_j\lambda^{-j},\tag{43}$$

which is different from (2). Substituting (43) into equation (3) yields

$$\begin{cases} s_n\Delta a_j+r_n\Delta b_j=q_nc_j-r_nb_j,\\ q_n\Delta c_{j+2}-\Delta a_j=-q_nc_{j+2}+r_nb_{j+2},\\ q_n\Delta a_j+\Delta b_j=s_nb_{j+2}-2q_na_j-b_j,\\ s_n\Delta c_{j+2}-r_n\Delta a_j=-s_nc_{j+2}+c_j+2q_na_j,\end{cases}\tag{44}$$

which is similar to equation (5), but the terms of odd numbers in (44) are all taken to be zero. Equation (6) has various degrees of elements of loop algebra which are $-1,0,1$, different from the case where we took (2). Hence, one infers that

$$\begin{aligned} &-(\Delta V_n)_+U_n+\bigl[U_n,(V_n)_+\bigr]\\ &\quad=(s_n\Delta a_{m+2}+r_n\Delta b_{m+2}+r_nb_{m+2})h_1(-1)\\ &\qquad+(q_n\Delta c_{m+2}-r_nb_{m+2}+q_nc_{m+2})h_2(-1)+(s_n\Delta c_{m+2}+s_nc_{m+2})f(0)-s_nb_{m+2}e(0)\\ &\quad\equiv P_n.\end{aligned}\tag{45}$$

Remark 4 Equation (45) could have terms such as $s_n\Delta c_{m+1}f(1),q_n\Delta c_{m+1}h_2(1),\ldots$, here we omit them due to equation (44).

Take

$$V_n^{(m)}=(V_n)_++(a_m+\sigma)h_2(0)-a_mh_1(0),$$

where

$$(V_n)_+=\sum_{j=0}^{m}\lambda^mV_n=\sum_{j=0}^{m}\bigl[a_j\bigl(h_1(m-j)-h_2(m-j)\bigr)+b_je(m-j+1)+c_jf(m-j+1)\bigr],$$

σ is an arbitrary constant. A direct calculation reads

$$-\bigl(\Delta V_n^{(m)}\bigr)U_n+\bigl[U_n,V_n^{(m)}\bigr]=s_n\Delta a_mh_1(1)+(c_m-\sigma r_n)f(0)+(-Eb_m+\sigma q_n)e(0)\equiv\Gamma_n.$$

Hence, the zero curvature equation

$$U_{n,t_m}-\bigl(\Delta V_n^{(m)}\bigr)U_n+\bigl[U_n,V_n^{(m)}\bigr]=0$$

admits an integrable discrete hierarchy

$$\begin{cases} s_{n,t_m} = -s_n \Delta a_m, \\ q_{n,t_m} = Eb_m - \sigma q_n, \\ r_{n,t_m} = -c_m + \sigma r_n. \end{cases} \tag{46}$$

Comparing equation (46) with equation (8), there is no difference except for the parameter σ as regards the forms. In the following, we only deduce a simple discrete integrable coupling system of equation (46). A loop algebra of the enlarging Lie algebra Q can be given by

$$\tilde{Q} = \operatorname{span}\{H_1(n), H_2(n), E(n), F(n), T_i(n), i = 1, 2, 3, 4\},$$

where

$$\begin{aligned} &H_j(n) = H_j\lambda^n, \qquad E(n) = E\lambda^n, \qquad F(n) = F\lambda^n, \\ &T_i(n) = T_i\lambda^n, \quad j = 1, 2; i = 1, 2, 3, 4. \end{aligned}$$

Applying the loop algebra $\tilde{Q}$ we introduce a Lax pair as follows:

$$\begin{cases} \bar{U}_n = s_n H_1(1) + H_2(-1) + q_n E(0) + r_n F(0) + u_1 T_1(1) + u_2 T_3(0) + u_3 T_4(0), \\ \bar{V}_n = A_n(H_1(0) - H_2(0)) + B_n E(1) + C_n F(1) + F_n T_3(1) + G_n T_4(1), \end{cases} \tag{47}$$

where

$$\begin{aligned} &A_n = \sum_{j\geq 0} a_j \lambda^{-j}, \qquad B_n = \sum_{j\geq 0} b_j \lambda^{-j}, \qquad C_n = \sum_{j\geq 0} c_j \lambda^{-j}, \\ &F_n = \sum_{j\geq 0} f_j \lambda^{-j}, \qquad G_n = \sum_{j\geq 0} g_j \lambda^{-j}. \end{aligned}$$

Solving the discrete stationary zero curvature equation

$$(\Delta \bar{V}_n) U_n = [\bar{U}_n, \bar{V}_n] \tag{48}$$

shows that the first part is equation (44), the second part is as follows:

$$\begin{cases} u_1 \Delta a_j + u_3 \Delta b_j + r_n \Delta f_j + u_3 \Delta f_j = q_n g_j - r_n f_j + u_2 c_j + u_2 g_j - u_3 b_j - u_3 f_j, \\ u_2 \Delta c_j + q_n \Delta g_j + u_2 \Delta g_j = -q_n g_j + r_n f_j - u_2 c_j - u_2 g_j + u_3 b_j + u_3 f_j, \\ u_2 \Delta a_j - \Delta f_j = s_n f_{j+2} - f_j + u_1 b_{j+2} + u_1 f_{j+2} - 2u_2 a_j, \\ -u_3 \Delta a_j + u_1 \Delta c_{j+2} + s_n \Delta g_{j+2} + u_1 \Delta g_{j+2} \\ \quad = -s_n g_{j+2} + g_j - u_1 c_{j+2} - u_1 g_{j+2} + 2u_3 g_j. \end{cases} \tag{49}$$

Equation (48) decomposes into two parts

$$-(\Delta \bar{V}_n)_+ \bar{U}_n + [\bar{U}_n, \bar{V}_n] = (\Delta \bar{V}_n)_- \bar{U}_n - [\bar{U}_n, \bar{V}_n]. \tag{50}$$

Similar to the discussion as above, one infers that

$$\begin{aligned}
&-(\Delta\bar{V}_n)_+\bar{U}_n+[\bar{U}_n,\bar{V}_n]\\
&\quad=P_n+[u_1\Delta a_{m+2}+u_3\Delta b_{m+2}+r_n\Delta f_{m+2}+2u_3f_{m+2}-q_ng_{m+2}\\
&\qquad+r_nf_{m+2}-u_2c_{m+2}-u_2g_{m+2}+u_3b_{m+2}]T_1(-1)+[u_2\Delta c_{m+2}+2q_n\Delta g_{m+2}\\
&\qquad+u_2\Delta g_{m+2}-r_nf_{m+2}+u_2c_{m+2}+u_2g_{m+2}-u_3b_{m+2}-u_3f_{m+2}]T_2(-1)\\
&\qquad-(s_nf_{m+2}+u_1b_{m+2}+u_1f_{m+2})T_3(0)\\
&\qquad+(u_1\Delta c_{m+2}+s_n\Delta g_{m+2}+2u_1\Delta g_{m+2}+s_ng_{m+2}+u_1c_{m+2})T_4(0).
\end{aligned}\tag{51}$$

Thus, the discrete zero curvature equation

$$\bar{U}_{n,t_m}-\big(\Delta\bar{V}_n^{(m)}\big)\bar{U}_n+\big[\bar{U}_n,\bar{V}_n^{(m)}\big]=0\tag{52}$$

admits a discrete integrable coupling of equation (46):

$$\begin{cases}
s_{n,t_m}=-s_n\Delta a_m,\\
q_{n,t_m}=Eb_m-\sigma q_n,\\
r_{n,t_m}=-c_m+\sigma r_n,\\
u_{1,t_m}=-u_1\Delta a_m,\\
u_{2,t_m}=-Ef_m-\sigma u_2,\\
u_{3,t_m}=u_3Ea_m+\sigma u_3.
\end{cases}\tag{53}$$

3 Applications of the second loop algebra

In the section we shall apply the Tu scheme and the second loop algebra $\bar{A}_1$ to deduce a new integrable discrete hierarchy whose quasi-Hamiltonian form will be derived from the trace identity proposed by Tu [16] when $\alpha=0$. This is a new application of the Tu scheme.

3.1 A new integrable discrete hierarchy and its reductions

Consider the following isospectral problems:

$$\psi_{n+1}=U_n\psi_n,\qquad U_n=p_nh_1(1)+\alpha h_1(0)+s_nh_2(0)+q_ne(0)+r_nf(0),\tag{54}$$

$$\frac{d}{dt}\psi_n=\big(Ah_1(1)+Dh_2(1)+Be(0)+Cf(0)\big)\psi_n,\tag{55}$$

where

$$\begin{aligned}
&A=\sum_{j\geq 0}a_j(n,t)\lambda^{-2j},\qquad B=\sum_{j\geq 0}b_j(n,t)\lambda^{-2j},\\
&C=\sum_{j\geq 0}c_j(n,t)\lambda^{-2j},\qquad D=\sum_{j\geq 0}d_j(n,t)\lambda^{-2j}.
\end{aligned}\tag{56}$$

The stationary discrete zero curvature equation

$$(\Delta V_n)U_n=[U_n,V_n]\tag{57}$$

admits

$$\begin{cases} (\lambda^2 p_n + \alpha)\Delta A\lambda^2 + r_n \Delta B\lambda^2 = q_n C\lambda^2 - r_n B\lambda^2, \\ q_n \Delta A\lambda^3 + s_n \Delta B\lambda = B\lambda(\lambda^2 p_n + \alpha) + q_n D\lambda^3 - q_n A\lambda^3 - s_n B\lambda, \\ (\lambda^3 p_n + \lambda\alpha)\Delta C + r_n \Delta D\lambda^3 = r_n A\lambda^3 + s_n C\lambda - C(\lambda^3 p_n + \alpha\lambda) - r_n D\lambda^3, \\ q_n \Delta C\lambda^2 + s_n \Delta D\lambda^2 = r_n B\lambda^2 - q_n C\lambda^2. \end{cases} \tag{58}$$

Substituting (56) into (58) yields

$$\begin{cases} p_n \Delta a_{j+1} + \alpha \Delta a_j + r_n \Delta b_j = q_n c_j - r_n b_j, \\ q_n \Delta a_{j+1} + s_n \Delta b_j = p_n b_{j+1} + \alpha b_j + q_n d_{j+1} - q_n a_{j+1} - s_n b_j, \\ p_n \Delta c_{j+1} + \alpha \Delta c_j + r_n \Delta d_{j+1} = r_n a_{j+1} + s_n c_j - p_n c_{j+1} - \alpha c_j - r_n d_{j+1}, \\ q_n \Delta c_{j+1} + s_n \Delta d_{j+1} = r_n b_{j+1} - q_n c_{j+1}. \end{cases} \tag{59}$$

Taking $a_0 = 0$, solving the above equations, we find that

$$\Delta d_0 = 0 \to d_0 = 1, \quad b_0 = -\frac{q_n}{p_n}, \quad c_0 = -\frac{r_{n-1}}{p_{n-1}}, \quad a_1 = \frac{q_n r_{n-1}}{p_n p_{n-1}}, \quad d_1 = -\frac{q_n r_{n-1}}{p_n p_{n-1}} + \delta,$$

from

$$p_n b_1 = q_n E a_1 + s_n E b_0 - q_n d_1 - \alpha b_0,$$

we have

$$b_1 = \frac{q_n r_n q_{n+1}}{p_n^2 p_{n+1}} + \frac{q_n^2 r_{n-1}}{p_n^2 p_{n-1}} - \frac{q_{n+1} s_n}{p_n p_{n+1}} + \frac{q_n(\alpha - \delta p_n)}{p_n^2},$$

$q_n E c_1 = r_n b_1 - s_n \Delta d_1 \to c_1 = \frac{q_n r_{n-1}^2}{p_n p_{n-1}^2} + \frac{q_{n-1} r_{n-1} r_{n-2}}{p_{n-2} p_{n-1}^2} + \frac{\alpha r_{n-1} - \delta p_{n-1} r_{n-1}}{p_{n-1}^2} - \frac{s_{n-1} r_{n-2}}{p_{n-1} p_{n-2}}, \ldots$ equation (57) can be decomposed into

$$-\left(\Delta V_n^{(m)}\right)_+ U_n + \left[U_n, \left(V_n^{(m)}\right)_+\right] = \left(\Delta V_n^{(m)}\right)_- U_n - \left[U_n, \left(V_n^{(m)}\right)_-\right], \tag{60}$$

where

$$\begin{aligned} \left(V_n^{(m)}\right)_+ &= \sum_{j=0}^{m} \left(a_j(n,t)h_1(m+1-j) + d_j(n,t)h_2(m+1-j) + b_j(n,t)e(m-j) + c_j f(m-j)\right) \\ &= \lambda^{2m} V - \left(V_n^{(m)}\right)_-. \end{aligned}$$

It is easy to see that the degrees of the left-hand side of (60) are higher than 1, while for the right-hand side they are smaller than 2. Therefore, the degrees of both sides are 1,2. Thus, we have

$$\begin{aligned} &-\left(\Delta V_n^{(m)}\right)_+ U_n + \left[U_n, \left(V_n^{(m)}\right)_+\right] \\ &\quad = p_n \Delta a_{m+1} h_1(1) + (q_n \Delta a_{m+1} - p_n b_{m+1} - q_n d_{m+1} + q_n a_{m+1}) e(0) \end{aligned}$$

$$
\begin{aligned}
&+ (p_n\Delta c_{m+1} + r_n\Delta d_{m+1} - r_n a_{m+1} + p_n c_{m+1} + r_n d_{m+1})f(0) \\
&= p_n\Delta a_{m+1}h_1(1) + (-s_n\Delta b_m + \alpha b_m - s_n b_m)e(0) + (-\alpha\Delta c_m + s_n c_m - \alpha c_m)f(0).
\end{aligned}
$$

Letting $V_{(n)} = (V_n^{(m)})_+ + d_{m+1}h_2(0)$, a direct calculation gives

$$
\begin{aligned}
&-(\Delta V_{(n)})U_n + [U_n, V_{(n)}] \\
&\quad = p_n\Delta a_{m+1}h_1(1) - s_n\Delta d_{m+1}h_2(0) + (q_n d_{m+1} + \alpha b_m - s_n E b_m)e(0) \\
&\qquad + (-r_n E d_{m+1} + s_n c_m - \alpha E c_m)f(0).
\end{aligned}
$$

Hence, the discrete zero curvature equation

$$
U_{n,t_m} - (\Delta V_{(n)})U_n + [U_n, V_{(n)}] = 0
$$

admits the following integrable discrete hierarchy of evolution equations:

$$
\begin{cases}
p_{n,t_m} = -p_n\Delta a_{m+1}, \\
s_{n,t_m} = s_n\Delta d_{m+1}, \\
q_{n,t_m} = p_n b_{m+1} - q_n E a_{m+1}, \\
r_{n,t_m} = r_n a_{m+1} - p_n E c_{m+1}.
\end{cases} \tag{61}
$$

When $m = 0$, we get a reduction of equation (61) which is a generalized Toda lattice equation

$$
\begin{cases}
p_{n,t} = \frac{q_n r_{n-1}}{p_{n-1}} - \frac{q_{n+1} r_n}{p_{n+1}}, \\
s_{n,t} = \frac{q_n s_n r_{n-1}}{p_n p_{n-1}} - \frac{s_n r_n q_{n+1}}{p_n p_{n+1}}, \\
q_{n,t} = \frac{q_n^2 r_{n-1}}{p_n p_{n+1}} - \frac{q_{n+1} s_n}{p_{n+1}} + \frac{q_n(\alpha - \delta p_n)}{p_n}, \\
r_{n,t} = \frac{q_n r_n r_{n-1}}{p_n p_{n-1}} - \frac{q_{n+1} r_n^2 + q_n r_n r_{n-1}}{p_n p_{n+1}} + \frac{\delta p_n r_n - \alpha r_n}{p_n} + \frac{s_n r_{n-1}}{p_{n-1}}.
\end{cases} \tag{62}
$$

When $\alpha = \delta = s_n = 0$, equation (62) reduces to a simpler nonlinear integrable discrete system

$$
\begin{cases}
p_{n,t} = \frac{q_n r_{n-1}}{p_{n-1}} - \frac{q_{n+1} r_n}{p_{n+1}}, \\
q_{n,t} = \frac{q_n^2 r_{n-1}}{p_n p_{n+1}}, \\
r_{n,t} = \frac{q_n r_n r_{n-1}}{p_n p_{n-1}} - \frac{q_{n+1} r_n^2 + q_n r_n r_{n-1}}{p_n p_{n+1}}.
\end{cases} \tag{63}
$$

In the following, we deduce a quasi-Hamiltonian form of the integrable discrete hierarchy (61). It is easy to see that

$$
\begin{aligned}
W = V_n U_n^{-1} = \frac{1}{M}\big[&(s_n A - r_n B)h_1(1) + (-q_n C + \alpha D)h_2(1) + p_n D h_2(2) \\
&+ (p_n B - q_n A)e(1) + \alpha B e(0) + s_n C f(0) - r_n D f(1)\big],
\end{aligned}
$$

where

$$
M = \alpha s_n + (p_n s_n - q_n r_n)\lambda^2.
$$

A direct calculation reads

$$\begin{aligned}
\operatorname{tr}\left(W\frac{\partial U_n}{\partial\lambda}\right) &= M^{-1}\bigl[2\lambda p_n\bigl(As_n\lambda^2 - r_nB\lambda^2\bigr) + r_n\bigl(-q_nA\lambda^3 + \alpha B\lambda + p_nB\lambda^3\bigr) \\
&\quad + q_n\bigl(s_nC\lambda - r_nD\lambda^3\bigr)\bigr], \\
\operatorname{tr}\left(W\frac{\partial U_n}{\partial p_n}\right) &= M^{-1}\lambda^2(s_nA - r_nB)\lambda^2, \\
\operatorname{tr}\left(W\frac{\partial U_n}{\partial s_n}\right) &= M^{-1}\bigl(-q_nC\lambda^2 + \alpha D\lambda^2 + p_nD\lambda^4\bigr), \\
\operatorname{tr}\left(W\frac{\partial U_n}{\partial q_n}\right) &= M^{-1}\lambda\bigl(s_nC\lambda - r_nD\lambda^3\bigr), \\
\operatorname{tr}\left(W\frac{\partial U_n}{\partial r_n}\right) &= M^{-1}\lambda\bigl(-q_nA\lambda^3 + \alpha B\lambda + p_nB\lambda^3\bigr).
\end{aligned}$$

When $\alpha = 0$, substituting the above results and (56) into the trace identity shows that

$$\begin{aligned}
&\frac{\delta}{\delta u}\left(\frac{2p_ns_na_{m+1} - p_nr_nb_{m+1} - q_nr_na_{m+1} + q_ns_nc_m - q_nr_nd_{m+1}}{p_ns_n - q_nr_n}\right) \\
&\quad = (-2m+\gamma)\begin{pmatrix} \frac{s_na_{m+1} - r_nb_{m+1}}{p_ns_n - q_nr_n} \\ \frac{p_nd_{m+1} - q_nc_m}{p_ns_n - q_nr_n} \\ \frac{s_nc_m - r_nd_{m+1}}{p_ns_n - q_nr_n} \\ \frac{p_nb_{m+1} - q_na_{m+1}}{p_ns_n - q_nr_n} \end{pmatrix}.
\end{aligned}$$

Therefore, equation (61) can be written when $\alpha = 0$:

$$\begin{aligned}
u_{t_m} &= \begin{pmatrix} p_n \\ s_n \\ q_n \\ r_n \end{pmatrix}_{t_m} = \begin{pmatrix} -p_n\Delta a_{m+1} \\ s_n\Delta d_{m+1} \\ p_nb_{m+1} - q_nEa_{m+1} \\ r_na_{m+1} - p_nEc_{m+1} \end{pmatrix} = \begin{pmatrix} -p_n\Delta a_{m+1} \\ s_n\Delta d_{m+1} \\ p_nb_{m+1} - q_nEa_{m+1} \\ r_nEd_{m+1} - s_nc_m \end{pmatrix} \\
&= \begin{pmatrix} -p_n^2\Delta & 0 & 0 & -p_nr_n\Delta \\ 0 & s_n^2\Delta & q_ns_n\Delta & 0 \\ q_np_n - q_np_nE & 0 & 0 & p_ns_n - q_nr_nE \\ 0 & -s_nr_n + s_nr_nE & -p_ns_n + q_nr_nE & 0 \end{pmatrix} \begin{pmatrix} s_na_{m+1} - r_nb_{m+1} \\ p_nd_{m+1} - q_nc_m \\ s_nc_m - r_nd_{m+1} \\ p_nb_{m+1} - q_na_{m+1} \end{pmatrix} \\
&= J\begin{pmatrix} s_na_{m+1} - r_nb_{m+1} \\ p_nd_{m+1} - q_nc_m \\ s_nc_m - r_nd_{m+1} \\ p_nb_{m+1} - q_na_{m+1} \end{pmatrix}. \qquad (64)
\end{aligned}$$

Therefore, equation (64) can be written as

$$u_{t_m} = \begin{pmatrix} p_n \\ s_n \\ q_n \\ r_n \end{pmatrix}_{t_m} = J\begin{pmatrix} s_na_{m+1} - r_nb_{m+1} \\ p_nd_{m+1} - q_nc_m \\ s_nc_m - r_nd_{m+1} \\ p_nb_{m+1} - q_na_{m+1} \end{pmatrix} = J\frac{\delta H_{m+1}}{\delta u}, \qquad (65)$$

where

$$H_{m+1} = \frac{2p_ns_na_{m+1} - p_nr_nb_{m+1} - q_nr_na_{m+1} + q_ns_nc_m - q_nr_nd_{m+1}}{(-2m+\gamma)(p_ns_n - q_nr_n)},$$

the constant γ can be determined by some initial values of equation (59).

3.2 A Darboux transformation of equation (63)

In order to conveniently deduce the Darboux transformation of equation (63), we first recall the general scheme for Darboux transformations. For spectral problems

$$\psi_{n+1} = U_n\psi, \qquad \frac{\psi_n}{dt} = V_n\psi_n,$$

one makes a transformation of the eigenfunction

$$\tilde{\psi}_n = T_n\psi_n,$$

then the above spectral problems are transformed to

$$E\tilde{\psi}_n(\lambda) = T_{n+1}U_nT_n^{-1}\tilde{\psi}_n(\lambda), \qquad \frac{d\tilde{\psi}_n}{dt} = (T_{n,t} + T_nV_n)T_n^{-1}\tilde{\psi}_n.$$

Denote

$$\tilde{U}_n(\tilde{p}_n, \tilde{q}_n) = T_{n+1}U_nT_n^{-1}, \qquad \tilde{V}_n(\tilde{p}_n, \tilde{q}_n) = (T_{n,t} + T_nV_n)T_n^{-1}.$$

We hope to construct the matrix T_n by the use of such the eigenfunctions so that $T_{n+1}U_nT_n^{-1}$ and $(T_{n,t} + T_nV_n)T_n^{-1}$ have the same structures as U_n and V_n. With this purpose, we should take various matrices T_n according to the given different spectral problems.

To obtain the Darboux transformations of equation (63), we rewrite its Lax pair as follows:

$$\psi_{n+1} = U_n\psi_n, \qquad U_n = p_nh_1(1) + q_ne(0) + r_nf(0), \tag{66}$$

$$\frac{d\psi_n(\lambda)}{dt} = V_{(n)}\psi_n, \qquad V_{(n)} = a_0h_1(1) + d_0h_2(1) + b_0e(0) + c_0f(0)) + d_1h_2(0). \tag{67}$$

We first make a transformation of the eigenfunction

$$\tilde{\psi}_n = T_n\psi_n. \tag{68}$$

By equation (68), equations (66) and (67) can be transformed into

$$\tilde{\psi}_{n+1} = T_{n+1}U_nT_n^{-1}\tilde{\psi}_n \equiv \tilde{U}_n\tilde{\psi}_n, \tag{69}$$

$$\frac{d\tilde{\psi}_n}{dt} = (T_{n,t} + T_nV_n)T_n^{-1}\tilde{\psi}_n \equiv \tilde{V}_n\tilde{\psi}_n. \tag{70}$$

Suppose $\psi_n = (\psi_{1n}, \psi_{2n})^T, \phi_n = (\phi_{1n}, \phi_{2n})^T$ are two linear independent eigenfunctions of the spectral problems (66) and (67) corresponding to the solutions p_n, q_n, r_n. We want to construct the matrix T_n by using such the two eigenfunctions so that $\tilde{U}_n$ and $\tilde{V}_n$ have the same structures as U_n and V_n. For this purpose, we take the matrix T_n as follows:

$$T_n = \begin{pmatrix} \lambda^2 + a_n & b_n\lambda \\ c_n\lambda & \lambda^2 + d_n \end{pmatrix},$$

where a_n, b_n, c_n, and d_n will be expressed by ψ_n, ϕ_n. Assume that λ_1, λ_2 are two arbitrary distinct solutions of $\det T_n = 0$. Set

$$\Phi_n = \begin{pmatrix} \phi_{1n} & \psi_{1n} \\ \phi_{2n} & \psi_{2n} \end{pmatrix}, \qquad \tilde{\Phi}_n = T_n\Phi_n,$$

then when λ takes the values λ_i ($i = 1, 2$) the two column vectors in T_n and $\tilde{\Phi}_n$ are linear dependent, which means that

$$\begin{cases} a_n = \frac{\lambda_1\lambda_2(\alpha_2(n)\lambda_1 - \alpha_1(n)\lambda_2)}{\alpha_1(n)\lambda_1 - \alpha_2(n)\lambda_2}, \\ b_n = \frac{\lambda_1^2 - \lambda_2^2}{\alpha_1(n)\lambda_1 - \alpha_2(n)\lambda_2}, \\ c_n = \frac{(\lambda_1^2 - \lambda_2^2)\alpha_1(n)\alpha_2(n)}{\alpha_2(n)\lambda_1 - \alpha_1(n)\lambda_2}, \\ d_n = \frac{\lambda_1\lambda_2(\alpha_1(n)\lambda_1 - \alpha_2(n)\lambda_2)}{\alpha_2(n)\lambda_1 - \alpha_1(n)\lambda_2}, \end{cases} \tag{71}$$

here

$$\alpha_i(n) = -\frac{\gamma_i\psi_{2n}(\lambda_i) - \phi_{2n}(\lambda_i)}{\gamma_i\psi_{1n}(\lambda_i) - \phi_{1n}(\lambda_i)}, \quad i = 1, 2,$$

where γ_i are suitable constants chosen. From (66), we can easily have

$$\alpha_i(n+1) = \frac{r_n\lambda_i}{-\lambda_i^2 p_n + \alpha_i(n)q_n\lambda_i} \equiv \frac{\nu_i(n)}{\mu_i(n)}, \quad i = 1, 2. \tag{72}$$

Thus, one infers that

$$\begin{cases} a_{n+1} = \frac{\lambda_1\lambda_2(\nu_2\mu_1\lambda_1 - \mu_2\nu_1\lambda_2)}{\nu_1\mu_2\lambda_1 - \mu_1\nu_2\lambda_2}, \\ b_{n+1} = \frac{\mu_1\mu_2(\lambda_1^2 - \lambda_2^2)}{\nu_1\mu_2\lambda_1 - \mu_1\nu_2\lambda_2}, \\ c_{n+1} = \frac{\nu_1\nu_2(\lambda_1^2 - \lambda_2^2)}{\nu_2\mu_1\lambda_1 - \mu_2\nu_1\lambda_2}, \\ d_{n+1} = \frac{\lambda_1\lambda_2(\mu_2\nu_1\lambda_1 - \mu_1\nu_2\lambda_2)}{\nu_2\mu_1\lambda_1 - \mu_2\nu_1\lambda_2}. \end{cases} \tag{73}$$

Theorem *Assume* $\tilde{U}_n = \tilde{p}_n h_1(1) + \tilde{q}_n e(0) + \tilde{r}_n f(0)$, *then we have*

$$\tilde{p}_n = p_n, \qquad \tilde{q}_n = q_n - p_n b_n, \qquad \tilde{r}_n = r_n + p_n c_{n+1}, \tag{74}$$

which is a set of new solutions of equations (66) *and* (67). *The proof of the theorem is similar to that presented in* [15, 30] *and* [31] *by using* (71)-(73), *here we omit it.*

Remark 5 Just like discussions for the applications of the loop algebra $\tilde{A}_1$, we could also investigate the integrable couplings of the integrable discrete hierarchy (61) and the associated $(2+1)$-dimensional integrable discrete systems for further applications of the second loop algebra $\bar{A}_1$, here we do not again go into details in this paper.

4 Reductions of the isospectral problem (54) and some applications

In this subsection, we shall deform the isospectral problem (54) to obtain the well-known parametrized Toda lattice equation and other new lattice integrable systems including $(2+1)$-dimensional lattice equations and their Lax pairs by applying the r-matrix method [23, 30]. In the following, we recall the notion on r-matrix. A r-matrix from g to itself is defined by [20]

$$r_k : g \to g, \quad r_k = P_{\geq k} - P_{<k}, \tag{75}$$

where $k = 0, 1$. $P_{\leq k}$ represents a projection operator from g to a Lie subalgebra

$$g_{\leq k} = \left\{ \sum_{i \geq k} u_i E^i \right\}.$$

Similarly,

$$P_{<k} = 1 - P_{\leq k}$$

stands for a projection operator from g to a Lie subalgebra $g_{<k} = \{\sum_{i<k} u_i E^i\}$. In addition, we have the fact

$$g = g_{\leq k} \oplus g_{<k}.$$

According to the general scheme in [23], we obtain two hierarchies of flows on g:

$$L_{t_q} = \left[P_{\leq k}\left(L^q\right), L\right], \quad k = 0, 1. \tag{76}$$

Equation (54) can be written as

$$E\phi_1 = \lambda p_n \psi_1 + \alpha \psi_1 + q_n \psi_2, \qquad E\psi_2 = r_n \psi_1 + s_n \psi_2. \tag{77}$$

When $s_n = 0$, we have

$$\psi_2 = E^{-1} r_n \psi_1. \tag{78}$$

Substituting (78) into the first equation of (77) yields

$$E p_n^{-1} \psi_1 = \lambda \psi_1 + \alpha p_n^{-1} + \frac{q_n}{p_n} r_{n-1} E^{-1} \psi_1,$$

which can be simplified to

$$u_n E\psi + v_n \psi + w_n E^{-1} \psi = \lambda \psi, \tag{79}$$

where $\psi = \psi_1, u_n = p_{n+1}^{-1}, v_n = -\alpha p_n^1, w_n = -\frac{q_n r_{n-1}}{p_n}$.

Denote

$$L = u_n E + v_n + w_n E^{-1}. \tag{80}$$

It can be verified that all the operators like (80) consist of a Lie algebra g if $u_n = 1$ and if equipped with a commutator

$$[L_1, L_2] = L_1L_2 - L_2L_1. \tag{81}$$

Now we take $k = 0, q = 1$; equation (76) gives rise to the simplest lattice system

$$\begin{cases} w_{n,t_1} = (1 - E^{-1})v_n, \\ v_{n,t_1} = (E - 1)w_n. \end{cases}$$

Taking $s_n = -E$, the second equation of (77) gives

$$\psi_2 = \frac{1}{2}r_{n-1}E^{-1}\psi_1. \tag{82}$$

Inserting (82) into

$$E^2\psi_1 = \lambda p_n\psi_1 + \alpha E\psi_1 + q_nr_n\psi_1 + q_ns_n\psi_2$$

leads to the following form:

$$E^2\psi = u_nE\psi + \alpha E\psi + v_n\psi + w_nE^{-1}\psi, \tag{83}$$

where $\psi = \psi_1$.

Denote

$$L = E^2 - u_nE - v_n - w_nE^{-1},$$

then (83) becomes

$$E^{-1}L\psi = \alpha\psi. \tag{84}$$

Denote

$$\bar{L} = E^{-1}L = E + u_n + v_nE^{-1} + w_nE^{-2},$$

then equation (84) becomes

$$\bar{L}\psi = \alpha\psi. \tag{85}$$

If we regard the parameter α as a spectral parameter and let $\alpha = \lambda$, then (85) is just right an isospectral problem of the spatial part

$$\bar{L}\psi = \lambda\psi. \tag{86}$$

When $k = 0$, equation (76) reduces to

$$\bar{L}_{t_q} = \left[P_{\leq 0}\left(\bar{L}^q\right), \bar{L}\right]. \tag{87}$$

Set $q = 1$, it is easy to calculate that

$$\begin{cases} u_{n,t_1} = (E-1)v_n, \\ v_{n,t_1} = (E-1)w_n + v_n(1-E^{-1})u_{n-1}, \\ w_{n,t_1} = w_n(u_n - u_{n-2}), \end{cases} \tag{88}$$

which is a three-field integrable system. When taking $w_n = 0$, equation (88) reduces to the well-known reparameterized Toda lattice equation:

$$u_{n,t_1} = (E-1)v_n, \qquad v_{n,t_1} = v_n\big(1-E^{-1}\big)u_{n-1}.$$

Taking $q = 2$, one infers that

$$P_{\geq 0}\big(\bar{L}^2\big) = E^2 + (u_n + u_{n+1})E + u_n^2 + v_n + v_{n+1}.$$

Equation (87) admits the following new three-field lattice system:

$$\begin{cases} u_{n,t_2} = w_{n+2} - w_n + (E-1)[v_n(u_n + u_{n+1})], \\ v_{n,t_2} = w_{n+1}(u_n + u_{n+1}) - w_n(u_{n-1} + u_{n-2}) + v_n(E-1)(v_{n-1} + v_n + u_{n-1}^2), \\ w_{n,t_2} = w_n[(E^2-1)(v_{n-1} + v_{n-2} + u_{n-2}^2)]. \end{cases} \tag{89}$$

In the following, we shall deduce the Lax pairs of the lattice systems (88) and (89). Set $\psi_1 = E^{-2}\psi$, $\psi_2 = E^{-1}\psi$, $\psi_3 = \psi$, then the spectral equation (86) gives

$$\begin{cases} E\psi_3 = (\lambda - u_n)\psi_3 - v_n\psi_2 - w_n\psi_1, \\ E\psi_1 = \psi_2, \\ E\psi_2 = \psi_3, \end{cases}$$

which is equivalent to the following spectral problem:

$$\Psi_{n+1} = U\Psi, \tag{90}$$

where $\Psi = (\psi_1\psi_2, \psi_3)^T$, $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -w_n & -v_n & \lambda-u_n \end{pmatrix}$.

When $q = 1$, we represent $A_1 = P_{\geq 0}(\bar{L})$, one infers that

$$A_1\psi_1 = \psi_2 + u_{n-2}\psi_2,$$

$$A_1\psi_2 = \psi_3 + u_{n-1}\psi_2,$$

$$A_1\psi_3 = \lambda\psi_3 - v_n\psi_2 - w_n\psi_1,$$

which conclude that the temporal part of the Lax pair for equation (88) is presented as

$$V_1 = \begin{pmatrix} u_{n-2} & 1 & 0 \\ 0 & u_{n-1} & 1 \\ -w_n & -v_n & \lambda \end{pmatrix}. \tag{91}$$

As for $q = 2$, similarly we can obtain the time part of the Lax pair for equation (89) as follows:

$$V_2 = \begin{pmatrix} v_{n-2} + v_{n-1} + u_{n-2}^2 & u_{n-1} + u_{n-2} & 1 \\ -w_n & v_{n-1} + u_{n-1}^2 & \lambda + u_{n-1} \\ -(\lambda + u_n)w_n & -\lambda v_n - u_n v_n - w_{n+1} & \lambda^2 + v_n \end{pmatrix}. \tag{92}$$

4.1 (2 + 1)-Dimensional lattice systems and Lax pairs

In the following, we want to deduce $(2+1)$-dimensional integrable lattice equations which correspond to the $(1+1)$-dimensional lattice systems (88) and (89). Set

$$\nabla C_i = \sum_{i\geq j} a_j(n)E^j, \quad i = 1, 2, \ldots, \tag{93}$$

where $a_j(n)$ are to be determined from the following equation via the recurrent procedure [21]:

$$[\nabla C_i, \tilde{L} - \partial_y] = 0, \tag{94}$$

then we have the following $(2+1)$-dimensional lattice hierarchy:

$$\tilde{L}_{t_i} = \left[P_{\geq 0}(\nabla C_i), \tilde{L} - \partial_y\right], \tag{95}$$

where $P_{\geq 0}(\nabla C_i) = \sum_{j\geq 0} a_j(n)E^j$.

We take

$$\tilde{L} = \bar{L} - \partial_y, \qquad \bar{L} = E + u_n + v_n E^{-1} + w_n E^{-2}, \qquad \nabla C_1 = a_0(n) + a_1(n)E + a_2(n)E^2,$$

then from (94) we have

$$a_1(n) = u_n + u_{n+1}, \qquad a_0 = Hu_n + v_n + v_{n+1} + Hu_{ny}, \qquad a_2(n) = 1,$$

where $H = (E-1)^{-1}(E+1)$. Therefore, equation (95) admits the following $(2+1)$-dimensional lattice system:

$$\begin{cases} u_{n,t_1} = v_{n+1}(u_n + u_{n+1}) + w_{n+2} - w_n - v_n(u_n + u_{n-1}) \\ \qquad\qquad + Hu_{ny} + v_{ny} + v_{n+1,y} + Hu_{nyy}, \\ v_{n,t_1} = v_n(E-1)[u_{n-1} + (E+1)v_{n-1} + u_{n-1,y}] + (E-1)(u_n w_n + w_n u_{n-1}), \\ w_{n,t_1} = w_n(E-1)(E^{-1} + E^{-2})[Hu_n + v_n v_{n+1} + Hu_{ny}]. \end{cases} \tag{96}$$

Similar to the previous calculations, we obtain a Lax pair of equation (96) as follows:

$$\begin{cases} U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -v_n & -w_n & \lambda - u_n + \partial_y \end{pmatrix}, \\ \tilde{V}_1 = \begin{pmatrix} a_0(n-1) - v_n & -w_n & a_1(n-1) + \lambda - u_n + \partial_y \\ a_1(n-2) & a_0(n-2) & 1 \\ a_1(n)v_n - w_{n+1} & -a_1(n)w_n & (a_1(n)+1)(\lambda - u_{n+1} + \partial_y) - v_{n+1} \end{pmatrix}. \end{cases} \tag{97}$$

According to [23, 30], we can also derive a (2 + 1)-dimensional lattice system corresponding to the (1 + 1)-dimensional lattice equation (89) as follows:

$$\begin{cases} u_{n,t_2} = (E-1)v_n a_1(n-1) + a_2(n)w_{n+2} - w_n a_2(n-2) + a_{0y}, \\ v_{n,t_2} = v_n(E-1)a_0(n-1) + a_1(n)w_{n+1} - w_n a_1(n-2), \\ w_{n,t_2} = w_n(a_0(n) - a_0(n-2)), \end{cases} \tag{98}$$

where

$$\begin{aligned} a_2(n) &= u_n + u_{n+1} + u_{n+2}, \\ a_1(n) &= v_n + v_{n+1} + u_{n+1}(E-1)^{-1}u_{n+2} - \big[(E-1)^{-1}u_n\big](E-1)^{-1}(E+1)u_n \\ &\quad + (E-1)^{-1}(u_n + u_{n+1} + u_{n+2})_y, \\ a_0(n) &= u_n(E-1)^{-1}a_1(n) + w_n + w_{n+1} + w_{n+2} + u_{n+1}v_{n+1} + u_n v_n \\ &\quad + (E-1)^{-1}(u_{n+1} + v_{n+2} + u_n v_{n+2} - v_n u_{n+1} - v_n u_{n-1}) + (E-1)^{-1}\big(a_1(n)\big)_y. \end{aligned}$$

It is easy to obtain the Lax pair of equations (98)

$$\begin{aligned} U &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -v_n & -w_n & \lambda - u_n + \partial_y \end{pmatrix}, \\ \tilde{V}_2 &= \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ a_1(n-2) & a_0(n-2) - w_n & a_2(n) + \lambda - u_n + \partial_y \\ V_{31} & V_{32} & V_{33} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} V_{11} &= a_0(n-1) + a_2(n-1)v_n - w_{n+1} - (\lambda - u_{n+1} + \partial_y)v_n, \\ V_{12} &= a_2(n-1)w_n - (\lambda - u_{n+1} + \partial_y)w_n, \\ V_{13} &= a_1(n-1) + a_2(n-2)(\lambda - u_n + \partial_y) + (\lambda - u_{n+1} + \partial_y)^2 - v_{n+1}, \\ V_{31} &= a_1(n)v_n - a_2(n)(\lambda - u_{n+1} + \partial_y)v_n - a_2(n)w_{n+1} - (\lambda - u_{n+2} + \partial_y)(\lambda - u_{n+1} + \partial_y)v_n \\ &\quad + v_n v_{n+2} - (\lambda - u_{n+2} + \partial_y)w_{n+1}, \\ V_{32} &= -a_1(n)w_n - a_2(n)(\lambda - u_{n+1} + \partial_y)w_n + w_n v_{n+2} - (\lambda - u_{n+2} + \partial_y)(\lambda - u_{n+1} + \partial_y)w_n, \\ V_{33} &= a_0(n) + a_1(n)(\lambda - u_n + \partial_y) + a_2(n)(\lambda - u_{n+1} + \partial_y)(\lambda - u_n + \partial_y) - a_2(n)v_{n+1} \\ &\quad - w_{n+2} - v_{n+2}(\lambda - u_n + \partial_y) + (\lambda - u_{n+2} + \partial_y)(\lambda - u_{n+1} + \partial_y)(\lambda - u_n + \partial_y) \\ &\quad - (\lambda - u_{n+2} + \partial_y)v_{n+1}. \end{aligned}$$

Remark 6 We have obtained the Lax pairs of equations (88), (89), (96), and (98), from which we could investigate their infinite conservation laws and different Darboux transformations just like in the ways presented before. Hence we do not want to go into a discussion of them again in this paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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