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REVIEW

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On difference equations concerning Schwarzian equation

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Abstract

Consider the difference equation

$$\left[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta^2 f(z)}{\Delta f(z)}\right)^2\right]^k = \frac{P(z,f(z))}{Q(z,f(z))},$$

where $P(z,f)$ and $Q(z,f)$ are prime polynomials in $f(z)$ with $\deg_f P = p, \deg_f Q = q$, and $d = \max\{p,q\} > 0$. We give the supremum of d, an estimation of the sum of Nevanlinna exceptional values of meromorphic solution $f(z)$ of the equation, and study the value distributions of their difference $\Delta f(z)$ and divided difference $\frac{\Delta f(z)}{f(z)}$.

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1 Introduction and main results

In this paper, we use the basic notions of Nevanlinna theory, such as $T(r,f), m(r,f), N(r,f)$, and so on; see [1–3]. Let $S(r,f)$ denote any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside a set of finite logarithmic measure. We call f *an admissible solution* of a difference (or differential) equation if all coefficients α of the equation satisfy $T(r,\alpha) = S(r,f)$. In addition, we denote by $\sigma(f)$ the order of growth of a meromorphic function $f(z)$ and by $\lambda(f)$ and $\lambda(\frac{1}{f})$, respectively, the exponents of convergence of zeros and poles of $f(z)$, which are defined by

$$\sigma(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r}, \qquad \lambda(f) = \overline{\lim_{r\to\infty}} \frac{\log N(r,\frac{1}{f})}{\log r}, \qquad \lambda\left(\frac{1}{f}\right) = \overline{\lim_{r\to\infty}} \frac{\log N(r,f)}{\log r}.$$

If $\lambda(f-a) < \sigma(f)$, then a is called a Borel exceptional value of f.

For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $\delta(a,f)$ the deficiency of a to $f(z)$, which is defined by

$$\delta(a,f) = \underline{\lim_{r\to\infty}} \frac{m(r,\frac{1}{f-a})}{T(r,f)} \quad (\text{if } a \in \mathbb{C}), \qquad \delta(\infty,f) = \underline{\lim_{r\to\infty}} \frac{m(r,f)}{T(r,f)}.$$

Obviously, $\delta(a,f) \geq 0$. If $\delta(a,f) > 0$, then a is called a Nevanlinna exceptional value of f.

The forward differences $\Delta^n f(z), n \in \mathbb{N}^+$, are defined in the standard way [4] by

$$\Delta f(z) = f(z+1) - f(z), \qquad \Delta^{n+1} f(z) = \Delta^n f(z+1) - \Delta^n f(z).$$

Ishizaki [5] studied Schwarzian differential equations and obtained the following:

Theorem A *Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be distinct constants. If*

$$\left[\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right]^k = \frac{P(z,f)}{Q(z,f)}$$

possesses an admissible solution, then we have

$$\sum_{j=1}^{s} \delta(\alpha_j, f) \le 2 - \frac{d}{2k},$$

where $d = \max\{\deg_f P(z,f), \deg_f Q(z,f)\}$.

Chen and Li [6] investigated difference equations and obtained the following theorem, which can be regarded as a difference analogue of Theorem A.

Theorem B *Let $f(z)$ be an admissible solution of the difference equation*

$$\left[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta^2 f(z)}{\Delta f(z)}\right)^2\right]^k = R\bigl(z, f(z)\bigr) = \frac{P(z, f(z))}{Q(z, f(z))} \tag{1.1}$$

such that $\sigma_2(f) < 1$, where k (≥ 1) is an integer, $P(z,f)$ and $Q(z,f)$ are polynomials with $\deg_f P(z,f) = p, \deg_f Q(z,f) = q$, *and* $d = \max\{p,q\}$. *Let $\alpha_1, \ldots, \alpha_s$ be s (≥ 2) distinct complex constants. Then*

$$\sum_{j=1}^{s} \delta(\alpha_j, f) \le 4 - \frac{q}{2k}. \tag{1.2}$$

In particular, if $N(r,f) = S(r,f)$, then

$$\sum_{j=1}^{s} \delta(\alpha_j, f) \le 2 - \frac{d}{2k}. \tag{1.3}$$

Lan and Chen [7] studied the value distribution of transcendental meromorphic solutions and their differences of some difference equation concerning a Schwarzian equation. From Theorem B we do a rough calculation: if $f(z)$ has two finite Borel exceptional values, (1.2) shows that $q \le 4k$; if $f(z)$ has only one finite Borel exceptional value, (1.2) shows $q \le 6k$ ((1.3) shows that $d \le 2k$); if $f(z)$ has no finite Nevanlinna exceptional values, (1.2) shows that $q \le 8k$ ((1.3) shows that $d \le 4k$). The upper bound of d seems to bear on k. We ask the following interesting questions: what is the supremum of d? whether inequalities (1.2) and (1.3) can be improved? Once we fix d and k, can we get the number of Nevanlinna exceptional values of meromorphic solutions of (1.1)? In this paper, we answer these questions and obtain the following theorem.

Theorem 1.1 *Suppose that $f(z)$ is an admissible solution of difference equation* (1.1) *of finite order, where k (≥ 1) is an integer,* $\deg_f P = p, \deg_f Q = q$, *and* $d = \max\{p,q\} > 0$. *Let α_j $(j = 1, \ldots, s)$ be distinct constants. Then*

(i) $d \leq 7k$;
(ii) $\sum_{j=1}^{s} \delta(\alpha_j, f) \leq \frac{7}{2} - \frac{d}{2k}$;
(iii) *if* $N(r,f) = S(r,f)$, *then* $q = d \leq 2k$, *and* $\sum_{j=1}^{s} \delta(\alpha_j, f) \leq 1 - \frac{d}{2k}$.

Example 1.1 The function $f(z) = z\tan\frac{\pi z}{4}$ satisfies the difference equation

$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta^2 f(z)}{\Delta f(z)}\right)^2 = \frac{P(z,f)}{Q(z,f)},$$

where

$$\begin{aligned} P(z,f) &= 16f^7 + 2(9z+44)f^6 + \bigl(61z^2 + 90z + 82\bigr)f^5 + z\bigl(99z^2 + 116z + 56\bigr)f^4 \\ &\quad + 2z^2\bigl(43z^2 + 84z + 26\bigr)f^3 + 4z^3\bigl(21z^2 + 16z + 6\bigr)f^2 + z^4\bigl(53z^2 - 6z - 32\bigr)f \\ &\quad + 3z^5(3z+2)^2, \\ Q(z,f) &= -8f^2(f+z)\bigl(f^2 + f + z^2 + z\bigr)^2. \end{aligned}$$

We see that $k = 1$ and $p = q = 7$. So, $d = \max\{p,q\} = 7 = 7k$. Obviously, $f(z) = z\tan\frac{\pi z}{4}$ has no Nevanlinna exceptional value, that is, for any distinct constants α_j ($j = 1, \ldots, s$), we have $\sum_{j=1}^{s} \delta(\alpha_j, f) = 0 = \frac{7}{2} - \frac{d}{2k}$.

Example 1.2 The function $f(z) = e^z + z^2$ satisfies the difference equation

$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta^2 f(z)}{\Delta f(z)}\right)^2 = \frac{P(z,f)}{Q(z,f)},$$

where

$$\begin{aligned} P(z,f) &= -(e-1)^4 f^2 + 2(e-1)^2\bigl((e-1)^2 z^2 + 2(e-1)z + e - 7\bigr)f \\ &\quad - (e-1)^4 z^2 - (e-1)^4 z - 4(e-1)^3 z^3 - 2(e-7)(e-1)^2 z^2 - 12, \\ Q(z,f) &= 2\bigl((e-1)f + \bigl((1+e)z + 1\bigr)\bigl((1-e)z + 1\bigr)\bigr)^2. \end{aligned}$$

We see that $k = 1$ and $p = q = 2$. So, $d = \max\{p,q\} = 2 = 2k$. Obviously, $f(z)$ has no finite Nevanlinna exceptional value, that is, for any distinct constants α_j ($j = 1, \ldots, s$), we have $\sum_{j=1}^{s} \delta(\alpha_j, f) = 0 = 1 - \frac{d}{2k}$.

Remark 1.1
(i) Comparing Theorem 1.1 and Theorem B, we see that the result of Theorem 1.1 is better than that of Theorem B.
(ii) Under conditions of Theorem 1.1(iii), the result shows that $\sum_{j=1}^{s} \delta(\alpha_j, f) \leq 1 - \frac{d}{2k} < 1$, that is, $f(z)$ has no finite Borel exceptional value.
(iii) Examples 1.1 and 1.2 show the equalities in Theorem 1.1 can be reached. So, the result of Theorem 1.1 is precise.

In general, $\lambda(\Delta f) \leq \sigma(f)$, where $f(z)$ is a meromorphic function. For example, if $f_1(z) = e^z + 1$ and $f_2(z) = e^z + z$, by calculation we see that $\Delta f_1(z) = (e-1)e^z$ has no zeros, whereas

$\Delta f_2(z) = (e-1)e^z + 1$ satisfies $\lambda(\Delta f_2) = \sigma(f_2)$. Even for a meromorphic function of small lower order, $\Delta f(z)$ may have finitely many zeros. To show this, Bergweiler and Langley [8] constructed a meromorphic function like this and obtained the following:

Theorem C *Let $\phi(r)$ be a positive nondecreasing function on $[1,\infty)$ that satisfies $\lim_{r\to\infty}\phi(r) = \infty$. Then there exists a transcendental and meromorphic function f in the plane with*

$$\overline{\lim_{r\to\infty}}\frac{T(r,f)}{r} < \infty, \qquad \underline{\lim_{r\to\infty}}\frac{T(r,f)}{\phi(r)\log r} < \infty$$

such that $g(z) = \Delta f(z)$ has only one zero. Moreover, the function g satisfies

$$\overline{\lim_{r\to\infty}}\frac{T(r,g)}{\phi(r)\log r} < \infty.$$

Many authors tried hard to investigate zeros of difference and divided difference of meromorphic solutions of difference equation and obtained profound results (see [7, 9, 10], etc.). Fixing the connection of d and k, we get the number of Nevanlinna exceptional values of a meromorphic solution f of (1.1) and discover some very good properties of f. These results are stated as follows.

Theorem 1.2 *Suppose that $f(z)$ is an admissible meromorphic solution of difference equation* (1.1) *of finite order, where $k \in \mathbb{N}^+$ and $d = \max\{\deg_f P, \deg_f Q\} = 7k$. Then*

(i) *$f(z)$ has no Nevanlinna exceptional value*;
(ii) $\lambda(\Delta f) = \lambda(\frac{1}{\Delta f}) = \sigma(f)$, $\lambda(\frac{\Delta f}{f}) = \lambda(\frac{1}{\frac{\Delta f}{f}}) = \sigma(f)$;
(iii) $T(r,\Delta f) = 2T(r,f) + S(r,f)$.

Set $d = 2k$ in Theorem 1.1(iii). By the proof of Theorem 1.1(iii) we easily obtain the following corollary.

Corollary 1.1 *Suppose that $f(z)$ is an admissible entire solution of difference equation* (1.1) *of finite order, where $k \in \mathbb{N}^+$ and $d = \max\{\deg_f P, \deg_f Q\} = 2k$. Then*

(i) *$f(z)$ has no finite Nevanlinna exceptional value*;
(ii) $T(r,\Delta f) = T(r,f) + S(r,f)$.

The following Example 1.3 satisfies the conditions and results of Theorem 1.2.

Example 1.3 In Example 1.1, we see that $f(z) = z\tan\frac{\pi z}{4}$ is also a transcendental meromorphic solution of finite order of the equation

$$f(z+1)f(z-1) = 1 - z^2.$$

By the following Lemma 2.6, $f(z)$ has no Nevanlinna exceptional value, and

$$\lambda(\Delta f) = \lambda\left(\frac{1}{\Delta f}\right) = \sigma(f), \lambda\left(\frac{\Delta f}{f}\right) = \lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) = \sigma(f).$$

By calculation we have

$$\Delta f(z) = \frac{z\tan^2\frac{\pi z}{4} + \tan\frac{\pi z}{4} + z + 1}{1 - \tan\frac{\pi z}{4}}.$$

Thus,

$$T(r, \Delta f) = 2T\left(r, \tan\frac{\pi z}{4}\right) + S\left(r, \tan\frac{\pi z}{4}\right) = 2T(r,f) + S(r,f).$$

2 Lemmas for the proofs of theorems

Lemma 2.1 ([11, 12]) *Let $f(z)$ be a meromorphic function of finite order σ, and let η be a nonzero complex constant. Then, for each ε $(0 < \varepsilon < 1)$, we have*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

Lemma 2.2 ([11]) *Let $f(z)$ be a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < \infty$, and let $\eta \neq 0$ be fixed. Then, for each ε $(0 < \varepsilon < 1)$,*

$$N\bigl(r, f(z+\eta)\bigr) = N\bigl(r, f(z)\bigr) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r).$$

Lemma 2.3 ([11]) *Let $f(z)$ be a meromorphic function of order $\sigma = \sigma(f), \sigma < \infty$, and let η be a fixed nonzero complex number. Then, for each $\varepsilon > 0$,*

$$T\bigl(r, f(z+\eta)\bigr) = T\bigl(r, f(z)\bigr) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r).$$

Lemma 2.4 ([2], Lemma 1.3) *Suppose that $f(z)$ is a nonconstant meromorphic function in $|z| < R$. Let a_j $(j = 1, 2, \ldots, q)$ be distinct finite complex numbers. Then, for $0 < r < R$, we have*

$$m\left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) = \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + O(1).$$

Lemma 2.5 ([13] (Valiron-Mohon'ko)) *Let $f(z)$ be a meromorphic function. Then, for all irreducible rational functions in f,*

$$R\bigl(z, f(z)\bigr) = \frac{a_n(z)f(z)^n + \cdots + a_0(z)}{b_m(z)f(z)^m + \cdots + b_0(z)}$$

with meromorphic coefficients $a_i(z)$ and $b_j(z)$ being small with respect to f, the characteristic function of $R(z, f(z))$ satisfies

$$T\bigl(r, R\bigl(z, f(z)\bigr)\bigr) = \max\{m, n\}T(r,f) + S(r,f).$$

Lemma 2.6 ([10]) *Suppose that $h(z)$ is a nonconstant rational function. If $w(z)$ is a transcendental meromorphic solution of finite order of the equation*

$$w(z+1)w(z-1) = h(z).$$

Then

(i) $w(z)$ *has no Nevanlinna exceptional value*;
(ii) $\lambda(\Delta w) = \lambda(\frac{1}{\Delta w}) = \sigma(w)$, $\lambda(\frac{\Delta w}{w}) = \lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$.

Lemma 2.7 ([14]) *Let* $g : (0,+\infty) \to R, h : (0,+\infty) \to R$ *be nondecreasing functions. Suppose that*
(i) $g(r) \le h(r)$ *outside of an exceptional set of finite linear measure, or*
(ii) $g(r) \le h(r), r \notin H \cup (0,1]$, *where* $H \subset (1,\infty)$ *is a set of finite logarithmic measure.*
Then, for any $\alpha > 1$, *there exists* $r_0 > 0$ *such that* $g(r) \le h(\alpha r)$ *for all* $r > r_0$.

From Lemma 2.7 we easily obtain the following:

Lemma 2.8 *Let* $g(r)$ *be a nondecreasing function, and* E *be a set of finite logarithmic measure or finite linear measure. Then*

$$\varliminf_{\substack{r\to\infty\\ r\in(0,\infty)}} \frac{\log g(r)}{\log r} = \varliminf_{\substack{r\to\infty\\ r\notin E}} \frac{\log g(r)}{\log r}$$

and

$$\varlimsup_{\substack{r\to\infty\\ r\in(0,\infty)}} \frac{\log g(r)}{\log r} = \varlimsup_{\substack{r\to\infty\\ r\notin E}} \frac{\log g(r)}{\log r}.$$

3 Proofs of theorems

Proof of Theorem 1.1 (i) We first prove that $d \le 7k$. Lemma 2.1 shows that

$$m(r,R) = km\left(r, \frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta^2 f(z)}{\Delta f(z)}\right)^2\right) = S(r,f).$$

Combining this with Lemma 2.5, we have

$$T(r,R) = m(r,R) + N(r,R) = N(r,R) + S(r,f) = dT(r,f) + S(r,f). \tag{3.1}$$

Rewrite (1.1) in the form

$$\begin{aligned} R(z,f) &= \left[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta^2 f(z)}{\Delta f(z)}\right)^2\right]^k \\ &= \left[\frac{\Delta f(z+2) - 2\Delta f(z+1) + \Delta f(z)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta f(z+1) - \Delta f(z)}{\Delta f(z)}\right)^2\right]^k \\ &= \left[\frac{\Delta f(z+2)}{\Delta f(z)} + \frac{\Delta f(z+1)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta f(z+1)}{\Delta f(z)}\right)^2 - \frac{1}{2}\right]^k \\ &= \left[\frac{f(z+3) - f(z+2)}{f(z+1) - f(z)} + \frac{f(z+2) - f(z+1)}{f(z+1) - f(z)} - \frac{3}{2}\left(\frac{f(z+2) - f(z+1)}{f(z+1) - f(z)}\right)^2 - \frac{1}{2}\right]^k \\ &= \left[\frac{f(z+3) - f(z+1)}{f(z+1) - f(z)} - \frac{3}{2}\left(\frac{f(z+2) - f(z+1)}{f(z+1) - f(z)}\right)^2 - \frac{1}{2}\right]^k. \end{aligned} \tag{3.2}$$

If z_0 is a pole of $f(z+1)$ and not a pole of $f(z+3), f(z+2)$, or $f(z)$, then z_0 cannot be a pole of $\frac{f(z+3)-f(z+2)}{f(z+1)-f(z)}$ or a pole of $\frac{f(z+2)-f(z+1)}{f(z+1)-f(z)}$. By (3.2) we see that z_0 is not a pole of $R(z,f(z))$.

Similarly, if z_1 is a pole of $f(z)$and not a pole of $f(z+3), f(z+2)$, or $f(z+1)$, then z_1 cannot be a pole of $R(z,f(z))$. By this and (3.2), the poles of $R(z,f(z))$ come from poles of $f(z+3)$ and $f(z+2)$ and zeros of $\Delta f(z)$. By (3.1), (3.2), and Lemmas 2.2 and 2.3 we obtain

$$\begin{aligned} dT(r,f) &= N(r,R) + S(r,f) \\ &\leq kN\bigl(r,f(z+3)\bigr) + 2kN\bigl(r,f(z+2)\bigr) + 2kN\left(r,\frac{1}{\Delta f(z)}\right) + S(r,f) \\ &\leq 3kN\bigl(r,f(z)\bigr) + 2kN\left(r,\frac{1}{\Delta f(z)}\right) + S(r,f) \qquad (3.3) \\ &\leq 3kT\bigl(r,f(z)\bigr) + 2kN\left(r,\frac{1}{\Delta f(z)}\right) + S(r,f) \qquad (3.4) \\ &\leq 3kT\bigl(r,f(z)\bigr) + 2kT\bigl(r,\Delta f(z)\bigr) + S(r,f) \\ &\leq 7kT(r,f) + S(r,f). \qquad (3.5) \end{aligned}$$

From this it follows that $d \leq 7k$.

(ii) By (3.4) we have

$$N\left(r,\frac{1}{\Delta f(z)}\right) \geq \frac{d-3k}{2k}T(r,f) + S(r,f).$$

Combining this with Lemma 2.3, we obtain

$$\begin{aligned} m\left(r,\frac{1}{\Delta f(z)}\right) &= T\bigl(r,\Delta f(z)\bigr) - N\left(r,\frac{1}{\Delta f(z)}\right) \\ &\leq 2T\bigl(r,f(z)\bigr) - \frac{d-3k}{2k}T(r,f) + S(r,f) \\ &= \left(\frac{7}{2} - \frac{d}{2k}\right)T(r,f) + S(r,f). \qquad (3.6) \end{aligned}$$

From (3.6) and Lemmas 2.1 and 2.4 we deduce that

$$\begin{aligned} \sum_{j=1}^{s} m\left(r,\frac{1}{f-\alpha_j}\right) &= m\left(r,\sum_{j=1}^{s}\frac{1}{f-\alpha_j}\right) + O(1) \\ &\leq m\left(r,\sum_{j=1}^{s}\frac{\Delta f}{f-\alpha_j}\right) + m\left(r,\frac{1}{\Delta f}\right) + O(1) \\ &= m\left(r,\frac{1}{\Delta f}\right) + S(r,f) \\ &\leq \left(\frac{7}{2} - \frac{d}{2k}\right)T(r,f) + S(r,f). \qquad (3.7) \end{aligned}$$

By the definition of $S(r,f)$, (3.7) shows that

$$\sum_{j=1}^{s} m\left(r,\frac{1}{f-\alpha_j}\right) \leq \left(\frac{7}{2} - \frac{d}{2k} + o(1)\right)T(r,f), \quad r \notin E,$$

where $E \subset (1,\infty)$ is a set of finite logarithmic measure. Thus,

$$\begin{aligned}
\sum_{j=1}^{s} \delta(\alpha_j,f) &= \sum_{j=1}^{s} \underline{\lim}_{r\to\infty} \frac{m(r,\frac{1}{f-\alpha_j})}{T(r,f)} \le \underline{\lim}_{r\to\infty} \frac{\sum_{j=1}^{s} m(r,\frac{1}{f-\alpha_j})}{T(r,f)} \\
&\le \underline{\lim}_{\substack{r\to\infty \\ r\notin E}} \frac{\sum_{j=1}^{s} m(r,\frac{1}{f-\alpha_j})}{T(r,f)} \\
&\le \underline{\lim}_{\substack{r\to\infty \\ r\notin E}} \frac{(\frac{7}{2}-\frac{d}{2k}+o(1))T(r,f)}{T(r,f)} \\
&= \frac{7}{2}-\frac{d}{2k},
\end{aligned}$$

that is,

$$\sum_{j=1}^{s} \delta(\alpha_j,f) \le \frac{7}{2}-\frac{d}{2k}.$$

(iii) We still have (3.1)-(3.3). By $N(r,f) = S(r,f)$ and Lemmas 2.1 and 2.2 we have

$$\begin{aligned}
N\left(r,\frac{1}{\Delta f}\right) &\le T(r,\Delta f) = m(r,\Delta f) + N(r,\Delta f) \\
&\le m(r,f) + m\left(r,\frac{\Delta f}{f}\right) + N\bigl(r,f(z+1)\bigr) + N\bigl(r,f(z)\bigr) \\
&= m(r,f) + 2N\bigl(r,f(z)\bigr) + S(r,f) \\
&= T(r,f) + S(r,f).
\end{aligned} \tag{3.8}$$

By $N(r,f) = S(r,f)$, (3.3), (3.8), and Lemmas 2.1 and 2.5 we obtain

$$\begin{aligned}
dT(r,f) &= N(r,R) + S(r,f) \\
&\le 2kN\left(r,\frac{1}{\Delta f}\right) + 3kN\bigl(r,f(z)\bigr) + S(r,f) \\
&= 2kN\left(r,\frac{1}{\Delta f}\right) + S(r,f) \\
&\le 2kT(r,f) + S(r,f),
\end{aligned}$$

that is,

$$dT(r,f) = N(r,R) + S(r,f) \le 2kN\left(r,\frac{1}{\Delta f}\right) + S(r,f) \le 2kT(r,f) + S(r,f). \tag{3.9}$$

It follows that $d \le 2k$.

By (3.1) and Lemma 2.5 we have

$$\begin{aligned}
dT(r,f) &= N(r,R) + S(r,f) \\
&= N\left(r,\frac{1}{Q}\right) + S(r,f)
\end{aligned}$$

$$\begin{aligned}
&\le T(r,Q) + S(r,f)\\
&= qT(r,f) + S(r,f),
\end{aligned}$$

which yields $q \ge d$. So, $q = d \le 2k$.

By (3.9) we have

$$N\left(r, \frac{1}{\Delta f}\right) \ge \frac{d}{2k} T(r,f) + S(r,f).$$

Combining this with Lemma 2.1, we obtain

$$\begin{aligned}
m\left(r, \frac{1}{\Delta f}\right) &= T(r, \Delta f) - N\left(r, \frac{1}{\Delta f}\right)\\
&= m(r, \Delta f) - N\left(r, \frac{1}{\Delta f}\right)\\
&\le m(r,f) + m\left(r, \frac{\Delta f}{f}\right) - \frac{d}{2k} T(r,f) + S(r,f)\\
&= \left(1 - \frac{d}{2k}\right) T(r,f) + S(r,f).
\end{aligned} \tag{3.10}$$

By (3.10) and Lemmas 2.1 and 2.4 we have

$$\begin{aligned}
\sum_{j=1}^{s} m\left(r, \frac{1}{f - \alpha_j}\right) &= m\left(r, \sum_{j=1}^{s} \frac{1}{f - \alpha_j}\right) + O(1)\\
&\le m\left(r, \sum_{j=1}^{s} \frac{\Delta f}{f - \alpha_j}\right) + m\left(r, \frac{1}{\Delta f}\right) + O(1)\\
&= m\left(r, \frac{1}{\Delta f}\right) + S(r,f)\\
&\le \left(1 - \frac{d}{2k}\right) T(r,f) + S(r,f),
\end{aligned}$$

that is,

$$\sum_{j=1}^{s} m\left(r, \frac{1}{f - \alpha_j}\right) \le \left(1 - \frac{d}{2k} + o(1)\right) T(r,f), \quad r \notin E_1,$$

where $E_1 \subset (1, \infty)$ is a set of finite logarithmic measure. Thus,

$$\begin{aligned}
\sum_{j=1}^{s} \delta(\alpha_j, f) &= \sum_{j=1}^{s} \varliminf_{r\to\infty} \frac{m(r, \frac{1}{f-\alpha_j})}{T(r,f)} \le \varliminf_{r\to\infty} \frac{\sum_{j=1}^{s} m(r, \frac{1}{f-\alpha_j})}{T(r,f)}\\
&\le \varliminf_{\substack{r\to\infty\\ r\notin E_1}} \frac{\sum_{j=1}^{s} m(r, \frac{1}{f-\alpha_j})}{T(r,f)} \le \varliminf_{\substack{r\to\infty\\ r\notin E_1}} \frac{(1 - \frac{d}{2k} + o(1)) T(r,f)}{T(r,f)}\\
&= 1 - \frac{d}{2k}.
\end{aligned}$$

So,

$$\sum_{j=1}^{s}\delta(\alpha_j,f)\le 1-\frac{d}{2k}. \qquad \square$$

Proof of Theorem 1.2 (i) Using the same method as in Theorem 1.1(i), we still have (3.1)-(3.5). By the fact $d=7k$ and (3.5) we have

$$\begin{aligned}3kN(r,f)+2kN\left(r,\frac{1}{\Delta f(z)}\right)&=3kT(r,f)+2kT\bigl(r,\Delta f(z)\bigr)+S(r,f)\\&=7kT(r,f)+S(r,f).\end{aligned}$$

From this it follows that

$$N(r,f)=T(r,f)+S(r,f) \tag{3.11}$$

and

$$N\left(r,\frac{1}{\Delta f(z)}\right)=T\bigl(r,\Delta f(z)\bigr)+S(r,f)=2T(r,f)+S(r,f). \tag{3.12}$$

By (3.11) we see that

$$m(r,f)=T(r,f)-N(r,f)=S(r,f), \tag{3.13}$$

Hence,

$$m(r,f)=o\bigl(T(r,f)\bigr),\quad r\notin E,$$

where $E\subset(1,\infty)$ is a set of finite logarithmic measure. Therefore,

$$\delta(\infty,f)=\varliminf_{r\to\infty}\frac{m(r,f)}{T(r,f)}\le\varliminf_{\substack{r\to\infty\\ r\notin E}}\frac{m(r,f)}{T(r,f)}=\varliminf_{\substack{r\to\infty\\ r\notin E}}\frac{o(T(r,f))}{T(r,f)}=0.$$

So, $\delta(\infty,f)=0$.

We deduce from (3.12) that

$$m\left(r,\frac{1}{\Delta f(z)}\right)=T\bigl(r,\Delta f(z)\bigr)-N\left(r,\frac{1}{\Delta f(z)}\right)=S(r,f). \tag{3.14}$$

For any given $a\in\mathbb{C}$, by (3.14) and Lemma 2.1 we have

$$m\left(r,\frac{1}{f(z)-a}\right)\le m\left(r,\frac{\Delta f(z)}{f(z)-a}\right)+m\left(r,\frac{1}{\Delta f(z)}\right)=S(r,f),$$

that is,

$$m\left(r,\frac{1}{f(z)-a}\right)=o\bigl(T(r,f)\bigr),\quad r\notin E_1,$$

where $E_1 \subset (1,\infty)$ is a set of finite logarithmic measure. Hence,

$$\delta(a,f) = \varliminf_{r\to\infty} \frac{m(r, \frac{1}{f(z)-a})}{T(r,f)} \leq \varliminf_{\substack{r\to\infty \\ r\notin E_1}} \frac{m(r, \frac{1}{f(z)-a})}{T(r,f)} = \varliminf_{\substack{r\to\infty \\ r\notin E_1}} \frac{o(T(r,f))}{T(r,f)} = 0.$$

So, $\delta(a,f) = 0$.

Thus, for any $a \in \mathbb{C} \cup \{\infty\}$, we have $\delta(a,f) = 0$. So, $f(z)$ has no Nevanlinna exceptional value.

(ii) By (3.12) we easily obtain that $\lambda(\Delta f) = \sigma(\Delta f) = \sigma(f)$. By (3.13) and Lemma 2.1 we see

$$m(r, \Delta f) \leq m\left(r, \frac{\Delta f}{f}\right) + m(r,f) = S(r,f).$$

Together with (3.12), we have

$$N(r, \Delta f) = T(r, \Delta f) - m(r, \Delta f) = T(r, \Delta f) + S(r,f) = 2T(r,f) + S(r,f)$$

or

$$N(r, \Delta f) = \bigl(2 + o(1)\bigr) T(r,f), \quad r \notin E_2,$$

where $E_2 \subset (1,\infty)$ is a set of finite logarithmic measure.

By Lemmas 2.7 and 2.8 we see

$$\begin{aligned} \lambda\left(\frac{1}{\Delta f}\right) &= \varlimsup_{r\to\infty} \frac{\log N(r, \Delta f)}{\log r} = \varlimsup_{\substack{r\to\infty \\ r\notin E_2}} \frac{\log N(r, \Delta f)}{\log r} \\ &= \varlimsup_{\substack{r\to\infty \\ r\notin E_2}} \frac{\log(2 + o(1)) T(r,f)}{\log r} = \varlimsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \sigma(f). \end{aligned}$$

So, $\lambda(\frac{1}{\Delta f}) = \sigma(f)$.

Finally, we prove that $\lambda(\frac{\Delta f}{f}) = \lambda(\frac{1}{\frac{\Delta f}{f}}) = \sigma(f)$. Set $g(z) = \frac{\Delta f(z)}{f(z)}$. Then

$$\begin{cases} f(z+1) = (g(z)+1)f(z), \\ f(z+2) = (g(z+1)+1)f(z+1) = (g(z+1)+1)(g(z)+1)f(z), \\ f(z+3) = (g(z+2)+1)f(z+2) = (g(z+2)+1)(g(z+1)+1)(g(z)+1)f(z). \end{cases}$$

Substituting this into (3.2), we have

$$R(z,f) = \left[\frac{(g(z)+1)((g(z+2)+1)(g(z+1)+1)-1)}{g(z)} - \frac{3}{2}\left(\frac{g(z+1)(g(z)+1)}{g(z)}\right)^2\right]^k.$$

Combining this with (3.1) and Lemma 2.3, we obtain

$$\begin{aligned} dT(r,f) &= T\bigl(r, R(z,f)\bigr) + S(r,f) \\ &\leq M\bigl[T\bigl(r, g(z)\bigr) + T\bigl(r, g(z+1)\bigr) + T\bigl(r, g(z+2)\bigr)\bigr] + S(r,f) \\ &= 3MT\bigl(r, g(z)\bigr) + S(r,g) + S(r,f), \end{aligned} \tag{3.15}$$

where M is some nonzero constant.

By the definition of $S(r,f)$ and $S(r,g)$, (3.15) can be rewritten as

$$\bigl(d+o(1)\bigr)T(r,f)\le\bigl(3M+o(1)\bigr)T(r,g),\quad r\notin E_3,$$

where $E_3\subset(1,\infty)$ is a set of finite logarithmic measure.

Applying Lemmas 2.7 and 2.8 to the last inequality, we have

$$\begin{aligned}\sigma(f)&=\varlimsup_{r\to\infty}\frac{\log T(r,f)}{\log r}=\varlimsup_{\substack{r\to\infty\\ r\notin E_3}}\frac{\log T(r,f)}{\log r}\\&\le\varlimsup_{\substack{r\to\infty\\ r\notin E_3}}\frac{\log(\frac{3M+o(1)}{d+o(1)})+\log T(r,g)}{\log r}\\&=\varlimsup_{r\to\infty}\frac{\log T(r,g)}{\log r}=\sigma(g),\end{aligned}$$

that is, $\sigma(f)\le\sigma(g)$. So, $\sigma(\frac{\Delta f}{f})=\sigma(g)=\sigma(f)$.

If the quantity $\phi(r)$ satisfies $\phi(r)=S(r,f)$, then we have $\phi(r)=S(r,g)$ by (3.15). Hence,

$$S(r,f)=o\bigl(T(r,f)\bigr)\le o\bigl(T(r,g)\bigr)=S(r,g)=S\left(r,\frac{\Delta f}{f}\right).\tag{3.16}$$

Lemma 2.1 and (3.16) show that

$$m\left(r,\frac{\Delta f}{f}\right)=S(r,f)\le S\left(r,\frac{\Delta f}{f}\right),$$

from which it follows that

$$N\left(r,\frac{\Delta f}{f}\right)=T\left(r,\frac{\Delta f}{f}\right)-m\left(r,\frac{\Delta f}{f}\right)=T\left(r,\frac{\Delta f}{f}\right)+S\left(r,\frac{\Delta f}{f}\right).$$

By the definition of $S(r,\frac{\Delta f}{f})$, the last equality shows that

$$N\left(r,\frac{\Delta f}{f}\right)=\bigl(1+o(1)\bigr)T\left(r,\frac{\Delta f}{f}\right),\quad r\notin E_4,\tag{3.17}$$

where $E_4\subset(1,\infty)$ is a set of finite logarithmic measure.

Applying Lemmas 2.7 and 2.8 to (3.17), we see that

$$\begin{aligned}\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right)&=\varlimsup_{r\to\infty}\frac{\log N(r,\frac{\Delta f}{f})}{\log r}=\varlimsup_{\substack{r\to\infty\\ r\notin E_4}}\frac{\log N(r,\frac{\Delta f}{f})}{\log r}\\&=\varlimsup_{\substack{r\to\infty\\ r\notin E_4}}\frac{\log(1+o(1))T(r,\frac{\Delta f}{f})}{\log r}\\&=\varlimsup_{r\to\infty}\frac{\log T(r,\frac{\Delta f}{f})}{\log r}=\sigma\left(\frac{\Delta f}{f}\right).\end{aligned}$$

So, $\lambda(\frac{1}{\frac{\Delta f}{f}})=\sigma(\frac{\Delta f}{f})=\sigma(f)$.

By (3.13), (3.14), and (3.16) we see that

$$m\left(r, \frac{1}{\frac{\Delta f}{f}}\right) = m\left(r, \frac{f}{\Delta f}\right) \leq m(r,f) + m\left(r, \frac{1}{\Delta f}\right) = S(r,f) \leq S\left(r, \frac{\Delta f}{f}\right).$$

Thus,

$$N\left(r, \frac{1}{\frac{\Delta f}{f}}\right) = T\left(r, \frac{\Delta f}{f}\right) - m\left(r, \frac{1}{\frac{\Delta f}{f}}\right) = T\left(r, \frac{\Delta f}{f}\right) + S\left(r, \frac{\Delta f}{f}\right)$$

or

$$N\left(r, \frac{1}{\frac{\Delta f}{f}}\right) = \bigl(1 + o(1)\bigr) T\left(r, \frac{\Delta f}{f}\right), \quad r \notin E_5, \tag{3.18}$$

where $E_5 \subset (1, \infty)$ is a set of finite logarithmic measure.

Applying Lemmas 2.7 and 2.8 to (3.18), we obtain

$$\begin{aligned}
\lambda\left(\frac{\Delta f}{f}\right) &= \overline{\lim_{r\to\infty}} \frac{\log N(r, \frac{1}{\frac{\Delta f}{f}})}{\log r} = \overline{\lim_{\substack{r\to\infty\\ r\notin E_4}}} \frac{\log N(r, \frac{1}{\frac{\Delta f}{f}})}{\log r} \\
&= \overline{\lim_{\substack{r\to\infty\\ r\notin E_4}}} \frac{\log(1 + o(1))T(r, \frac{\Delta f}{f})}{\log r} \\
&= \overline{\lim_{r\to\infty}} \frac{\log T(r, \frac{\Delta f}{f})}{\log r} = \sigma\left(\frac{\Delta f}{f}\right).
\end{aligned}$$

So, $\lambda(\frac{\Delta f}{f}) = \sigma(\frac{\Delta f}{f}) = \sigma(f)$.

(iii) From (3.12) we see that

$$T(r, \Delta f) = 2T(r,f) + S(r,f). \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the manuscript and jointly worked on the results. Both authors typed, read, and approved the final manuscript.

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