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New delay-range-dependent exponential stability criteria for certain neutral differential equations with interval discrete and distributed time-varying delays

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Abstract

In this research, we investigate the problem of delay-dependent exponential stability analysis for certain neutral differential equations with discrete and distributed time-varying delays. The time-varying delays are continuous functions belonging to the given interval delays, which mean that the lower and upper bounds for the time-varying delays are available. The restrictions on the derivative of interval time-varying delays are needed. Based on a class of novel augmented Lyapunov-Krasovskii functionals, a model transformation, the decomposition technique of constant coefficients, the Leibniz-Newton formula, and utilization of a zero equation, new delay-range-dependent exponential stability criteria are derived in terms of the linear matrix inequality (LMI) for the equations considered. Numerical examples suggest for the results given to illustrate the effectiveness and improvement over some existing methods.

1 Introduction

The neutral differential equation is a retarded system that often appears in many scientific and engineering fields such as aircraft, chemical and process control systems, and biological systems [1–3]. The problem of various stability analyses for dynamical systems with state delays has been intensively studied in the past years by several researchers in mathematics [1–37]. However, delay-dependent stability criteria for neutral differential equations have been attracting the attention of several researchers. Delay-dependent stability criteria make use of information on the length of delays. A certain neutral differential equation (CNDE) with constant delays is of the form

$$\frac{d}{dt}\big[x(t) + px(t-\tau)\big] = -ax(t) + b\tanh x(t-\sigma), \quad t \geq 0, \tag{1.1}$$

where a, b, τ, σ are positive real constants and $|p| < 1$. For each solution $x(t)$ of (1.1), we assume the initial condition

$$x_0(t) = \phi(t), \quad t \in [-r, 0],$$

where $\phi \in C([-r,0];R)$ denotes the space of all continuous vector functions mapping $[-r,0]$ into R with $r = \max\{\tau,\sigma\}$. Over the past decades, the problem of asymptotic stability analysis for (1.1) has been discussed in [14, 21, 29, 30, 34, 35] by using several model transformation methods and the Lyapunov-Krasovskii functional approach, while the problem of exponential stability analysis has been studied with the use of the model transformation technique and the Lyapunov-Krasovskii functional approach in [34]. In [7, 8, 23], the authors studied the problem of exponential stability analysis for CNDE with time-varying delays of the form

$$\frac{d}{dt}\bigl[x(t) + px\bigl(t - \tau(t)\bigr)\bigr] = -ax(t) + b\tanh x\bigl(t - \sigma(t)\bigr), \quad t \geq 0, \tag{1.2}$$

where a, b are positive real constants and $|p| < 1$. $\tau(t)$ and $\sigma(t)$ are neutral and discrete time-varying delays, respectively,

$$0 \leq \tau(t) \leq \tau, \qquad \dot{\tau}(t) < \tau_d,$$
$$0 \leq \sigma(t) \leq \sigma, \qquad \dot{\sigma}(t) < \sigma_d,$$

where τ, σ, τ_d, and σ_d are given positive real constants. For each solution $x(t)$ of (1.2), we assume the initial condition

$$x_0(t) = \phi(t), \quad t \in [-r,0],$$

where $\phi \in C([-r,0];R)$. In [7], the results are derived without the use of the model transformation method and the bounding technique, while the authors have used the model transformation method, radially unboundedness, and the Lyapunov-Krasovskii functional approach in [23]. Stability analysis of uncertain neutral stochastic systems with time-varying delays has received the attention of a lot of theoreticians and engineers in this field over the last few decades [9–13]. Moreover, the authors have studied the problem of stability for systems with discrete and distributed delays such as [36], which presented some stability conditions for uncertain neutral systems with discrete and distributed delays. The robust stability of uncertain dynamical systems with discrete and distributed delays has been studied in [12, 17, 26, 27].

This research presents new criteria based on new methods with mixed model transformation techniques. We investigate the problem of exponential stability criteria for CNDE with discrete and distributed time-varying delays. The time-varying delays are assumed to belong to the given lower and upper bound delays and restrictions on the derivative of the time-varying delays are needed. Based on the combination of a mixed model transformation, decomposition technique of constant coefficients, utilization of a zero equation, and a new Lyapunov-Krasovskii functional, sufficient conditions for exponential stability are obtained and formulated in terms of LMIs for the systems. Finally, numerical examples suggest that the proposed criteria are effective and an improvement over previous ones.

2 Problem formulation and preliminaries

Consider the CNDE with mixed interval time-varying delays of the form

$$\frac{d}{dt}\bigl[x(t) + px\bigl(t - \tau(t)\bigr)\bigr] = -ax(t) + b\tanh x\bigl(t - \sigma(t)\bigr) + c\int_{t-\rho(t)}^{t} x(s)\,ds, \quad t \geq 0, \tag{2.1}$$

where a, b, c are positive real constants and $|p| < 1$. $\tau(t)$, $\sigma(t)$, and $\rho(t)$ are neutral, discrete, and distributed interval time-varying delays, respectively,

$$0 \le \tau_1 \le \tau(t) \le \tau_2, \qquad \dot{\tau}(t) \le \tau_d < \infty, \tag{2.2}$$

$$0 \le \sigma_1 \le \sigma(t) \le \sigma_2, \qquad \dot{\sigma}(t) \le \sigma_d < \infty, \tag{2.3}$$

$$0 \le \rho_1 \le \rho(t) \le \rho_2, \tag{2.4}$$

where τ_1, τ_2, σ_1, σ_2, ρ_1, ρ_2, τ_d, and σ_d are given positive real constants. For each solution $x(t)$ of (2.1), we assume the initial condition

$$x_0(t) = \phi(t), \quad t \in [-\omega, 0],$$

where $\phi \in C([-\omega, 0]; R)$ and $\omega = \max\{\tau_2, \sigma_2, \rho_2\}$.

Definition 2.1 ([20]) Equation (2.1) is *exponentially stable*, if there exist positive real constants α, β such that, for each $\phi(t) \in C([-\omega, 0], R)$, the solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \le \beta \|\phi\| e^{-\alpha t}, \quad t \ge 0.$$

Lemma 2.2 ([16] (*Jensen's inequality*)) *For any symmetric positive definite matrix Q, positive real number h, and vector function $\dot{x}(t) : [-h, 0] \to R^n$ the following integral is well defined*:

$$-h \int_{-h}^{0} \dot{x}^T(s+t) Q \dot{x}(s+t)\, ds \le -\left(\int_{-h}^{0} \dot{x}(s+t)\, ds\right)^T Q \left(\int_{-h}^{0} \dot{x}(s+t)\, ds\right).$$

Lemma 2.3 *For any constant symmetric positive definite matrix $Q \in R^{n\times n}$, $h(t)$ a discrete time-varying delay with* (2.4), *the vector function $\omega : [-h_2, 0] \to R^n$ such that the integrations concerned are well defined, we have*

$$\begin{aligned} &-[h_2 - h_1] \int_{-h_2}^{-h_1} \omega^T(s) Q \omega(s)\, ds \\ &\quad \le -\int_{-h(t)}^{-h_1} \omega^T(s)\, ds Q \int_{-h(t)}^{-h_1} \omega(s)\, ds - \int_{-h_2}^{-h(t)} \omega^T(s)\, ds Q \int_{-h_2}^{-h(t)} \omega(s)\, ds. \end{aligned}$$

Proof It is easy to see that

$$\begin{aligned} &[h_2 - h_1] \int_{-h_2}^{-h_1} \omega^T(s) Q \omega(s)\, ds \\ &\quad = [h_2 - h_1] \int_{-h(t)}^{-h_1} \omega^T(s) Q \omega(s)\, ds + [h_2 - h_1] \int_{-h_2}^{-h(t)} \omega^T(s) Q \omega(s)\, ds \\ &\quad \ge \bigl[h(t) - h_1\bigr] \int_{-h(t)}^{-h_1} \omega^T(s) Q \omega(s)\, ds + \bigl[h_2 - h(t)\bigr] \int_{-h_2}^{-h(t)} \omega^T(s) Q \omega(s)\, ds \\ &\quad = \frac{1}{2} \int_{-h(t)}^{-h_1} \int_{-h(t)}^{-h_1} \omega^T(s) Q \omega(s) + \omega^T(\xi) Q \omega(\xi)\, ds\, d\xi \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{-h_2}^{-h(t)}\int_{-h_2}^{-h(t)}\omega^T(s)Q\omega(s)+\omega^T(\xi)Q\omega(\xi)\,ds\,d\xi \\
&\geq \frac{1}{2}\int_{-h(t)}^{-h_1}\int_{-h(t)}^{-h_1}2\omega^T(s){Q^{1/2}}^T Q^{1/2}\omega(\xi)\,ds\,d\xi \\
&\quad+\frac{1}{2}\int_{-h_2}^{-h(t)}\int_{-h_2}^{-h(t)}2\omega^T(s){Q^{1/2}}^T Q^{1/2}\omega(\xi)\,ds\,d\xi \\
&=\int_{-h(t)}^{-h_1}\omega^T(s)\,dsQ\int_{-h(t)}^{-h_1}\omega(s)\,ds+\int_{-h_2}^{-h(t)}\omega^T(s)\,dsQ\int_{-h_2}^{-h(t)}\omega(s)\,ds.
\end{aligned}
$$

This completes the proof. □

Remark 2.4 In Lemma 2.3, we have modified the method of [19].

3 Main results

In this section, we investigate the exponential stability problem for equation (2.1) with interval time-varying delays satisfying (2.2)-(2.4). From the model transformation method, we have the Leibniz-Newton formula of the form

$$0=x(t)-x\bigl(t-\tau(t)\bigr)-\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds, \tag{3.1}$$

$$0=x(t)-x\bigl(t-\gamma\tau(t)\bigr)-\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds, \tag{3.2}$$

where γ is a given positive real constant. We utilize the zero equations and obtain

$$0=r_1x(t)-r_1x\bigl(t-\tau(t)\bigr)-r_1\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds, \tag{3.3}$$

$$0=r_2x(t)-r_2x\bigl(t-\gamma\tau(t)\bigr)-r_2\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds, \tag{3.4}$$

where $r_1, r_2 \in R$ will be chosen to guarantee the exponential stability of equation (2.1). By (3.1)-(3.4), equation (2.1) can be represented by the form

$$
\begin{aligned}
&\frac{d}{dt}\left[p_1x(t)+p_2x\bigl(t-\tau(t)\bigr)+x\bigl(t-\gamma\tau(t)\bigr)+\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds-p_1\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\right] \\
&\quad=-(a_1-r_1-r_2)x(t)-(a_2+r_1)x\bigl(t-\tau(t)\bigr)-(a_2+r_1)\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds \\
&\qquad-r_2x\bigl(t-\gamma\tau(t)\bigr)-r_2\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds \\
&\qquad+b\tanh x\bigl(t-\sigma(t)\bigr)+c\int_{t-\rho(t)}^{t}x(s)\,ds.
\end{aligned} \tag{3.5}
$$

For convenience, we define a new variable,

$$D(t)=p_1x(t)+p_2x\bigl(t-\tau(t)\bigr)+x\bigl(t-\gamma\tau(t)\bigr)+\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds-p_1\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds. \tag{3.6}$$

Rewrite equation (3.5) in the following equation:

$$\begin{aligned}\dot{D}(t) &= -(a_1 - r_1 - r_2)x(t) - (a_2 + r_1)x\big(t - \tau(t)\big) - (a_2 + r_1)\int_{t-\tau(t)}^{t} \dot{x}(s)\,ds \\ &\quad - r_2 x\big(t - \gamma\tau(t)\big) - r_2\int_{t-\gamma\tau(t)}^{t} \dot{x}(s)\,ds \\ &\quad + b\tanh x\big(t - \sigma(t)\big) + c\int_{t-\rho(t)}^{t} x(s)\,ds. \end{aligned} \tag{3.7}$$

We introduce the following notations for later use:

$$\Sigma = [\Omega_{(i,j)}]_{25\times 25}, \tag{3.8}$$

where $\Omega_{(i,j)} = \Omega_{(j,i)}$,

$$\begin{aligned}
\Omega_{(1,1)} &= 2k_1\alpha - 2q_1, \\
\Omega_{(1,2)} &= q_1p_1 - q_2 + m_1 + n_1 + k_1r_2 + k_1r_1 - k_1a_1, \\
\Omega_{(1,3)} &= q_1p_2 - q_3 - m_1 - k_1a_2 - k_1r_1, \\
\Omega_{(1,4)} &= -q_1p_1 - q_4 - m_1 - k_1a_2 - k_1r_1, \\
\Omega_{(1,5)} &= q_1 - q_5 - n_1 - k_1r_2, \\
\Omega_{(1,6)} &= q_1 - q_6 - n_1 - k_1r_2, \\
\Omega_{(1,7)} &= \Omega_{(1,8)} = \Omega_{(1,9)} = \Omega_{(1,10)} = 0, \\
\Omega_{(1,11)} &= k_1b, \\
\Omega_{(1,12)} &= \Omega_{(1,13)} = \Omega_{(1,14)} = \Omega_{(1,15)} = \Omega_{(1,16)} = \Omega_{(1,17)} = \Omega_{(1,18)} = \Omega_{(1,19)} = \Omega_{(1,20)} = \Omega_{(1,21)} = 0, \\
\Omega_{(1,22)} &= -m_9, \\
\Omega_{(1,23)} &= \Omega_{(1,24)} = 0, \\
\Omega_{(1,25)} &= k_1c, \\
\Omega_{(2,2)} &= 2q_2p_1 + k_2 + k_3 + k_4 + k_5 + k_6\tau_2^2 + k_7\gamma^2\tau_2^2 + k_8 + k_9\sigma_2^2 + 2m_2 + 2n_2 \\
&\quad + w_1 + w_2 + w_5\tau_1^2 + w_6\gamma^2\tau_1^2 + w_7(\tau_2 - \tau_1)^2 + w_8\gamma^2(\tau_2 - \tau_1)^2 + w_{10}(\sigma_2 - \sigma_1)^2 \\
&\quad + k_{10} + \omega(\rho_2 - \rho_1)^2, \\
\Omega_{(2,3)} &= q_2p_2 + q_3p_1 + m_3 + n_3 - m_2, \\
\Omega_{(2,4)} &= -q_2p_1 + q_4p_1 + m_5 + n_5 - m_2, \\
\Omega_{(2,5)} &= q_2 + q_5p_1 + m_4 + n_4 - n_2, \\
\Omega_{(2,6)} &= q_2 + q_6p_1 + m_6 + n_6 - n_2, \\
\Omega_{(2,7)} &= \Omega_{(2,8)} = \Omega_{(2,9)} = \Omega_{(2,10)} = 0, \\
\Omega_{(2,11)} &= m_7 + n_7, \\
\Omega_{(2,12)} &= m_8 + n_8,
\end{aligned}$$

$$\Omega_{(2,13)} = -q_7 a_1,$$

$$\Omega_{(2,14)} = q_7 r_1 + q_7 r_2 - q_7 a_1,$$

$$\Omega_{(2,15)} = \Omega_{(2,16)} = \Omega_{(2,17)} = \Omega_{(2,18)} = \Omega_{(2,19)} = \Omega_{(2,20)} = \Omega_{(2,21)} = 0,$$

$$\Omega_{(2,22)} = m_9 + n_9,$$

$$\Omega_{(2,23)} = \Omega_{(2,24)} = \Omega_{(2,25)} = 0,$$

$$\Omega_{(3,3)} = 2q_3 p_2 - k_3 e^{-2\alpha\tau_2} + k_3 t_d - 2m_3,$$

$$\Omega_{(3,4)} = -q_3 p_1 + q_4 p_2 - m_5 - m_3,$$

$$\Omega_{(3,5)} = q_3 + q_5 p_2 - m_4 - n_3,$$

$$\Omega_{(3,6)} = q_3 + q_6 p_2 - m_6 - n_3,$$

$$\Omega_{(3,7)} = \Omega_{(3,8)} = \Omega_{(3,9)} = \Omega_{(3,10)} = 0,$$

$$\Omega_{(3,11)} = -m_7,$$

$$\Omega_{(3,12)} = -m_8,$$

$$\Omega_{(3,13)} = -q_7 a_2,$$

$$\Omega_{(3,14)} = \Omega_{(3,15)} = \Omega_{(3,16)} = \Omega_{(3,17)} = \Omega_{(3,18)} = \Omega_{(3,19)} = \Omega_{(3,20)} = \Omega_{(3,21)} = 0,$$

$$\Omega_{(3,22)} = -m_9,$$

$$\Omega_{(3,23)} = \Omega_{(3,24)} = \Omega_{(3,25)} = 0,$$

$$\Omega_{(4,4)} = -2q_4 p_1 - 2m_5,$$

$$\Omega_{(4,5)} = q_4 - q_5 p_1 - m_4 - n_5,$$

$$\Omega_{(4,6)} = q_4 - q_6 p_1 - m_6 - n_5,$$

$$\Omega_{(4,7)} = \Omega_{(4,8)} = \Omega_{(4,9)} = \Omega_{(4,10)} = 0,$$

$$\Omega_{(4,11)} = -m_7,$$

$$\Omega_{(4,12)} = -m_8,$$

$$\Omega_{(4,13)} = -q_7 a_2,$$

$$\Omega_{(4,14)} = \Omega_{(4,15)} = \Omega_{(4,16)} = \Omega_{(4,17)} = \Omega_{(4,18)} = \Omega_{(4,19)} = \Omega_{(4,20)} = \Omega_{(4,21)} = 0,$$

$$\Omega_{(4,22)} = -m_9,$$

$$\Omega_{(4,23)} = \Omega_{(4,24)} = \Omega_{(4,25)} = 0,$$

$$\Omega_{(5,5)} = 2q_5 - k_5 e^{-2\alpha\gamma\tau_2} + k_5 t_d - n_4,$$

$$\Omega_{(5,6)} = q_5 + q_6 - n_6 - n_4,$$

$$\Omega_{(5,7)} = \Omega_{(5,8)} = \Omega_{(5,9)} = \Omega_{(5,10)} = 0,$$

$$\Omega_{(5,11)} = -n_7,$$

$$\Omega_{(5,12)} = -n_8,$$

$$\Omega_{(5,13)} = 0,$$

$$\Omega_{(5,14)} = \Omega_{(5,15)} = \Omega_{(5,16)} = \Omega_{(5,17)} = \Omega_{(5,18)} = \Omega_{(5,19)} = \Omega_{(5,20)} = \Omega_{(5,21)} = 0,$$

$$\Omega_{(5,22)} = -n_9,$$

$$\Omega_{(5,23)} = \Omega_{(5,24)} = \Omega_{(5,25)} = 0,$$

$$\Omega_{(6,6)} = 2q_6 - 2n_6,$$

$$\Omega_{(6,7)} = \Omega_{(6,8)} = \Omega_{(6,9)} = \Omega_{(6,10)} = 0,$$

$$\Omega_{(6,11)} = -n_7,$$

$$\Omega_{(6,12)} = -n_8,$$

$$\Omega_{(6,13)} = 0,$$

$$\Omega_{(6,14)} = \Omega_{(6,15)} = \Omega_{(6,16)} = \Omega_{(6,17)} = \Omega_{(6,18)} = \Omega_{(6,19)} = \Omega_{(6,20)} = \Omega_{(6,21)} = 0,$$

$$\Omega_{(6,22)} = -n_9,$$

$$\Omega_{(6,23)} = \Omega_{(6,24)} = \Omega_{(6,25)} = 0,$$

$$\Omega_{(7,7)} = -(k_2 + w_3)e^{-2\alpha\tau_2},$$

$$\begin{aligned}\Omega_{(7,8)} &= \Omega_{(7,9)} = \Omega_{(7,10)} = \Omega_{(7,11)} = \Omega_{(7,12)} = \Omega_{(7,13)} = \Omega_{(7,14)} = \Omega_{(7,15)} = \Omega_{(7,16)} = \Omega_{(7,17)} \\ &= \Omega_{(7,18)} = \Omega_{(7,19)} = \Omega_{(7,20)} = \Omega_{(7,21)} = \Omega_{(7,22)} = \Omega_{(7,23)} = \Omega_{(7,24)} = \Omega_{(7,25)} = 0,\end{aligned}$$

$$\Omega_{(8,8)} = -k_6 e^{-2\alpha\tau_2},$$

$$\begin{aligned}\Omega_{(8,9)} &= \Omega_{(8,10)} = \Omega_{(8,11)} = \Omega_{(8,12)} = \Omega_{(8,13)} = \Omega_{(8,14)} = \Omega_{(8,15)} = \Omega_{(8,16)} = \Omega_{(8,17)} = \Omega_{(8,18)} \\ &= \Omega_{(8,19)} = \Omega_{(8,20)} = \Omega_{(8,21)} = \Omega_{(8,22)} = \Omega_{(8,23)} = \Omega_{(8,24)} = \Omega_{(8,25)} = 0,\end{aligned}$$

$$\Omega_{(9,9)} = -(k_4 + w_4)e^{-2\alpha\gamma\tau_2},$$

$$\begin{aligned}\Omega_{(9,10)} &= \Omega_{(9,11)} = \Omega_{(9,12)} = \Omega_{(9,13)} = \Omega_{(9,14)} = \Omega_{(9,15)} = \Omega_{(9,16)} = \Omega_{(9,17)} = \Omega_{(9,18)} = \Omega_{(9,19)} \\ &= \Omega_{(9,20)} = \Omega_{(9,21)} = \Omega_{(9,22)} = \Omega_{(9,23)} = \Omega_{(9,24)} = \Omega_{(9,25)} = 0,\end{aligned}$$

$$\Omega_{(10,10)} = -k_7 e^{-2\alpha\gamma\tau_2},$$

$$\begin{aligned}\Omega_{(10,11)} &= \Omega_{(10,12)} = \Omega_{(10,13)} = \Omega_{(10,14)} = \Omega_{(10,15)} = \Omega_{(10,16)} = \Omega_{(10,17)} = \Omega_{(10,18)} = \Omega_{(10,19)} \\ &= \Omega_{(10,20)} = \Omega_{(10,21)} = \Omega_{(10,22)} = \Omega_{(10,23)} = \Omega_{(10,24)} = \Omega_{(10,25)} = 0,\end{aligned}$$

$$\Omega_{(11,11)} = -k_8 e^{-2\alpha\sigma_2} + k_8 s_d - k_{10},$$

$$\Omega_{(11,12)} = 0,$$

$$\Omega_{(11,13)} = q_7 b,$$

$$\begin{aligned}\Omega_{(11,14)} &= \Omega_{(11,15)} = \Omega_{(11,16)} = \Omega_{(11,17)} = \Omega_{(11,18)} = \Omega_{(11,19)} = \Omega_{(11,20)} = \Omega_{(11,21)} = \Omega_{(11,22)} \\ &= \Omega_{(11,23)} = \Omega_{(11,24)} = \Omega_{(11,25)} = 0,\end{aligned}$$

$$\Omega_{(12,12)} = -k_9 e^{-2\alpha\sigma_2},$$

$$\begin{aligned}\Omega_{(12,13)} &= \Omega_{(12,14)} = \Omega_{(12,15)} = \Omega_{(12,16)} = \Omega_{(12,17)} = \Omega_{(12,18)} = \Omega_{(12,19)} = \Omega_{(12,20)} = \Omega_{(12,21)} \\ &= \Omega_{(12,22)} = \Omega_{(12,23)} = \Omega_{(12,24)} = \Omega_{(12,25)} = 0,\end{aligned}$$

$$\Omega_{(13,13)} = -2q_7,$$

$$\begin{aligned}\Omega_{(13,14)} &= \Omega_{(13,15)} = \Omega_{(13,16)} = \Omega_{(13,17)} = \Omega_{(13,18)} = \Omega_{(13,19)} = \Omega_{(13,20)} = \Omega_{(13,21)} = \Omega_{(13,22)} \\ &= \Omega_{(13,23)} = \Omega_{(13,24)} = \Omega_{(13,25)} = 0,\end{aligned}$$

$$\Omega_{(14,14)} = (w_3 - w_1)e^{-2\alpha\tau_1},$$

$$\Omega_{(14,15)} = \Omega_{(14,16)} = \Omega_{(14,17)} = \Omega_{(14,18)} = \Omega_{(14,19)} = \Omega_{(14,20)} = \Omega_{(14,21)} = \Omega_{(14,22)} = \Omega_{(14,23)} = \Omega_{(14,24)} = \Omega_{(14,25)} = 0,$$

$$\Omega_{(15,15)} = (w_4 - w_2)e^{-2\alpha\gamma\tau_1},$$

$$\Omega_{(15,16)} = \Omega_{(15,17)} = \Omega_{(15,18)} = \Omega_{(15,19)} = \Omega_{(15,20)} = \Omega_{(15,21)} = \Omega_{(15,22)} = \Omega_{(15,23)} = \Omega_{(15,24)} = \Omega_{(15,25)} = 0,$$

$$\Omega_{(16,16)} = -w_5 e^{-2\alpha\tau_1},$$

$$\Omega_{(16,17)} = \Omega_{(16,18)} = \Omega_{(16,19)} = \Omega_{(16,20)} = \Omega_{(16,21)} = \Omega_{(16,22)} = \Omega_{(16,23)} = \Omega_{(16,24)} = \Omega_{(16,25)} = 0,$$

$$\Omega_{(17,17)} = -w_6 e^{-2\alpha\gamma\tau_1},$$

$$\Omega_{(17,18)} = \Omega_{(17,19)} = \Omega_{(17,20)} = \Omega_{(17,21)} = \Omega_{(17,22)} = \Omega_{(17,23)} = \Omega_{(17,24)} = \Omega_{(17,25)} = 0,$$

$$\Omega_{(18,18)} = -w_7 e^{-2\alpha\tau_2},$$

$$\Omega_{(18,19)} = \Omega_{(18,20)} = \Omega_{(18,21)} = \Omega_{(18,22)} = \Omega_{(18,23)} = \Omega_{(18,24)} = \Omega_{(18,25)} = 0,$$

$$\Omega_{(19,19)} = -w_7 e^{-2\alpha\tau_2},$$

$$\Omega_{(19,20)} = \Omega_{(19,21)} = \Omega_{(19,22)} = \Omega_{(19,23)} = \Omega_{(19,24)} = \Omega_{(19,25)} = 0,$$

$$\Omega_{(20,20)} = -w_8 e^{-2\alpha\gamma\tau_2},$$

$$\Omega_{(20,21)} = \Omega_{(20,22)} = \Omega_{(20,23)} = \Omega_{(20,24)} = \Omega_{(20,25)} = 0,$$

$$\Omega_{(21,21)} = -w_8 e^{-2\alpha\gamma\tau_2},$$

$$\Omega_{(21,22)} = \Omega_{(21,23)} = \Omega_{(21,24)} = \Omega_{(21,25)} = 0,$$

$$\Omega_{(22,22)} = -w_9 e^{-2\alpha\sigma_1},$$

$$\Omega_{(22,23)} = \Omega_{(22,24)} = \Omega_{(22,25)} = 0,$$

$$\Omega_{(23,23)} = -w_{10} e^{-2\alpha\sigma_2},$$

$$\Omega_{(23,24)} = \Omega_{(23,25)} = 0,$$

$$\Omega_{(24,24)} = -w_{10} e^{-2\alpha\sigma_2},$$

$$\Omega_{(24,25)} = 0,$$

$$\Omega_{(25,25)} = -\omega e^{-2\alpha\rho_2}.$$

The exponential stability for the CNDE with time-varying delays in equation (2.1) will be represented as follows.

Theorem 3.1 *For given positive real constants* σ_1, σ_2, σ_d, τ_1, τ_2, τ_d, ρ_1, ρ_2 *and* γ, *equation* (2.1) *is exponentially stable with a decay rate* α *if there exist positive real constants* ω, k_i, w_i *where* $i = 1, 2, \dots, 10$, *and real constants* r_1, r_2, m_k, n_k *where* $k = 1, 2, \dots, 9$ *such that the following symmetric linear matrix inequality holds*:

$$\Sigma < 0. \tag{3.9}$$

Proof For ω, k_i and w_i are positive real constants where $i = 1, 2, \ldots, 10$, we consider the Lyapunov-Krasovskii functional candidate for equation (3.7) of the form

$$V(t, x_t) = \sum_{i=1}^{5} V_i(t, x_t), \tag{3.10}$$

where

$$\begin{aligned}
V_1(t, x_t) &= k_1 D^2(t), \\
V_2(t, x_t) &= k_2 \int_{t-\tau_2}^{t} e^{2\alpha(s-t)} x^2(s)\, ds + k_3 \int_{t-\tau(t)}^{t} e^{2\alpha(s-t)} x^2(s)\, ds \\
&\quad + k_4 \int_{t-\gamma\tau_2}^{t} e^{2\alpha(s-t)} x^2(s)\, ds + k_5 \int_{t-\gamma\tau(t)}^{t} e^{2\alpha(s-t)} x^2(s)\, ds \\
&\quad + w_1 \int_{t-\tau_1}^{t} e^{2\alpha(s-t)} x^2(s)\, ds + w_2 \int_{t-\gamma\tau_1}^{t} e^{2\alpha(s-t)} x^2(s)\, ds \\
&\quad + w_3 \int_{t-\tau_2}^{t-\tau_1} e^{2\alpha(s-t)} x^2(s)\, ds + w_4 \int_{t-\gamma\tau_2}^{t-\gamma\tau_1} e^{2\alpha(s-t)} x^2(s)\, ds, \\
V_3(t, x_t) &= k_6 \tau_2 \int_{-\tau_2}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} x^2(\theta)\, d\theta\, ds + k_7 \gamma \tau_2 \int_{-\gamma\tau_2}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} x^2(\theta)\, d\theta\, ds \\
&\quad + w_5 \tau_1 \int_{-\tau_1}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} x^2(\theta)\, d\theta\, ds \\
&\quad + w_6 \gamma \tau_1 \int_{-\gamma\tau_1}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} x^2(\theta)\, d\theta\, ds \\
&\quad + w_7 (\tau_2 - \tau_1) \int_{-\tau_2}^{-\tau_1} \int_{t+s}^{t} e^{2\alpha(\theta-t)} x^2(\theta)\, d\theta\, ds \\
&\quad + w_8 \gamma (\tau_2 - \tau_1) \int_{-\gamma\tau_2}^{-\gamma\tau_1} \int_{t+s}^{t} e^{2\alpha(\theta-t)} x^2(\theta)\, d\theta\, ds, \\
V_4(t, x_t) &= k_8 \int_{t-\sigma(t)}^{t} e^{2\alpha(s-t)} \tanh^2 x(s)\, ds \\
&\quad + k_9 \sigma_2 \int_{-\sigma_2}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} \tanh^2 x(\theta)\, d\theta\, ds \\
&\quad + w_9 \sigma_1 \int_{-\sigma_1}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} \tanh^2 x(\theta)\, d\theta\, ds \\
&\quad + w_{10} (\sigma_2 - \sigma_1) \int_{-\sigma_2}^{-\sigma_1} \int_{t+s}^{t} e^{2\alpha(\theta-t)} \tanh^2 x(\theta)\, d\theta\, ds, \\
V_5(t, x_t) &= \omega (\rho_2 - \rho_1) \int_{-\rho_2}^{-\rho_1} \int_{t+s}^{t} e^{2\alpha(\theta-t)} x^2(\theta)\, d\theta\, ds.
\end{aligned}$$

Calculating the time derivatives of $V(t, x_t)$ along the solution of equation (3.7) yields

$$\dot{V}(t, x_t) = \sum_{i=1}^{5} \dot{V}_i(t, x_t). \tag{3.11}$$

The time derivatives of $V_1(t,x_t)$ and $V_2(t,x_t)$ are calculated as

$$
\begin{aligned}
\dot{V}_1(t,x_t) &= 2k_1D(t)\dot{D}(t) \\
&= 2k_1D(t)\biggl[-(a_1-r_1-r_2)x(t)-(a_2+r_1)x\bigl(t-\tau(t)\bigr)-(a_2+r_1)\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds \\
&\quad -r_2x\bigl(t-\gamma\tau(t)\bigr)-r_2\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds+b\tanh x\bigl(t-\sigma(t)\bigr)+c\int_{t-\rho(t)}^{t}x(s)\,ds\biggr] \\
&\quad +2q_1D(t)\biggl[-D(t)+p_1x(t)+p_2x\bigl(t-\tau(t)\bigr)+x\bigl(t-\gamma\tau(t)\bigr) \\
&\quad +\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds-p_1\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\biggr] \\
&\quad +2q_2x(t)\biggl[-D(t)+p_1x(t)+p_2x\bigl(t-\tau(t)\bigr)+x\bigl(t-\gamma\tau(t)\bigr) \\
&\quad +\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds-p_1\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\biggr] \\
&\quad +2q_3x\bigl(t-\tau(t)\bigr)\biggl[-D(t)+p_1x(t)+p_2x\bigl(t-\tau(t)\bigr)+x\bigl(t-\gamma\tau(t)\bigr) \\
&\quad +\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds-p_1\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\biggr] \\
&\quad +2q_4\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\biggl[-D(t)+p_1x(t)+p_2x\bigl(t-\tau(t)\bigr)+x\bigl(t-\gamma\tau(t)\bigr) \\
&\quad +\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds-p_1\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\biggr] \\
&\quad +2q_5x\bigl(t-\gamma\tau(t)\bigr)\biggl[-D(t)+p_1x(t)+p_2x\bigl(t-\tau(t)\bigr)+x\bigl(t-\gamma\tau(t)\bigr) \\
&\quad +\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds-p_1\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\biggr] \\
&\quad +2q_6\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds\biggl[-D(t)+p_1x(t)+p_2x\bigl(t-\tau(t)\bigr)+x\bigl(t-\gamma\tau(t)\bigr) \\
&\quad +\int_{t-\gamma\tau(t)}^{t}\dot{x}(s)\,ds-p_1\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\biggr] \\
&\quad +2q_7\dot{D}(t)\biggl[-\dot{D}(t)-a_1x(t)-a_2x\bigl(t-\tau(t)\bigr) \\
&\quad -a_2\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds+b\tanh x\bigl(t-\sigma(t)\bigr)+c\int_{t-\rho(t)}^{t}x(s)\,ds\biggr] \\
&\quad +2\alpha k_1D^2(t)-2\alpha V_1(t), \qquad (3.12) \\
\dot{V}_2(t,x_t) &= (k_2+k_3+k_4+k_5+w_1+w_2)x^2(t)-(k_2+w_3)e^{-2\alpha\tau_2}x^2(t-\tau_2) \\
&\quad -k_3\bigl(1-\dot{\tau}(t)\bigr)e^{-2\alpha\tau(t)}x^2\bigl(t-\tau(t)\bigr)-(k_4+w_4)e^{-2\alpha\gamma\tau_2}x^2(t-\gamma\tau_2) \\
&\quad -k_5\bigl(1-\gamma\dot{\tau}(t)\bigr)e^{-2\alpha\gamma\tau(t)}x^2\bigl(t-\gamma\tau(t)\bigr) \\
&\quad +(w_3-w_1)e^{-2\alpha\tau_1}x^2(t-\tau_1)(w_4-w_2)e^{-2\alpha\gamma\tau_1}x^2(t-\gamma\tau_1)-2\alpha V_2(t)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (k_2+k_3+k_4+k_5+w_1+w_2)x^2(t)-(k_2+w_3)e^{-2\alpha\tau_2}x^2(t-\tau_2)\\
&\quad -k_3e^{-2\alpha\tau_2}x^2\big(t-\tau(t)\big)+k_3\tau_d x^2\big(t-\tau(t)\big)\\
&\quad -(k_4+w_4)e^{-2\alpha\gamma\tau_2}x^2(t-\gamma\tau_2)-k_5e^{-2\alpha\gamma\tau_2}x^2\big(t-\gamma\tau(t)\big)\\
&\quad +k_5\gamma\tau_d x^2\big(t-\gamma\tau(t)\big)\\
&\quad +(w_3-w_1)e^{-2\alpha\tau_1}x^2(t-\tau_1)(w_4-w_2)e^{-2\alpha\gamma\tau_1}x^2(t-\gamma\tau_1)-2\alpha V_2(t).
\end{aligned}
\tag{3.13}
$$

Obviously, for any a scalar $s\in[t-\tau_2,t]$, we get $e^{-2\alpha\tau_2}\leq e^{2\alpha(s-t)}\leq 1$ and $e^{-2\alpha\gamma\tau_2}\leq e^{2\alpha(s-t)}\leq 1$, for any a scalar $s\in[t-\gamma\tau_2,t]$. Together with Lemma 2.2 and 2.3, we obtain

$$
\begin{aligned}
\dot{V}_3(t,x_t) &= k_6\tau_2\int_{-\tau_2}^{0}x^2(t)\,ds-k_6\tau_2\int_{-\tau_2}^{0}e^{2s\alpha}x^2(t+s)\,ds\\
&\quad +k_7\gamma\tau_2\int_{-\gamma\tau_2}^{0}x^2(t)\,ds-k_7\gamma\tau_2\int_{-\gamma\tau_2}^{0}e^{2s\alpha}x^2(t+s)\,ds\\
&\quad +w_5\tau_1\int_{-\tau_1}^{0}x^2(t)\,ds-w_5\tau_1\int_{-\tau_1}^{0}e^{2s\alpha}x^2(t+s)\,ds\\
&\quad +w_6\gamma\tau_1\int_{-\gamma\tau_1}^{0}x^2(t)\,ds-w_6\gamma\tau_1\int_{-\gamma\tau_1}^{0}e^{2s\alpha}x^2(t+s)\,ds\\
&\quad +w_7(\tau_2-\tau_1)\int_{-\tau_2}^{-\tau_1}x^2(t)\,ds-w_7(\tau_2-\tau_1)\int_{-\tau_2}^{-\tau_1}e^{2s\alpha}x^2(t+s)\,ds\\
&\quad +w_8(\gamma\tau_2-\gamma\tau_1)\int_{-\gamma\tau_2}^{-\gamma\tau_1}x^2(t)\,ds-w_8(\gamma\tau_2-\gamma\tau_1)\int_{-\gamma\tau_2}^{-\gamma\tau_1}e^{2s\alpha}x^2(t+s)\,ds\\
&\quad -2\alpha V_3(t)\\
&\leq k_6(\tau_2)^2x^2(t)-k_6e^{-2\alpha\tau_2}\left(\int_{t-\tau_2}^{t}x(s)\,ds\right)^2\\
&\quad +k_7(\gamma\tau_2)^2x^2(t)-k_7e^{-2\alpha\gamma\tau_2}\left(\int_{t-\gamma\tau_2}^{t}x(s)\,ds\right)^2\\
&\quad +w_5(\tau_1)^2x^2(t)-w_5e^{-2\alpha\tau_1}\left(\int_{t-\tau_1}^{t}x(s)\,ds\right)^2\\
&\quad +w_6(\gamma\tau_1)^2x^2(t)-w_6e^{-2\alpha\gamma\tau_1}\left(\int_{t-\gamma\tau_1}^{t}x(s)\,ds\right)^2\\
&\quad +w_7(\tau_2-\tau_1)^2x^2(t)+w_8(\gamma\tau_2-\gamma\tau_1)^2x^2(t)\\
&\quad -w_7e^{-2\alpha\tau_2}\left(\int_{t-\tau(t)}^{t-\tau_1}x(s)\,ds\right)^2-w_7e^{-2\alpha\tau_2}\left(\int_{t-\tau_2}^{t-\tau(t)}x(s)\,ds\right)^2\\
&\quad -w_8e^{-2\gamma\alpha\tau_2}\left(\int_{t-\gamma\tau(t)}^{t-\gamma\tau_1}x(s)\,ds\right)^2-w_8e^{-2\gamma\alpha\tau_2}\left(\int_{t-\gamma\tau_2}^{t-\gamma\tau(t)}x(s)\,ds\right)^2\\
&\quad -2\alpha V_3(t).
\end{aligned}
\tag{3.14}
$$

From Lemma 2.2, 2.3, and $\tanh^2 x(t)\leq x^2(t)$, we have

$$
\begin{aligned}
\dot{V}_4(t,x_t) &= k_8\tanh^2 x(t)-k_8\big(1-\dot{\sigma}(t)\big)e^{-2\alpha\sigma(t)}\tanh^2 x\big(t-\sigma(t)\big)\\
&\quad +k_9\sigma_2\int_{-\sigma_2}^{0}\tanh^2 x(t)\,ds-k_9\sigma_2\int_{-\sigma_2}^{0}e^{2\alpha s}\tanh^2 x(t+s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ w_9\sigma_1 \int_{-\sigma_1}^{0} \tanh^2 x(t)\,ds - w_9\sigma_1 \int_{-\sigma_1}^{0} e^{2\alpha s}\tanh^2 x(t+s)\,ds \\
&+ w_{10}(\sigma_2-\sigma_1) \int_{-\sigma_2}^{-\sigma_1} \tanh^2 x(t)\,ds \\
&- w_{10}(\sigma_2-\sigma_1) \int_{-\sigma_2}^{-\sigma_1} e^{2\alpha s}\tanh^2 x(t+s)\,ds \\
&- 2\alpha V_4(t) \\
\le{}& k_8 \tanh^2 x(t) - k_8 e^{-2\alpha\sigma_2}\tanh^2 x\bigl(t-\sigma(t)\bigr) + k_8 s_d \tanh^2 x\bigl(t-\sigma(t)\bigr) \\
&+ k_9\sigma_2^2 x^2(t) - k_9 e^{-2\alpha\sigma_2}\left(\int_{t-\sigma_2}^{t} \tanh x(s)\,ds\right)^2 \\
&+ w_9\sigma_1^2 x^2(t) - w_9 e^{-2\alpha\sigma_1}\left(\int_{t-\sigma_1}^{t} \tanh x(s)\,ds\right)^2 \\
&+ w_{10}(\sigma_2-\sigma_1)^2 x^2(t) - w_{10} e^{-2\alpha\tau_2}\left(\int_{t-\tau(t)}^{t-\tau_1} \tanh x(s)\,ds\right)^2 \\
&+ k_{10} x^2(t) - k_{10}\tanh^2 x(t) \\
&- w_{10} e^{-2\alpha\tau_2}\left(\int_{t-\tau_2}^{t-\tau(t)} \tanh x(s)\,ds\right)^2 \\
&- 2\alpha V_4(t),
\end{aligned}
\tag{3.15}
$$

$$
\dot{V}_5(t,x_t) \le \omega(\rho_2-\rho_1)^2 x^2(t) - \omega e^{-2\alpha\rho_2}\left(\int_{t-\rho(t)}^{t} x(s)\,ds\right)^2. \tag{3.16}
$$

From the Leibniz-Newton formula, the following equations are true for any real constants m_i, n_i, $i = 1, 2, \ldots, 9$ with appropriate dimensions:

$$
\begin{aligned}
&2\Bigl[m_1 D(t) + m_2 x(t) + m_3 x\bigl(t-\tau(t)\bigr) + m_4 x\bigl(t-\gamma\tau(t)\bigr) + m_5 \int_{t-\tau(t)}^{t} \dot{x}(s)\,ds \\
&\quad + m_6 \int_{t-\gamma\tau(t)}^{t} \dot{x}(s)\,ds + m_7 \tanh x\bigl(t-\sigma(t)\bigr) + m_8 \int_{t-\sigma_2}^{t} \tanh x(s)\,ds \\
&\quad + m_9 \int_{t-\sigma_1}^{t} \tanh x(s)\,ds\Bigr] \times \Bigl[x(t) - x\bigl(t-\tau(t)\bigr) - \int_{t-\tau(t)}^{t} \dot{x}(s)\,ds\Bigr] = 0,
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
&2\Bigl[n_1 D(t) + n_2 x(t) + n_3 x\bigl(t-\tau(t)\bigr) + n_4 x\bigl(t-\gamma\tau(t)\bigr) + n_5 \int_{t-\tau(t)}^{t} \dot{x}(s)\,ds \\
&\quad + n_6 \int_{t-\gamma\tau(t)}^{t} \dot{x}(s)\,ds + n_7 \tanh x\bigl(t-\sigma(t)\bigr) + n_8 \int_{t-\sigma_2}^{t} \tanh x(s)\,ds \\
&\quad + n_9 \int_{t-\sigma_1}^{t} \tanh x(s)\,ds\Bigr] \times \Bigl[x(t) - x\bigl(t-\gamma\tau(t)\bigr) - \int_{t-\gamma\tau(t)}^{t} \dot{x}(s)\,ds\Bigr] = 0.
\end{aligned}
\tag{3.18}
$$

According to (3.11)-(3.18), it is straightforward to see that

$$
\dot{V}(t,x_t) + 2\alpha V(t,x_t) \le \xi^T(t)\Sigma\xi(t), \tag{3.19}
$$

where

$$\begin{aligned}\zeta^T(t) = \Bigg[& D(t), x(t), x\big(t-\tau(t)\big), \int_{t-\tau(t)}^{t} \dot{x}(s)\,ds, x\big(t-\gamma\tau(t)\big), \int_{t-\gamma\tau(t)}^{t} \dot{x}(s)\,ds, x(t-\tau_2), \\ & \int_{t-\tau_2}^{t} x(s)\,ds, x(t-\gamma\tau_2), \int_{t-\gamma\tau_2}^{t} x(s)\,ds, \tanh x\big(t-\sigma(t)\big), \int_{t-\sigma_2}^{t} \tanh x(s)\,ds, \dot{D}(t), \\ & x(t-\tau_1), x(t-\gamma\tau_1), \int_{t-\tau_1}^{t} x(s)\,ds, \int_{t-\gamma\tau_1}^{t} x(s)\,ds, \int_{t-\tau(t)}^{t-\tau_1} x(s)\,ds, \int_{t-\tau_2}^{t-\tau(t)} x(s)\,ds, \\ & \int_{t-\gamma\tau(t)}^{t-\gamma\tau_1} x(s)\,ds, \int_{t-\gamma\tau_2}^{t-\gamma\tau(t)} x(s)\,ds, \int_{t-\sigma_1}^{t} \tanh x(s)\,ds, \int_{t-\sigma(t)}^{t-\sigma_1} \tanh x(s)\,ds, \\ & \int_{t-\sigma_2}^{t-\sigma(t)} \tanh x(s)\,ds, \int_{t-\rho_2}^{t} \dot{x}(s)\,ds \Bigg],\end{aligned}$$

and Σ is defined in (3.8). It is true that if condition (3.9) holds, then

$$\dot{V}(t, x_t) + 2\alpha V(t, x_t) \le 0, \quad \forall t \in R^+. \tag{3.20}$$

From (3.20), it is easy to see that

$$\big\| x(t,\phi) \big\| \le \beta \|\phi\| e^{-\alpha t}, \quad t \in R^+.$$

This means that equation (2.1) is exponentially stable. The proof of the theorem is complete. □

We now present the delay-dependent criteria for exponential stability of equation (2.1) where $\tau_1 = \sigma_1 = c = 0$. We introduce the following notations for later use:

$$\tilde{\Sigma} = [\tilde{\Omega}_{(i,j)}]_{13\times 13}, \tag{3.21}$$

where $\tilde{\Omega}_{(i,j)} = \Omega_{(i,j)}$, except

$$\begin{aligned}\tilde{\Omega}_{(2,2)} &= 2q_2p_1 + k_2 + k_3 + k_4 + k_5 + k_6\tau_2^2 + k_7\gamma^2\tau_2^2 + k_8 + k_9\sigma_2^2 + 2m_2 + 2n_2 + k_{10}, \\ \tilde{\Omega}_{(7,7)} &= -k_2 e^{-2\alpha\tau_2}, \\ \tilde{\Omega}_{(9,9)} &= -k_4 e^{-2\alpha\gamma\tau_2}.\end{aligned}$$

Corollary 3.2 *For given positive real constants σ_2, τ_2, σ_d, τ_d and γ, equation* (2.1) *where $\tau_1 = \sigma_1 = c = 0$ is exponentially stable with a decay rate α if there exist positive real constants k_i where $i = 1, 2, \dots, 10$ and we have the real constants r_1, r_2, m_k, n_k where $k = 1, 2, \dots, 8$ such that the following symmetric linear matrix inequality holds*:

$$\tilde{\Sigma} < 0. \tag{3.22}$$

Proof For k_i, $i = 1, 2, \dots, 9$ are positive real numbers, we consider the Lyapunov-Krasovskii functional candidate for equation (2.1) where $\tau_1 = \sigma_1 = c = 0$ of the form

$$V(t, x_t) = \sum_{i=1}^{4} V_i(t, x_t), \tag{3.23}$$

where

$$
\begin{aligned}
V_1(t,x_t) &= k_1 D^2(t),\\
V_2(t,x_t) &= k_2 \int_{t-\tau_2}^{t} e^{2\alpha(s-t)}x^2(s)\,ds + k_3 \int_{t-\tau(t)}^{t} e^{2\alpha(s-t)}x^2(s)\,ds\\
&\quad + k_4 \int_{t-\gamma\tau_2}^{t} e^{2\alpha(s-t)}x^2(s)\,ds + k_5 \int_{t-\gamma\tau(t)}^{t} e^{2\alpha(s-t)}x^2(s)\,ds,\\
V_3(t,x_t) &= k_6\tau_2 \int_{-\tau_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}x^2(\theta)\,d\theta\,ds\\
&\quad + k_7\gamma\tau_2 \int_{-\gamma\tau_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}x^2(\theta)\,d\theta\,ds,\\
V_4(t,x_t) &= k_8 \int_{t-\sigma(t)}^{t} e^{2\alpha(s-t)}\tanh^2 x(s)\,ds\\
&\quad + k_9\sigma_2 \int_{-\sigma_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}\tanh^2 x(\theta)\,d\theta\,ds.
\end{aligned}
$$

According to Theorem 3.1, we have the delay-dependent exponential stability criteria (3.22) of equation (2.1) where $\tau_1 = \sigma_1 = c = 0$. □

4 Numerical examples

In this section, four numerical examples are given to present the effectiveness of our main results by comparing the upper bounds of the delays σ and the parameter b as well as investigating the rate of convergence.

Example 4.1 Consider the following equation with mixed interval time-varying delays:

$$
\frac{d}{dt}\big[x(t) + 0.1x\big(t-\tau(t)\big)\big] = -1.5x(t) + b\tanh x\big(t-\sigma(t)\big) + 0.5\int_{t-\rho(t)}^{t} x(s)\,ds. \tag{4.1}
$$

Decompose the constants a and p as $a = a_1 + a_2$ and $p = p_1 + p_2$, respectively, where

$$
a_1 = 1, \qquad a_2 = 0.5, \qquad p_1 = 0.05, \qquad p_2 = 0.05.
$$

Solving the LMI (3.9) when $b = 0.2$, $\alpha = 0.4$, $\tau(t) = 0.1 + \frac{\sin^2(t)}{2}$, $\sigma(t) = 0.2 + \frac{\cos^2(t)}{2}$, and $\rho(t) = 0.2 + \frac{|\cos(t)|}{5}$, we can obtain a set of parameters guaranteeing exponential stability as follows:

$$
\begin{aligned}
&q_1 = 26.7604, \quad q_2 = -47.2411, \quad q_3 = 27.1931, \quad q_4 = 24.6455,\\
&q_5 = -7.5706, \quad q_6 = -9.9581, \quad q_7 = 1.8709, \quad k_1 = 36.9295, \quad k_2 = 3.8691,\\
&k_3 = 8.2250, \quad k_4 = 3.4622, \quad k_5 = 3.2629, \quad k_6 = 7.4445, \quad k_7 = 8.0889,\\
&k_8 = 3.4072, \quad k_9 = 6.9838, \quad \omega = 30.6074, \quad w_1 = 7.4709, \quad w_2 = 7.0912,\\
&w_3 = 3.9990, \quad w_4 = 3.6515, \quad w_5 = 8.2545, \quad w_6 = 8.0142, \quad w_7 = 9.2084,\\
&w_8 = 8.1243, \quad w_9 = 8.0686, \quad w_{10} = 9.2084.
\end{aligned}
$$

Table 1 Upper bounds of b for Example 4.1 when $\gamma = 0.05$

$\tau_d = \sigma_d$	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$
0.1	1.4906	1.3865	1.2747	1.1601	1.0433
0.2	1.4910	1.3852	1.2740	1.1580	1.0380
0.3	1.4883	1.3835	1.2710	1.1512	1.0271
0.4	1.4861	1.3812	1.2661	1.1450	1.0126
0.5	1.4834	1.3774	1.2610	1.1305	0.9830

Table 2 Upper bounds of α for Example 4.1 when $\gamma = 0.05$

$\tau_d = \sigma_d$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$b = 1$	$b = 1.2$
0.1	0.9105	0.7610	0.6015	0.4364	0.2660
0.2	0.8728	0.7382	0.5895	0.4302	0.2623
0.3	0.8095	0.7031	0.5730	0.4217	0.2591
0.4	0.7020	0.6414	0.5426	0.4094	0.2543
0.5	0.5549	0.5394	0.4897	0.3892	0.2477

Table 3 Upper bounds of b for Example 4.1 when $\gamma = 0.05$

c	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$
0.5	1.4834	1.3774	1.2610	1.1305	0.9830
1	1.4830	1.3761	1.2610	1.1288	0.9806
2	1.4791	1.3727	1.2580	1.1249	0.9781
3	1.4780	1.3704	1.2534	1.1228	0.9720
4	1.4750	1.3681	1.2511	1.1270	0.9704

Table 4 Upper bounds of α for Example 4.1 when $\gamma = 0.05$

c	$b = 0.4$	$b = 0.6$	$b = 0.8$	$b = 1$	$b = 1.2$
0.5	0.5549	0.5394	0.4897	0.3892	0.2477
1	0.5548	0.5391	0.4887	0.3877	0.2480
2	0.5546	0.5400	0.4880	0.3890	0.2460
3	0.5546	0.5400	0.4880	0.3850	0.2430
4	0.5546	0.5377	0.4843	0.3840	0.2401

Moreover, the upper bounds of the parameter b which guarantees the exponential and asymptotic stabilities are 0.9830 and 1.4834, respectively. The maximum upper bounds b of this example can be found in Table 1 for different values of α, τ_d, σ_d. The maximum upper bounds as regards the rate of convergent α for this example can be found in Table 2 with different values of b, τ_d, σ_d. The maximum upper bounds b for exponential and asymptotic stabilities of Example 4.1 are listed in Table 3 for different values of α, c. The maximum upper bounds α for exponential and asymptotic stabilities of Example 4.1 are listed in Table 4 for different values of c, b.

Example 4.2 Consider the following equation studied in [8, 23]:

$$\frac{d}{dt}\bigl[x(t) + 0.2x\bigl(t - \tau(t)\bigr)\bigr] = -0.6x(t) + 0.5\tanh x\bigl(t - \sigma(t)\bigr), \tag{4.2}$$

when $\sigma(t) = \frac{\sin^2(t)}{10}$ and $\tau_d = 0.2$.

Decompose the constants a and p thus: $a = a_1 + a_2$ and $p = p_1 + p_2$, respectively, where

$$a_1 = 0.3, \qquad a_2 = 0.3, \qquad p_1 = 0.1, \qquad p_2 = 0.1.$$

Table 5 The upper bound of time delay $\sigma(t)$ for Example 4.2 when $\gamma = 0.02$

Methods	$\alpha = 0.0038$	$\alpha = 0.02$	$\alpha = 0.028$
Chen and Meng (2011) [8]	Infeasible	Infeasible	Infeasible
Keadnarmol and Rojsiraphisal (2014) [23]	7.5231	0.5234	0.0321
Corollary 3.2	3.5×10^3	564.9979	448.9991

Table 6 Upper bounds of *b* for Example 4.3 when $\gamma = 0.5$

Methods $\sigma_2 = \tau_2 = 0.5$	*b*	
	A.S. ($\alpha = 0$)	E.S. ($\alpha = 0.177$)
Agarwal and Grace (2000) [4]	0.318	-
El-Morshedy and Gopalsamy (2000) [15]	0.424	-
Park and Kwon (2008) [32]	0.422	-
Kwon and Park (2008) [21]	1.49	-
Li (2009) [24]	0.699	0.722
Deng *et al.* (2009) [14]	0.889	-
Nam and Phat (2009) [29]	1.405	-
Rojsiraphisal and Niamsup (2010) [34]	1.405	0.478
Chen and Meng (2011) [8]	1.346	-
Chen (2012) [7]	1.405	1.092
Keadnarmol and Rojsiraphisal (2014) [23]	1.405	1.1089
Corollary 3.2	1.4051	1.2114

By solving the linear matrix inequality (3.22), the maximum upper bounds σ_2 for exponential stability of this example is listed fore comparison in Table 5, for different values of α. We can see that our results in Corollary 3.2 are much less conservative than those obtained in [8, 23].

Example 4.3 Consider the following equation, which is considered in [4, 7, 8, 14, 15, 21, 23, 24, 29, 32, 34]:

$$\frac{d}{dt}\bigl[x(t) + 0.35x(t-0.5)\bigr] = -1.5x(t) + b\tanh x(t-0.5). \tag{4.3}$$

Decompose constants a and p as $a = a_1 + a_2$ and $p = p_1 + p_2$, respectively, where

$$a_1 = 1.3, \qquad a_2 = 0.2, \qquad p_1 = 0.15, \qquad p_2 = 0.2.$$

Table 6 lists for comparison the upper bounds b for asymptotic stability ($\alpha = 0$) and exponential stability ($\alpha = 0.177$) of equation (4.3) by different methods. We can see from Table 6 that our result (Corollary 3.2) is better than other existing work.

Example 4.4 Consider the following equation in [7, 8, 21, 23, 24, 29, 30, 32, 34]:

$$\frac{d}{dt}\bigl[x(t) + 0.2x(t-0.1)\bigr] = -0.6x(t) + 0.3\tanh x(t-\sigma_2). \tag{4.4}$$

Decompose the matrix a and p in the same way as the decomposition in Example 4.3. Table 7 lists for comparison the upper bounds delay for asymptotic stability ($\alpha = 0$) and exponential stability ($\alpha = 0.0038$) of (4.4) by different methods. It is clear that our results (Corollary 3.2) are significantly better than some existing criteria.

Table 7 Upper bounds of σ_2 for Example 4.4 when $\gamma = 0.05$

Methods $\tau_2 = 0.1$	σ_2	
	A.S. ($\alpha = 0$)	E.S. ($\alpha = 0.0038$)
Park (2004) [30]	0.444	-
Park and Kwon (2008) [32]	1.90	-
Kwon and Park (2008) [21]	10^7	-
Li (2009) [24]	2.07	-
Nam and Phat (2009) [29]	2.32	-
Rojsiraphisal and Niamsup (2010) [34]	2.32	1.947
Chen and Meng (2011) [8]	10^{21}	-
Chen (2012) [7]	1.34×10^{21}	175.289
Corollary 3.2	3.15×10^9	3,700

5 Conclusions

In this paper, we proposed the delay-range-dependent exponential stability criteria for certain NDE with discrete and distributed interval time-varying delays. Then we presented the delay-dependent exponential stability criteria for certain NDE with time-varying delays. The method combining an augmented Lyapunov-Krasovskii functional, a mixed model transformation, the decomposition technique of constant coefficients, and utilization of zero equations has been adopted to study the paper. New stability criteria have been formulated in terms of LMI. Finally, four numerical examples are given to show that the proposed criteria are less conservative than some existing stability criteria.

Competing interests
The authors declare to have no competing interests.

Authors' contributions
The paper was carried out in collaboration between both authors. KM proposed the problem and initiated techniques used in this paper. WC presented the new method and numerical reasoning tests for analysis. Both authors wrote the manuscript and approved the final manuscript.

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