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On modified degenerate Carlitz q-Bernoulli numbers and polynomials

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Abstract

In a recent study by Kim (Bull. Korean Math. Soc. 53(4):1149-1156, 2016) an attempt was made to examine some of the identities and properties that are related to the degenerate Carlitz q-Bernoulli numbers and polynomials. In our paper we define the modified degenerate q-Bernoulli numbers and polynomials. As part of this we investigate some of the identities and properties that are associated with these numbers and polynomials which are derived from the generating functions and p-adic integral equations.

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1 Introduction

Let p be a fixed prime number. In our study, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ refer to the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$; meanwhile v_p will be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$. In terms of the q-extension, q is considered to be as indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we suppose that $|q| < 1$. If $q \in \mathbb{C}_p$, we suppose that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the notation $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

It is well known that the Bernoulli numbers are defined by the generating function to be

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = e^{Bt} \quad \text{(see [1–17])}. \tag{1}$$

By (1), we derive

$$\begin{aligned} t &= \left(e^t-1\right)e^{Bt} = e^{(B+1)t} - e^{Bt} \\ &= \sum_{n=0}^{\infty} \left\{(B+1)^n - B_n\right\} \frac{t^n}{n!}. \end{aligned} \tag{2}$$

For (2), we have

$$B_0 = 1, \qquad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{3}$$

In [2], Carlitz (1948) defined the recurrence relation as

$$\gamma_{0,q} = 1, \qquad (q\gamma_q + 1)^n - \gamma_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{4}$$

Observe that for $n = 1$, by (4), we have

$$\begin{aligned} 1 &= (q\gamma_q + 1)^1 - \gamma_{1,q} \\ &= \sum_{l=0}^{1} \binom{1}{l} q^l \gamma_{l,q} - \gamma_{1,q} \\ &= \gamma_{0,q} + (q-1)\gamma_{1,q}. \end{aligned} \tag{5}$$

By (5), we see that if $q \neq 1$, then $\gamma_1, q = 0$.

For $n = 2$, as stated in (4), we conclude that

$$\begin{aligned} 0 &= (q\gamma_q + 1)^2 - \gamma_{2,q} \\ &= \sum_{l=0}^{2} \binom{2}{l} q^l \gamma_{l,q} - \gamma_{2,q} \\ &= \gamma_{0,q} + 2q\gamma_{1,q} + q^2\gamma_{2,q} - \gamma_{2,q} \\ &= 1 + (q^2 - 1)\gamma_{2,q}. \end{aligned} \tag{6}$$

By (6), we find that $\gamma_{2,q} = -\frac{1}{q^2-1}$. Therefore we state that $\lim_{q\to 1} \gamma_{2,q} = \frac{1}{0} = \infty$. As a consequence we examine the following recurrence equation which remodels equation (4):

$$\beta_{0,q} = 1, \qquad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1 \end{cases} \qquad \text{(see [2])}. \tag{7}$$

For $n = 1$, by (7), we have

$$\begin{aligned} 1 &= q(q\beta_q + 1)^1 - \beta_{1,q} \\ &= q\sum_{l=0}^{1} \binom{1}{l} q^l \beta_{l,q} - \beta_{1,q} \\ &= q(\beta_{0,q} + q\beta_{1,q}) - \beta_{1,q} \\ &= q\beta_{0,q} + q^2\beta_{1,q} - \beta_{1,q}. \end{aligned} \tag{8}$$

By (8), we see that $1 - q = (q^2 - 1)\beta_{1,q}$ and hence $\beta_{1,q} = -\frac{1}{q+1} = -\frac{1}{[2]_q}$.

Therefore we state that $\lim_{q\to 1} \beta_{1,q} = -\frac{1}{2} = B_1$.

For $n = 2$, by (7), we have

$$\begin{aligned} 0 &= q(q\beta_q + 1)^2 - \beta_{2,q} \\ &= q\sum_{l=0}^{2} \binom{2}{l} q^l \beta_{l,q} - \beta_{2,q} \end{aligned}$$

$$\begin{aligned}&= q\bigl(\beta_{0,q} + 2q\beta_{1,q} + q^2\beta_{2,q}\bigr) - \beta_{2,q}\\&= q - 2q^2\frac{1}{1+q} + \bigl(q^3-1\bigr)\beta_{2,q}\\&= \frac{q-q^2}{1+q} + \bigl(q^3-1\bigr)\beta_{2,q}.\end{aligned} \tag{9}$$

By (9), we see that $\beta_{2,q} = \frac{q}{[2]_q[3]_q}$. Therefore we state that $\lim_{q\to 1}\beta_{2,q} = \frac{1}{6} = B_2$.

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of $\mathbb{C}_p$-valued uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ has been defined by Kim as being:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x \quad \text{(see [10])}. \tag{10}$$

The Carlitz's q-Bernoulli numbers are represented by the p-adic q-integrals on $\mathbb{Z}_p$ accordingly:

$$\int_{\mathbb{Z}_p} [x]_q^n\,d\mu_q(x) = \beta_{n,q} \quad (n \geq 0) \text{ (see [9–12])}. \tag{11}$$

Therefore, for (11), we get

$$\int_{\mathbb{Z}_p} e^{[x]_q t}\,d\mu_q(x) = \sum_{n=0}^{\infty}\beta_{n,q}\frac{t^n}{n!} \quad \text{(see [6–8, 13])}. \tag{12}$$

From (12), we are able to derive the following equation:

$$\beta_{n,q} = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{l+1}{[l+1]_q} \quad \text{(see [9–11, 14, 15])}. \tag{13}$$

Recently, Kim [9] studied some identities and properties of the degenerate Carlitz q-Bernoulli numbers and polynomials. In our paper we define the modified degenerate q-Bernoulli numbers and polynomials. As part of this we investigate some of the identities and properties that are associated with these numbers and polynomials which are calculated from the generating functions and p-adic integral equations.

2 Modified Carlitz q-Bernoulli numbers and polynomials

As a result of (11), we define the modified Carlitz q-Bernoulli numbers as follows:

$$\int_{\mathbb{Z}_p} q^{-x}[x]_q^n\,d\mu_q(x) = B_{n,q}. \tag{14}$$

Observe that, for $n = 0$, we state

$$\begin{aligned}B_{0,q} &= \int_{\mathbb{Z}_p} q^{-x}\,d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1}1\\&= \lim_{N\to\infty}\frac{1-q}{1-q^{p^N}}p^N = \frac{q-1}{\log q}.\end{aligned} \tag{15}$$

We also observe that if $f_1(x) = f(x+1)$, then

$$\begin{aligned} qI_q(f_1) &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x+1)q^{x+1} \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \left(\sum_{x=0}^{p^N-1} f(x)q^x - f(0) + f(p^N)q^{p^N} \right) \\ &= I_q(f) + \lim_{N\to\infty} \left((q-1)\frac{p^N}{q^{p^N}-1} \frac{f(p^N)-f(0)}{p^N} + (q-1)f(p^N) \right) \\ &= I_q(f) + (q-1)f(0) + \frac{q-1}{\log q} f'(0). \end{aligned} \tag{16}$$

Therefore, by (16), we are able to obtain the p-adic integral equation on $\mathbb{Z}_p$ as follows:

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0). \tag{17}$$

Therefore, examining (14) and (17), if we take $f(x) = q^{-x}[x]_q^n$, then we have

$$\begin{aligned} &q\int_{\mathbb{Z}_p} q^{-(x+1)}[x+1]_q^n \, d\mu_q(x) - \int_{\mathbb{Z}_p} q^{-x}[x]_q^n \, d\mu_q(x) \\ &\quad = (q-1)[0]_q^n + \frac{q-1}{\log q} f'(0). \end{aligned} \tag{18}$$

Hence, we are able to obtain the following recurrence relation results:

$$B_{0,q} = \frac{q-1}{\log q}, \qquad (qB_q+1)^n - B_{n,q} = \begin{cases} 1 & \text{if } n=1, \\ 0 & \text{if } n>1. \end{cases} \tag{19}$$

By (19), we state that

$$B_{n,q} = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l}{[l]_q}. \tag{20}$$

From (14), we are able to define the modified Carlitz's q-Bernoulli polynomials as follows:

$$\int_{\mathbb{Z}_p} q^{-y} e^{[x+y]_q t} \, d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}. \tag{21}$$

For (21), we are able to state

$$B_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-y} [x+y]_q^n \, d\mu_q(y). \tag{22}$$

For (22), we calculate

$$\begin{aligned} B_{n,q}(x) &= \int_{\mathbb{Z}_p} q^{-y}[x+y]_q^n \, d\mu_q(y) \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \int_{\mathbb{Z}_p} q^{-y} q^{ly} \, d\mu_q(y) \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l}{[l]_q}. \end{aligned} \tag{23}$$

3 Modified degenerate Carlitz q-Bernoulli numbers and polynomials

Here, we assume that $\lambda, t \in \mathbb{C}_p$, $0 < |\lambda|_p \leq 1$, $|t|_p < p^{-\frac{1}{p-1}}$.

In terms of (21), we define the modified degenerate Carlitz q-Bernoulli polynomials as

$$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda t)^{\frac{1}{\lambda}[x+y]_q} \, d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,\lambda,q}(x) \frac{t^n}{n!}, \tag{24}$$

when $x = 0$, $B_{n,\lambda,q} = B_{n,\lambda,q}(0)$ are called the modified degenerate Carlitz q-Bernoulli numbers.

We observe that

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{-y}(1+\lambda t)^{\frac{1}{\lambda}[x+y]_q} \, d\mu_q(y) &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{-y} \binom{\frac{[x+y]_q}{\lambda}}{n} \lambda^n t^n \, d\mu_q(y) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{-y} \left(\frac{[x+y]_q}{\lambda} \right)_n d\mu_q(y) \lambda^n \frac{t^n}{n!}, \end{aligned} \tag{25}$$

where $(\frac{[x+y]_q}{\lambda})_n = \frac{[x+y]_q}{\lambda} \times (\frac{[x+y]_q}{\lambda} - 1) \times \cdots \times (\frac{[x+y]_q}{\lambda} - n + 1)$. Note that $[x+y]_{q,n,\lambda} = [x+y]_q([x+y]_q - \lambda) \cdots ([x+y]_q - (n-1)\lambda)$ $(n \geq 1)$.

For (25), we are able to derive the following theorem.

Theorem 3.1 *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} q^{-y}[x+y]_{q,n,\lambda} \, d\mu_q(y) = B_{n,\lambda,q}(x). \tag{26}$$

Let $S_1(n,m)$ be the Stirling numbers of the first kind, which are defined by $(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l$ $(n \geq 0)$. Note that $\lim_{\lambda \to 0} B_{n,\lambda,q}(x) = B_{n,q}(x)$.

Then, by using (25), we are able to state

$$\begin{aligned} \lambda^n \int_{\mathbb{Z}_p} q^{-y} \left(\frac{[x+y]_q}{\lambda} \right)_n d\mu_q(y) &= \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} \int_{\mathbb{Z}_p} q^{-y}[x+y]_q^l \, d\mu_q(y) \\ &= \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} B_{l,q}(x). \end{aligned} \tag{27}$$

Therefore, by using (26) and (27), we are able to derive the following theorem.

Theorem 3.2 *For $n \geq 0$, we have*

$$B_{n,\lambda,q}(x) = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}B_{l,q}(x). \tag{28}$$

By using (23) and (27), we are able to present details of the following corollary.

Corollary 3.3 *For $n \geq 0$, we have*

$$B_{n,\lambda,q}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l} \frac{S_1(n,l)}{(1-q)^l}\binom{l}{k}(-1)^k q^{kx}\frac{k}{[k]_q}\lambda^{n-l}. \tag{29}$$

Observe that

$$\begin{aligned}
q^{-y}(1+\lambda t)^{\frac{[x+y]_q}{\lambda}} &= q^{-y}e^{\frac{[x+y]_q}{\lambda}\log(1+\lambda t)} \\
&= q^{-y}\sum_{m=0}^{\infty}\left(\frac{[x+y]_q}{\lambda}\right)^m \frac{1}{m!}\bigl(\log(1+\lambda t)\bigr)^m \\
&= \sum_{m=0}^{\infty} q^{-y}\left(\frac{[x+y]_q}{\lambda}\right)^m \frac{1}{m!}m!\sum_{n=m}^{\infty} S_1(n,m)\frac{\lambda^n t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)q^{-y}[x+y]_q^m\right)\frac{t^n}{n!}.
\end{aligned} \tag{30}$$

Therefore by using (30), we are able to state

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^{-y}(1+\lambda t)^{\frac{[x+y]_q}{\lambda}}\,d\mu_q(y) &= \sum_{n=0}^{\infty}\sum_{m=0}^{m}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p} q^{-y}[x+y]_q^m\,d\mu_q(y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)B_{m,q}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{31}$$

By substituting t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (24), we find

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^{-y}e^{[x+y]_q t}\,d\mu_q(y) &= \sum_{m=0}^{\infty} B_{m,\lambda,q}(x)\frac{1}{m!}\frac{1}{\lambda^m}\bigl(e^{\lambda t}-1\bigr)^m \\
&= \sum_{m=0}^{\infty} B_{m,\lambda,q}(x)\lambda^{-m}\sum_{n=m}^{\infty} S_2(n,m)\frac{\lambda^n t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} B_{m,\lambda,q}(x)\lambda^{n-m}S_2(n,m)\frac{t^n}{n!},
\end{aligned} \tag{32}$$

where $S_2(n,m)$ are the Stirling numbers of the second kind as follows:

$$\bigl(e^t-1\bigr)^m = m!\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}. \tag{33}$$

Note that the left-hand side of (32) is derived as

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{-y} e^{[x+y]_q t}\, d\mu_q(y) &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{-y}[x+y]_q^n\, d\mu_q(y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!}. \end{aligned} \tag{34}$$

Therefore, by using (32) and (34), the following theorem can be derived.

Theorem 3.4 *For $n \geq 0$, we have*

$$B_{n,q}(x) = \sum_{m=0}^{n} B_{m,\lambda,q}(x)\lambda^{n-m} S_2(n,m). \tag{35}$$

Note that

$$\begin{aligned} q^{-y}(1+\lambda t)^{\frac{[x+y]_q}{\lambda}} &= q^{-y}(1+\lambda t)^{\frac{[x]_q}{\lambda}}(1+\lambda t)^{\frac{q^x[y]_q}{\lambda}} \\ &= \left(\sum_{m=0}^{\infty}[x]_{q,m,\lambda}\frac{t^m}{m!}\right)\sum_{l=0}^{\infty} q^{-y}\frac{q^{lx}}{\lambda^l}\frac{[y]_q^l(\log(1+\lambda t))^l}{l!} \\ &= \left(\sum_{m=0}^{\infty}[x]_{q,m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\lambda^{k-l}q^{-y+lk}[y]_q^l S_1(k,l)\right)\frac{t^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}[x]_{q,n-k,\lambda}\lambda^{k-l}q^{-y+lk}[y]_q^l S_1(k,l)\binom{n}{k}\right)\frac{t^n}{n!}. \end{aligned} \tag{36}$$

Thus, by (36), we get

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,\lambda,q}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}[x]_{q,n-k,\lambda}\lambda^{k-l}q^{lx}\int_{\mathbb{Z}_p} q^{-y}[y]_q^l\, d\mu_q(y) S_1(k,l)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}[x]_{q,n-k,\lambda}\lambda^{k-l}q^{lx}B_{l,q}S_1(k,l)\right)\frac{t^n}{n!}. \end{aligned} \tag{37}$$

Therefore, by (37), we obtain the following theorem.

Theorem 3.5

$$B_{n,\lambda,q}(x) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}[x]_{q,n-k,\lambda}\lambda^{k-l}q^{lx}B_{l,q}S_1(k,l). \tag{38}$$

For $r \in \mathbb{N}$, we define the modified degenerate Carlitz q-Bernoulli polynomials of order r as follows:

$$\begin{aligned} &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}(1+\lambda t)^{\frac{[x_1+\cdots+x_r+x]_q}{\lambda}}\, d\mu_q(x_1)\cdots d\mu_q(x_r) \\ &\quad = \sum_{n=0}^{\infty} B_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!}. \end{aligned} \tag{39}$$

We observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}(1+\lambda t)^{\frac{[x_1+\cdots+x_r+x]_q}{\lambda}}\,d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&\quad=\sum_{m=0}^{\infty}\lambda^{-m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}[x_1+\cdots+x_r+x]_q^m\,d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&\qquad\times\frac{1}{m!}\bigl(\log(1+\lambda t)\bigr)^m\\
&\quad=\sum_{m=0}^{\infty}B_{m,\lambda,q}^{(r)}(x)\lambda^{-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^n}{n!}t^n\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}B_{m,q}^{(r)}(x)S_1(n,m)\right)\frac{t^n}{n!},
\end{aligned}\tag{40}$$

where $B_{m,q}^{(r)}(x)$ are the modified Carlitz q-Bernoulli polynomials of order r.

As a consequence of using (39) and (40), the following theorem can be derived.

Theorem 3.6 *For $n \geq 0$, we have*

$$B_{n,\lambda,q}^{(r)}(x)=\sum_{m=0}^{n}\lambda^{n-m}B_{m,q}^{(r)}(x)S_1(n,m).\tag{41}$$

Replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (39), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}e^{[x_1+\cdots+x_r+x]_q t}\,d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&\quad=\sum_{m=0}^{\infty}B_{m,\lambda,q}^{(r)}(x)\frac{1}{m!}\lambda^{-m}\bigl(e^{\lambda t}-1\bigr)^m\\
&\quad=\sum_{m=0}^{\infty}B_{m,\lambda,q}^{(r)}(x)\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n t^n}{n!}\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}B_{m,\lambda,q}^{(r)}(x)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{42}$$

By comparing the coefficients on the right hand sides of (42), the following theorem can be obtained.

Theorem 3.7 *For $n \geq 0$, we have*

$$B_{n,q}^{(r)}(x)=\sum_{m=0}^{n}\lambda^{n-m}B_{m,\lambda,q}^{(r)}(x)S_2(n,m).\tag{43}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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