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On existence of BVP's for impulsive fractional differential equations

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Abstract

In this research, the existence of the solutions for an impulsive fractional differential equation of order q with mixed boundary conditions is studied by using some well-known fixed point theorems. At last, an example is presented to illustrate our results.

1 Introduction

The boundary value problems of fractional differential equations have attracted the attention of many authors. Fractional differential equations are used in mathematical modelling, engineering, biology, chemistry, and many other fields of science; see the references. However, the impulsive fractional differential equations has become a new topic, therefore more researchers interest focused on the field of impulsive problems for fractional differential equations; see [1–28] and the references therein.

Tian and Bai, [7] used the Banach fixed point theorem and Schauder's fixed point theorem to obtain the existence of the solutions of the problem which is given as follows:

$$\begin{aligned}
&{}^cD^{\alpha}_{0+}u(t) = f\big(t,u(t)\big),\\
&\Delta u(t)_{t=t_k} = I_k\big(u(t)\big), \quad k = 1,2,\ldots,m,\\
&\Delta u'(t)_{t=t_k} = \bar{I}_k\big(u(t)\big), \quad k = 1,2,\ldots,m,\\
&u(0) + u'(0) = 0,\\
&u(1) + u'(\xi) = 0.
\end{aligned}$$

The existence and uniqueness of the solutions for an anti-periodic BVP of nonlinear impulsive differential equations of order $\alpha \in (2,3]$ were obtained, in 2010 [17], given in the following:

$$\begin{aligned}
&\big({}^cD^{\alpha}_{0+}u(t)\big) = f\big(t,u(t)\big), \quad 2 < \alpha \leq 3\\
&\Delta u(t_k) = Q_k\big(u(t_k)\big), \quad k = 1,2,\ldots,p,\\
&\Delta u'(t_k) = I_k\big(u(t_k)\big), \quad k = 1,2,\ldots,p,
\end{aligned}$$

$$\begin{aligned}
&\Delta u''(t_k) = I_k^*\big(u(t_k)\big), \quad k = 1,2,\dots,p, \\
&u(0) = -u(1), \\
&u'(0) = -u'(1), \\
&u''(0) = -u''(1),
\end{aligned}$$

with the Caputo fractional derivative ${}^cD^\alpha_{0+}$, $f \in C(J \times \mathbb{R}, \mathbb{R})$ and $Q_k, I_k, I_k^* \in C(\mathbb{R} \times \mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = 1$.

In 2011, Cao and Chen, [6], studied the following problem to give some existence results and a continuous version of Filippov's theorem of a fractional differential inclusion:

$$\begin{aligned}
&\big({}^CD^\alpha_{0+}u(t)\big) \in f\big(t, u(t)\big), \quad \text{a.e. } t \in J \\
&\Delta u(t)_{t=t_k} = I_k\big(u(t)\big), \quad k = 1,2,\dots,m, \\
&\big(\Delta D^\beta_{0+}u(t)_{t=t_k}\big) = \bar{I}_k\big(u(t)\big), \quad k = 1,2,\dots,m, \\
&u(0) + D^\beta_{0+}u(0) = A, \\
&u(1) + D^\beta_{0+}u(\zeta) = B.
\end{aligned}$$

Here, ${}^CD^\alpha_{0+}$ is the Caputo fractional derivative and multi-valued map with compact values $F : J \times \mathbb{R} \to P(\mathbb{R})$ where $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $1 < \alpha \le 2$ and $0 < \beta < \alpha - 1$ with real numbers A, B.

In 2012, the contraction mapping principle, Krasnoselskii's theorem, Schaefer's theorem, and the Leray-Schauder alternative were used, in [18], to find the existence of the solutions of the following problem:

$$\begin{aligned}
&{}^CD^q_{0+}u(t) = f\big(t, u(t)\big), \\
&\Delta u(t_k) = y_k, \\
&\Delta u'(t_k) = \bar{y}_k, \quad k = 1,\dots,m \\
&u(0) = u_0, \qquad u'(0) = \bar{u}_0; \quad y_k, \bar{y}_k, u_0, \bar{u}_0 \in \mathbb{R}.
\end{aligned}$$

By using fixed point theorems, the existence and uniqueness solutions for an impulsive mixed boundary value problem of nonlinear differential equations of fractional order were studied in 2016, [1], which is given as

$$\begin{aligned}
&{}^CD^q_{0+}u(t) = f\big(t, u(t)\big), \quad t \in J' \\
&\Delta u(t_k) = I_k\big(u(t_k)\big), \qquad \Delta u'(t_k) = J_k\big(u(t_k)\big), \\
&u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0,
\end{aligned}$$

where $q \in (1,2)$ and ${}^CD^q_{0+}$ is the Caputo derivative of order q.

Motivated by the above mentioned work, we focus on the existence of solutions of fractional differential equation:

$$ {}^CD^q_{0+}u(t) = f\big(t, u(t)\big), \quad t \in J', \tag{1}$$

with boundary conditions;

$$\begin{aligned}
&\Delta u(t_k) = I_k\big(u(t_k)\big) = u\big(t_k^+\big) - u\big(t_k^-\big),\\
&\Delta u'(t_k) = J_k\big(u(t_k)\big) = u'\big(t_k^+\big) - u'\big(t_k^-\big); \quad k = 1,\ldots,p\\
&u(0) + \mu_1 u'(1) = \sigma_1,\\
&u'(0) + \mu_2 u(1) = \sigma_2,
\end{aligned} \tag{2}$$

where ${}^C D^q_{0+}$ is the Caputo derivative of order $q \in (1,2)$, $J = [0,1]$, $J' = J\backslash\{t_1, t_2, \ldots, t_p\}$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ and $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$. Here, respectively, the right and the left limits of $u(t)$ at $t = t_k^+$ are represented by $u(t_k^+)$ and $u(t_k^-)$.

2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper. We have

$$\begin{aligned}
PC(J) = \big\{&u : [0,1] \to R, u \in C\big(J'\big), \text{and } u\big(t_k^+\big), u\big(t_k^-\big) \text{ exists},\\
&\text{and } u\big(t_k^-\big) = u(t_k), 1 \le k \le p\big\}.
\end{aligned}$$

Obviously, $PC(J)$ is a Banach space with the norm

$$\|u\|_{PC} = \sup_{0\le t\le 1} \big|u(t)\big|.$$

Definition 1 The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : [0,+\infty) \to R$ is defined as

$$I^\alpha_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

provided that the right hand side of the integral is pointwise defined on $(0,+\infty)$ and Γ is the gamma function.

Definition 2 The *Caputo* derivative of order $\alpha > 0$ for a function $f : [0,+\infty) \to R$ is written as

$$D^\alpha_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ is the integral part of α.

Lemma 3 *Let $\alpha > 0$. Then the differential equation $D^\alpha_{0+} f(t) = 0$ has solutions*

$$f(t) = k_0 + k_1 t + k_2 t^2 + \cdots + k_{n-1} t^{n-1}$$

and

$$I^\alpha_{0+} D^\alpha_{0+} f(t) = f(t) + k_0 + k_1 t + k_2 t^2 + \cdots + k_{n-1} t^{n-1},$$

where $k_i \in R$ and $i = 1, 2, \ldots, n = [\alpha] + 1$.

Lemma 4 ([9]) *The set $F \subset PC([0,1], R^n)$ is relatively compact if and only if F is bounded, that is, $\|x\| \leq C$ for each $x \in F$ and some $C > 0$, and/or F is quasi-equicontinuous in $[0,1]$. That is to say, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in F$, $k \in N$; $\tau_1, \tau_2 \in (t_{k-1}, t_k]$ and $|\tau_1 - \tau_2| < \delta$, we have $|x(\tau_1) - x(\tau_2)| < \varepsilon$.*

Lemma 5 ([19]) *Let M be a closed, convex, and nonempty subset of Banach space X, and the operators A and B be such that*

(i) *$Ax + By \in M$ whenever $x, y \in M$;*
(ii) *A is compact and continuous;*
(iii) *B is contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Lemma 6 *For $q \in (1,2)$, and the continuous function $f : J \to R$, we have the following impulsive fractional boundary value problem:*

$$\begin{aligned}
&{}^C D_{0+}^q u(t) = f\big(t, u(t)\big),\\
&\Delta u(t_k) = I_k\big(u(t_k)\big) = u\big(t_k^+\big) - u\big(t_k^-\big),\\
&\Delta u'(t_k) = J_k\big(u(t_k)\big) = u'\big(t_k^+\big) - u'\big(t_k^-\big); \quad k = 1, \ldots, p,\\
&u(0) + \mu_1 u'(1) = \sigma_1,\\
&u'(0) + \mu_2 u(1) = \sigma_2,
\end{aligned}$$

has a unique solution, and Green's function is given by

$$u(t) = \begin{cases}
\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds - \mu_2\omega_2(t)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\
\quad - \mu_1\omega_1(t)\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + \sigma_1\omega_1(t) + \sigma_2\omega_2(t), \quad t \in [0, t_1], \\
\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds - \mu_2\omega_2(t)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\
\quad - \mu_1\omega_1(t)\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + \sigma_1\omega_1(t) + \sigma_2\omega_2(t) \\
\quad - \omega_1(t)\sum_{j=1}^p J_j(u(t_j))t_j - \omega_2(t)\sum_{j=1}^p J_j(u(t_j)) \\
\quad + \omega_1(t)\sum_{j=1}^p I_j(u(t_j)) \\
\quad + \sum_{j=k+1}^p J_j(u(t_j))(t_j - t) - \sum_{j=k+1}^p I_j(u(t_j)); \quad t \in [t_k, t_{k+1}] \\
\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds - \mu_2\omega_2(t)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\
\quad - \mu_1\omega_1(t)\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + \sigma_1\omega_1(t) + \sigma_2\omega_2(t) \\
\quad - \omega_1(t)\sum_{j=1}^p J_j(u(t_j))t_j - \omega_2(t)\sum_{j=1}^p J_j(u(t_j)) \\
\quad + \omega_1(t)\sum_{j=1}^p I_j(u(t_j)), \quad t \in [t_p, t_{p+1}],
\end{cases}$$

where

$$\omega_1(t) = \frac{1 + \mu_2 - \mu_2 t}{1 + \mu_2 - \mu_1\mu_2} \quad \text{and} \quad \omega_2(t) = \frac{t - \mu_1}{1 + \mu_2 - \mu_1\mu_2}.$$

Proof A general solution ${}^C D_{0+}^q u(t) = f(t, u(t))$, on $(t_k, t_{k+1}]$, $k = 1, \ldots, p$,

$$u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds + a_k + b_k t, \quad \text{for } t \in (t_k, t_{k+1}],$$

where $t_0 = 0$, $t_{p+1} = 1$ and taking the derivative,

$$u'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + b_k, \quad \text{for } t \in (t_k, t_{k+1}].$$

We use the boundary conditions $u(0) + \mu_1 u'(1) = \sigma_1$ and $u'(0) + \mu_2 u(1) = \sigma_2$ to get

$$a_0 + \mu_1 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + \mu_1 b_p = \sigma_1$$

and

$$b_0 + \mu_2 \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \mu_2 a_p + \mu_2 b_p = \sigma_2,$$

where

$$\begin{aligned}
&u(0) = a_0, \qquad u'(0) = b_0, \\
&u(1) = \int_0^t \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds + a_p + b_p, \\
&u'(1) = \int_0^t \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + b_p.
\end{aligned}$$

That is,

$$\begin{aligned}
\Delta u'(t_k) &= J_k\big(u(t_k)\big) \\
&= u'\big(t_k^+\big) - u'\big(t_k^-\big) \\
&= b_k - b_{k-1}, \\
b_k &= b_{k-1} + J_k\big(u(t_k)\big), \\
b_{k+1} &= b_k + J_{k+1}\big(u(t_k+1)\big), \\
b_p &= b_{k-1} + \sum_{j=k}^{p} J_j\big(u(t_j)\big), \\
b_k &= b_p - \sum_{j=k+1}^{p} J_j\big(u(t_j)\big),
\end{aligned}$$

and

$$\begin{aligned}
&\Delta u(t_k) = I_k\big(u(t_k)\big) = u\big(t_k^+\big) - u\big(t_k^-\big), \\
&a_k + b_k t_k = a_{k-1} + b_{k-1} t_k + I_k\big(u(t_k)\big).
\end{aligned}$$

Since $b_k = b_{k-1} + J_k(u(t_k))$, we have

$$\begin{aligned}
&a_k + \big(b_{k-1} + J_k\big(u(t_k)\big)\big) t_k = a_{k-1} + b_{k-1} t_k + I_k\big(u(t_k)\big), \\
&a_k + b_{k-1} t_k + J_k\big(u(t_k)\big) t_k = a_{k-1} + b_{k-1} t_k + I_k\big(u(t_k)\big),
\end{aligned}$$

$$a_k + J_k\big(u(t_k)\big)t_k = a_{k-1} + I_k\big(u(t_k)\big),$$
$$a_k = a_{k-1} - J_k\big(u(t_k)\big)t_k + I_k\big(u(t_k)\big),$$
$$a_k = a_p + \sum_{j=k+1}^{p} J_j\big(u(t_j)\big)t_j - \sum_{j=k+1}^{p} I_j\big(u(tj)\big).$$

Then

$$a_0 + \mu_1 b_p + \mu_1 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds = \sigma_1 \tag{3}$$

and

$$b_0 + \mu_2 a_p + \mu_2 b_p + \mu_2 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds = \sigma_2. \tag{4}$$

Also we get

$$\begin{aligned} b_k &= b_{k-1} + J_k\big(u(t_k)\big), \\ b_k &= b_p - \sum_{j=k+1}^{p} J_j\big(u(t_j)\big), \end{aligned} \tag{5}$$

and

$$\begin{aligned} a_k &= a_{k-1} - J_k\big(u(t_k)\big)t_k + I_k\big(u(t_k)\big), \\ a_k &= a_p + \sum_{j=k+1}^{p} J_j\big(u(t_j)\big)t_j - \sum_{j=k+1}^{p} I_j\big(u(t_j)\big). \end{aligned} \tag{6}$$

By combining (3), (4), (5), and (6)

$$a_0 + \mu_1 b_p + \mu_1 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds = \sigma_1,$$
$$a_p + \mu_1 b_p + \mu_1 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + \sum_{j=1}^{p} J_j\big(u(t_j)\big)t_j - \sum_{j=1}^{p} I_j\big(u(t_j)\big) = \sigma_1,$$

and

$$b_0 + \mu_2 a_p + \mu_2 b_p + \mu_2 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds = \sigma_2,$$
$$\left[b_p - \sum_{j=1}^{p} J_k\big(u(t_j)\big) \right] + \mu_2 a_p + \mu_2 b_p + \mu_2 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds = \sigma_2,$$
$$\mu_2 a_p + (1+\mu_2) b_p + \mu_2 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds - \sum_{j=1}^{p} J_j\big(u(t_j)\big) = \sigma_2.$$

Then

$$a_p = \sigma_1 - \mu_1 b_p - \mu_1 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds - \sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)t_j + \sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr),$$
$$a_p = \frac{\sigma_2}{\mu_2} - \frac{(1+\mu_2)}{\mu_2} b_p - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \frac{1}{\mu_2} \sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr).$$

Also we have

$$\begin{aligned} &\sigma_1 - \mu_1 b_p - \mu_1 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds - \sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)t_j + \sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr) \\ &\quad = \frac{\sigma_2}{\mu_2} - \frac{(1+\mu_2)}{\mu_2} b_p - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \frac{1}{\mu_2} \sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr). \end{aligned}$$

Therefore a_p and b_p are found as follows:

$$\begin{aligned} b_p ={}& -\left(\frac{\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\sigma_1 + \left(\frac{1}{1+\mu_2-\mu_1\mu_2}\right)\sigma_2 \\ &- \left(\frac{\mu_2}{1+\mu_2-\mu_1\mu_2}\right) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ &+ \left(\frac{\mu_1\mu_2}{1+\mu_2-\mu_1\mu_2}\right) \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds \\ &+ \left(\frac{\mu_2}{1+\mu_2-\mu_1\mu_2}\right) \sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)t_j + \left(\frac{1}{1+\mu_2-\mu_1\mu_2}\right) \sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr) \\ &- \left(\frac{\mu_2}{1+\mu_2-\mu_1\mu_2}\right) \sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr) \end{aligned} \tag{7}$$

and

$$\begin{aligned} a_p ={}& \left(\frac{1+\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\sigma_1 - \left(\frac{\mu_1}{1+\mu_2-\mu_1\mu_2}\right)\sigma_2 \\ &+ \left(\frac{\mu_1\mu_2}{1+\mu_2-\mu_1\mu_2}\right) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ &- \mu_1\left(\frac{1+\mu_2}{1+\mu_2-\mu_1\mu_2}\right) \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds \\ &- \left(\frac{1+\mu_2}{1+\mu_2-\mu_1\mu_2}\right) \sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)t_j - \left(\frac{\mu_1}{1+\mu_2-\mu_1\mu_2}\right) \sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr) \\ &+ \left(\frac{1+\mu_2}{1+\mu_2-\mu_1\mu_2}\right) \sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr). \end{aligned} \tag{8}$$

By (5), (6), (7), and (8) since

$$b_k = b_p - \sum_{j=k+1}^{p} J_j\big(u(t_j)\big),$$

$$a_k = a_p + \sum_{j=k+1}^{p} J_j\big(u(t_j)\big)t_j - \sum_{j=k+1}^{p} I_j\big(u(t_j)\big),$$

are known. By (5),

$$\begin{aligned} b_k = {} & -\left(\frac{\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\sigma_1 + \left(\frac{1}{1+\mu_2-\mu_1\mu_2}\right)\sigma_2 \\ & - \left(\frac{\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \left(\frac{\mu_1\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds \\ & + \left(\frac{\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\sum_{j=1}^{p} J_j\big(u(t_j)\big)t_j + \left(\frac{1}{1+\mu_2-\mu_1\mu_2}\right)\sum_{j=1}^{p} J_j\big(u(t_j)\big) \\ & - \left(\frac{\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\sum_{j=1}^{p} I_j\big(u(t_j)\big) - \sum_{j=k+1}^{p} J_j\big(u(t_j)\big), \end{aligned} \tag{9}$$

with the help of (6),

$$\begin{aligned} a_k = {} & \left(\frac{1+\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\sigma_1 - \left(\frac{\mu_1}{1+\mu_2-\mu_1\mu_2}\right)\sigma_2 \\ & + \left(\frac{\mu_1\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ & - \mu_1\left(\frac{1+\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds \\ & - \left(\frac{1+\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\sum_{j=1}^{p} J_j\big(u(t_j)\big)t_j - \left(\frac{\mu_1}{1+\mu_2-\mu_1\mu_2}\right)\sum_{j=1}^{p} J_j\big(u(t_j)\big) \\ & + \left(\frac{1+\mu_2}{1+\mu_2-\mu_1\mu_2}\right)\sum_{j=1}^{p} I_j\big(u(t_j)\big) + \sum_{j=k+1}^{p} J_j\big(u(t_j)\big)t_j - \sum_{j=k+1}^{p} I_j\big(u(t_j)\big), \end{aligned} \tag{10}$$

for $k = 0,1,\ldots,p-1$. By using (9) and (10), we get

$$\begin{aligned} a_k + b_k t = {} & \left[\frac{1+\mu_2-\mu_2 t}{1+\mu_2-\mu_1\mu_2}\right]\sigma_1 + \left[\frac{t-\mu_1}{1+\mu_2-\mu_1\mu_2}\right]\sigma_2 \\ & + \left[\frac{-\mu_2(t-\mu_1)}{1+\mu_2-\mu_1\mu_2}\right]\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ & + \left[\frac{-\mu_1(1+\mu_2-\mu_2 t)}{1+\mu_2-\mu_1\mu_2}\right]\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds \\ & - \left[\frac{1+\mu_2(1-t)}{1+\mu_2-\mu_1\mu_2}\right]\sum_{j=1}^{p} J_j\big(u(t_j)\big)t_j \end{aligned}$$

$$+\left[\frac{-\mu_1+t}{1+\mu_2-\mu_1\mu_2}\right]\sum_{j=1}^{p} J_j\big(u(t_j)\big)$$
$$+\left[\frac{1+\mu_2(1-t)}{1+\mu_2-\mu_1\mu_2}\right]\sum_{j=1}^{p} I_j\big(u(t_j)\big)$$
$$+\sum_{j=k+1}^{p} J_j\big(u(t_j)\big)(t_j-t)-\sum_{j=k+1}^{p} I_j\big(u(t_j)\big).$$

Thus

$$\begin{aligned}
u(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \omega_1(t)\sigma_1 + \omega_2(t)\sigma_2 \\
& - \mu_2\omega_2(t)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\
& - \mu_1\omega_1(t)\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds - \omega_1(t)\sum_{j=1}^{p} J_j\big(u(t_j)\big)t_j \\
& - \omega_2(t)\sum_{j=1}^{p} J_j\big(u(t_j)\big) + \omega_1(t)\sum_{j=1}^{p} I_j\big(u(t_j)\big) \\
& + \sum_{j=k+1}^{p} J_j\big(u(t_j)\big)(t_j-t) - \sum_{j=k+1}^{p} I_j\big(u(t_j)\big),
\end{aligned}$$

where

$$\omega_1(t) = \frac{1+\mu_2-\mu_2 t}{1+\mu_2-\mu_1\mu_2} \quad \text{and} \quad \omega_2(t) = \frac{t-\mu_1}{1+\mu_2-\mu_1\mu_2}. \qquad \square$$

2.1 Existence and uniqueness results

In this section, we state and prove existence and uniqueness results of the fractional BVP (1)-(2) by using the Banach fixed point theorem. We use the following notations throughout this paper:

$$\omega_1(t) = \frac{1+\mu_2-\mu_2 t}{1+\mu_2-\mu_1\mu_2}, \qquad \omega_2(t) = \frac{t-\mu_1}{1+\mu_2-\mu_1\mu_2}$$

and

$$\omega_1(t) \le \omega_1 := \frac{1+2|\mu_2|}{|1+\mu_2-\mu_1\mu_2|}, \qquad \omega_2(t) \le \omega_2 := \frac{1+|\mu_1|}{|1+\mu_2-\mu_1\mu_2|}.$$

By using the following conditions, we state and prove our first result.

(A1) *The function $f:[0,1]\times R \to R$ is jointly continuous.*

(A2) *There exist positive constants L_1, L_2, L_3, M_1, M_2 such that*

$$\begin{aligned}
&\big|f(t,x)-f(t,y)\big| \le L_1|x-y|, \quad t\in[0,1], x,y\in R; \\
&\big|I_k(x)-I_k(y)\big| \le L_2|x-y|, \qquad \big|J_k(x)-J_k(y)\big| \le L_3|x-y|, \\
&\big|I_k(x)\big| \le M_1, \qquad \big|J_k(x)\big| \le M_2.
\end{aligned}$$

Also it is clear that

$$\begin{aligned}|f(t,x)| &\le |f(t,x)-f(t,0)| + |f(t,0)| \\ &\le L_1|x| + M,\end{aligned}$$

where $\sup_{t\in[0,1]}|f(t,0)| = M$.

Theorem 7 *Assume* (A1)-(A2) *holds. If*

$$L_1\left(\frac{1+|\mu_2|\omega_2}{\Gamma(q+1)} + \frac{|\mu_1|\omega_1}{\Gamma(q)}\right) + (\omega_1+1)p(L_2+L_3) + \omega_2 pL_3 < 1, \tag{11}$$

then our boundary value problem (1)-(2) *has a unique solution on* $[0,1]$.

Proof By using (11) r can be chosen as follows:

$$\begin{aligned}r > {}&\left(1 - \frac{L_1}{\Gamma(q+1)}\bigl(1+|\mu_2|\omega_2\bigr) - |\mu_1|\omega_1\frac{L_1}{\Gamma(q)}\right)^{-1}\left\{\frac{M}{\Gamma(q+1)} + \omega_1|\sigma_1| + \omega_2|\sigma_2|\right.\\ &+ |\mu_2|\omega_2\frac{M}{\Gamma(q+1)} + |\mu_1|\omega_1\frac{M}{\Gamma(q)}\\ &\left.+ (\omega_1+2)p(M_1+M_2) + \omega_2 pM_2\right\}.\end{aligned}$$

Define an operator $\mathcal{T}: PC([0,1],R) \to PC([0,1],R)$ to transform (1)-(2) into the fixed point problem

$$\begin{aligned}(\mathcal{T}u)(t) = {}&\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\bigl(s,u(s)\bigr)\,ds + \omega_1(t)\sigma_1 + \omega_2(t)\sigma_2\\ &- \mu_2\omega_2(t)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f\bigl(s,u(s)\bigr)\,ds\\ &- \mu_1\omega_1(t)\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f\bigl(s,u(s)\bigr)\,ds\\ &- \omega_1(t)\sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)t_j - \omega_2(t)\sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)\\ &+ \omega_1(t)\sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr)\\ &+ \sum_{j=k+1}^{p} J_j\bigl(u(t_j)\bigr)(t_j - t) - \sum_{j=k+1}^{p} I_j\bigl(u(t_j)\bigr),\end{aligned}$$

where $t_k < t < t_{k+1}$, $k = 0,\ldots,p$. Then

$$\begin{aligned}|Tu(t)| \le {}&\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \bigl|f\bigl(s,u(s)\bigr)\bigr|\,ds + |\omega_1(t)||\sigma_1| + |\omega_2(t)||\sigma_2|\\ &+ |\mu_2||\omega_2(t)|\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \bigl|f\bigl(s,u(s)\bigr)\bigr|\,ds\end{aligned}$$

$$
\begin{aligned}
&+|\mu_1|\left|\omega_1(t)\right|\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\left|f\big(s,u(s)\big)\right|ds\\
&+\left|\omega_1(t)\right|\sum_{j=1}^{p}\left|J_j\big(u(t_j)\big)\right|+\left|\omega_2(t)\right|\sum_{j=1}^{p}\left|J_j\big(u(t_j)\big)\right|\\
&+\left|\omega_1(t)\right|\sum_{j=1}^{p}\left|I_j\big(u(t_j)\big)\right|\\
&+2\sum_{j=k+1}^{p}\left|J_j\big(u(t_j)\big)\right|+\sum_{j=k+1}^{p}\left|I_j\big(u(t_j)\big)\right|
\end{aligned}
$$

and then

$$
\begin{aligned}
&\left|\mathcal{T}u(t)\right|\\
&\quad\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\left|f\big(s,u(s)\big)-f(s,0)\right|ds+\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\left|f(s,0)\right|ds\\
&\qquad+\left|\omega_1(t)\right||\sigma_1|+\left|\omega_2(t)\right||\sigma_2|\\
&\qquad+|\mu_2|\left|\omega_2(t)\right|\left[\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\left|f\big(s,u(s)\big)-f(s,0)\right|ds+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\left|f(s,0)\right|ds\right]\\
&\qquad+|\mu_1|\left|\omega_1(t)\right|\left[\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\left|f\big(s,u(s)\big)-f(s,0)\right|ds+\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\left|f(s,0)\right|ds\right]\\
&\qquad+\left|\omega_1(t)\right|\sum_{j=1}^{p}\left|J_j\big(u(t_j)\big)\right|+\left|\omega_2(t)\right|\sum_{j=1}^{p}\left|J_j\big(u(t_j)\big)\right|+\left|\omega_1(t)\right|\sum_{j=1}^{p}\left|I_j\big(u(t_j)\big)\right|\\
&\qquad+\sum_{j=k+1}^{p}\left|J_j\big(u(t_j)\big)\right|+\sum_{j=k+1}^{p}\left|I_j\big(u(t_j)\big)\right|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left|\mathcal{T}u(t)\right|\le{}&\frac{L_1r}{\Gamma(q+1)}+\frac{M}{\Gamma(q+1)}+\omega_1|\sigma_1|+\omega_2|\sigma_2|\\
&+|\mu_2|\omega_2\left[\frac{L_1r}{\Gamma(q+1)}+\frac{M}{\Gamma(q+1)}\right]\\
&+|\mu_1|\omega_1\left[\frac{L_1r}{\Gamma(q)}+\frac{M}{\Gamma(q)}\right]\\
&+(\omega_1+2)p(M_1+M_2)+\omega_2pM_2<r.
\end{aligned}
$$

For $t\in[0,1]$, the expression is well defined. The fixed point of the operator T is the solution of our boundary value problem (1)-(2). To show the existence and uniqueness of the solution, the Banach fixed point theorem is used and then it is shown that T is a contraction and we get

$$
\begin{aligned}
&\left|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)\right|\\
&\quad\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\left|f\big(s,x(s)\big)-f\big(s,y(s)\big)\right|ds
\end{aligned}
$$

$$\begin{aligned}
&+ |\mu_2|\big|\omega_2(t)\big|\left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\big|f(s,x(s)) - f(s,y(s))\big|\,ds\right] \\
&+ |\mu_1|\big|\omega_1(t)\big|\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\big|f(s,x(s)) - f(s,y(s))\big|\,ds \\
&+ \big|\omega_1(t)\big|\sum_{j=1}^{p}\big|J_j(x(t_j)) - J_j(y(t_j))\big| \\
&+ \big|\omega_2(t)\big|\sum_{j=1}^{p}\big|J_j(x(t_j)) - J_j(y(t_j))\big| \\
&+ \big|\omega_1(t)\big|\sum_{j=1}^{p}\big|I_j(x(t_j)) - I_j(y(t_j))\big| \\
&+ 2\sum_{j=k+1}^{p}\big|J_j(x(t_j)) - J_j(y(t_j))\big| + \sum_{j=k+1}^{p}\big|I_j(x(t_j)) - I_j(y(t_j))\big|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\big|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)\big| \\
&\quad \le \Bigg[L_1\left(\frac{1}{\Gamma(q+1)} + \frac{|\mu_2||\omega_2(t)|}{\Gamma(q+1)} + \frac{|\mu_1||\omega_1(t)|}{\Gamma(q)}\right) \\
&\qquad + \big(\big|\omega_1(t)\big| + 1\big)p(L_2 + L_3) + \big|\omega_2(t)\big|pL_3\Bigg]\|x - y\|.
\end{aligned} \tag{12}$$

T is contraction mapping. By condition (11), we have

$$\begin{aligned}
&\|Tx - Ty\| \\
&\quad \le \Bigg[L_1\left(\frac{1 + |\mu_2|\omega_2}{\Gamma(q+1)} + \frac{|\mu_1|\omega_1}{\Gamma(q)}\right) \\
&\qquad + (\omega_1 + 2)p(L_2 + L_3) + \omega_2 pL_3\Bigg]\|x - y\|.
\end{aligned}$$

Thus $\mathcal{T}$ is a contraction mapping. $\mathcal{T}$ has a fixed point, and that is the solution of the BVP by the Banach fixed point theorem. □

Theorem 8 *Assume $|f(t,u)| \le \rho(t)$ for $(t,u) \in J \times R$ where $\rho \in L^{\frac{1}{\sigma}}(J \times R)$ and $\sigma \in (0, q-1)$, moreover, there exist positive constants L_1, L_2, L_3, M_1, M_2 and M such that*

$$\begin{aligned}
&\big|f(t,x) - f(t,y)\big| \le L_1|x - y|, \quad t \in [0,1], x, y \in R; \\
&\big|I_k(x) - I_k(y)\big| \le L_2|x - y|, \qquad \big|J_k(x) - J_k(y)\big| \le L_3|x - y|, \\
&\big|I_k(x)\big| \le M_1, \qquad \big|J_k(x)\big| \le M_2,
\end{aligned}$$

with

$$(\omega_1 + 2)p(L_2 + L_3) + \omega_2 pL_3 < 1. \tag{13}$$

Then our boundary value problem has at least one solution on J.

Proof Let us choose

$$r \geq \left[\|\rho\|_{L^{\frac{1}{\sigma}}} \left(\frac{(1+|\mu_2|\omega_2)}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} + \frac{|\mu_1|\omega_1}{\Gamma(q-1)(\frac{q-\sigma-1}{1-\sigma})^{1-\sigma}} \right) + (\omega_1+2)p(M_1+M_2) + \omega_2 pM_2 \right],$$

and $B_r = \{u \in PC(J,R) \mid \|u\|_{PC} \leq r\}$. The operators S and N on B_r are defined as

$$(Su)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))\,ds - \mu_2\omega_2(t)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,u(s))\,ds - \mu_1\omega_1(t)\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s,u(s))\,ds$$

and

$$(Nu)(t) = -\omega_1(t)\sum_{j=1}^p J_j(u(t_j))t_j - \omega_2(t)\sum_{j=1}^p J_j(u(t_j)) + \omega_1(t)\sum_{j=1}^p I_j(u(t_j)) + \sum_{j=k+1}^p J_j(u(t_j))(t_j-t) - \sum_{j=k+1}^p I_j(u(t_j)).$$

For any $u, v \in B_r$ and $t \in J$, by using $|f(t,u)| \leq \rho(t)$ and the Hölder inequality,

$$\begin{aligned} \frac{1}{\Gamma(q)}\int_0^t \left|(t-s)^{q-1}f(s,u(s))\right|ds &\leq \frac{1}{\Gamma(q)}\left(\int_0^t (t-s)^{\frac{q-1}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^t (\rho(s))^{\frac{1}{\sigma}}\,ds\right)^{\sigma} \\ &\leq \frac{\|\rho\|_{L^{\frac{1}{\sigma}}}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}}, \\ \frac{1}{\Gamma(q)}\int_0^1 \left|(1-s)^{q-1}f(s,u(s))\right|ds &\leq \frac{1}{\Gamma(q)}\left(\int_0^1 (1-s)^{\frac{q-1}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^1 (\rho(s))^{\frac{1}{\sigma}}\,ds\right)^{\sigma} \\ &\leq \frac{\|\rho\|_{L^{\frac{1}{\sigma}}}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}}, \end{aligned}$$

and at last

$$\begin{aligned} \frac{1}{\Gamma(q-1)}\int_0^1 \left|(1-s)^{q-2}f(s,u(s))\right|ds &\leq \frac{1}{\Gamma(q-1)}\left(\int_0^1 (1-s)^{\frac{q-2}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^1 (\rho(s))^{\frac{1}{\sigma}}\,ds\right)^{\sigma} \\ &\leq \frac{\|\rho\|_{L^{\frac{1}{\sigma}}}}{\Gamma(q-1)(\frac{q-\sigma-1}{1-\sigma})^{1-\sigma}}. \end{aligned}$$

We get

$$\|Su+Nv\| \leq \frac{(1+|\mu_2|\omega_2)\|\rho\|_{L^{\frac{1}{\sigma}}}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} + \frac{|\mu_1|\omega_1\|\rho\|_{L^{\frac{1}{\sigma}}}}{\Gamma(q-1)(\frac{q-\sigma-1}{1-\sigma})^{1-\sigma}} + (\omega_1+2)p(M_1+M_2) + \omega_2 pM_2.$$

Thus $Su + Nv \in B_r$. By (13), it is obvious that N is a contraction mapping. Moreover, the continuity of f implies S is continuous. And the operator S is uniformly bounded on B_r where

$$\|Su\| \leq \frac{(1+|\mu_2|\omega_2)\|\rho\|_{L^{\frac{1}{\sigma}}}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} + \frac{|\mu_1|\omega_1\|\rho\|_{L^{\frac{1}{\sigma}}}}{\Gamma(q-1)(\frac{q-\sigma-1}{1-\sigma})^{1-\sigma}} \leq r.$$

Here the quasi-equicontinuity of the operator S is proved. Let $\Lambda = J \times B_r$, $f_{\sup} = \sup_{(t,u)\in\Lambda} |f(t,u)|$. For any $t_k < t_2 < t_1 < t_{k+1}$, we have

$$\begin{aligned}
&\bigl|(Su)(t_2) - (Su)(t_1)\bigr| \\
&\quad \leq \frac{f_{\sup}}{\Gamma(q)}\left|\int_0^{t_2}\bigl[(t_2-s)^{q-1} - (t_1-s)^{q-1}\bigr]\,ds + \int_{t_1}^{t_2}\frac{(t_1-s)^{q-1}}{\Gamma(q)}\,ds\right| \\
&\qquad + \left||\mu_2|(t_2-t_1)\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\,ds\right| \\
&\qquad + \left||\mu_1||\mu_2|(t_2-t_1)\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\,ds\right| \\
&\quad \leq f_{\sup}\left[\frac{(t_2-t_1)^q + t_2^q - t_1^q}{\Gamma(q+1)} + |\mu_2|\frac{[(t_2^q - t_1^q)]}{\Gamma(q+1)}\right. \\
&\qquad \left. + |\mu_1|\frac{|\mu_2|(t_2^q - t_1^q)}{\Gamma(q)}\right].
\end{aligned}$$

It tends to zero as $t_2 \to t_1$. On the interval $(t_k, t_{k+1}]$, S is quasi-equicontinuous. Also by lemma (4), S is compact and is relatively compact on B_r. Therefore our BVP has at least one solution on $J = [0,1]$. □

2.2 Examples

Example 9 Consider the following boundary value problem of fractional differential equation:

$$\begin{cases}
D_{0+}^{\frac{3}{2}}u(t) = \frac{\cos u(t)}{(t+10)^2(1+u^6(t))}, \\
\Delta u(\frac{1}{3}) = \frac{|u(\frac{1}{3})|}{100+|u(\frac{1}{3})|}, \\
\Delta u'(\frac{1}{3}) = \frac{|u(\frac{1}{3})|}{100+|u(\frac{1}{3})|}, \\
u(0) + u\prime(1) = 0, \\
u(1) + u\prime(0) = 0.
\end{cases}$$

Here $t \in [0,1]$, let

$$q = \frac{3}{2}, \qquad t = \frac{1}{3}, \qquad \mu_1 = \mu_2 = 1,$$

$$\sigma_1 = \sigma_2 = 0,$$

$$L_1 = L_2 = L_3 = 0.01,$$

and since $0.88 < \Gamma(1.5) < 0.89$ and $1.33 < \Gamma(2.5) < 1.34$, we found

$$\omega_1 \leq 3 \qquad \omega_2 \leq 2.$$

Therefore,

$$\begin{aligned}
&0.01\left(\frac{(1+1.99)}{\Gamma(2.5)}+\frac{2.99}{\Gamma(1.5)}\right)+(0.01+0.01)(2.99+2)+0.01(1.99)<1,\\
&0.01(2.24+3.38)+(0.02)(4.99)+(0.01)(1.99)<1,\\
&0.0562+0.0998+0.0199<1,\\
&0.1759<1.
\end{aligned}$$

Thus, by Theorem 7, the BVP has a unique solution on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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