Bao *Advances in Difference Equations* (2017) 2017:20
DOI 10.1186/s13662-017-1079-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Multiple solutions to impulsive differential equations

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Abstract

In this paper, we study the existence of a second-order impulsive differential equations depending on a parameter λ. By employing a critical point theorem, the existence of at least three solutions is obtained.

MSC: 34A37; 34K10

Keywords: multiple solutions; critical point theorem; impulses

1 Introduction

In recent years, the study of the existence of solutions to impulsive differential equation has aroused extensive interest, we refer the reader to [1–5] and the references therein.

In [1], by using some existing critical point theorems, Xie and Luo investigated the existence of multiple solutions of the following Neumann boundary value problem:

$$\begin{aligned} &-\big(p(t)u'(t)\big)' + q(t)u(t) = \lambda f\big(t,u(t)\big), \quad t \neq t_j, t \in [0,1], \\ &\Delta p(t_j)u'(t_j) = I_j\big(u(t_j)\big), \quad j = 1,2,\ldots,m, \\ &u'(0) = u'(1) = 0. \end{aligned} \tag{1.1}$$

In [2], Liang and Zhang considered the following boundary value problems:

$$\begin{aligned} &-\big(p(t)u'(t)\big)' = f\big(t,u(t)\big), \quad t \neq t_j, t \in [0,T], \\ &\Delta p(t_j)u'(t_j) = I_j\big(u(t_j)\big), \quad j = 1,2,\ldots,m, \\ &u(0) = u(T), \qquad p(0)u'(0) = p(T)u'(T). \end{aligned} \tag{1.2}$$

The authors gave some criteria to guarantee that the problem has at least one solution under some different conditions.

In [3], Li and Shen were concerned with the existence of three solutions for the following boundary value problems:

$$\begin{aligned} &-u''(t) = \lambda f\big(u(t)\big), \quad t \neq t_j, t \in [0,1], \\ &\Delta u'(t_j) = I_j\big(u(t_j)\big), \quad j = 1,2,\ldots,m, \\ &u(0) = u(1) = 0. \end{aligned} \tag{1.3}$$

Motivated by the previous mentioned paper, in this paper, we will study the existence of at least three solutions for the following boundary value problems:

$$\begin{aligned} &-\big(p(t)u'(t)\big)' + u(t) = \lambda f\big(t, u(t)\big), \quad t \neq t_j, t \in [0,1], \\ &\Delta p(t_j)u'(t_j) = I_j\big(u(t_j)\big), \quad j = 1, 2, \ldots, m, \\ &u'(0) = u'(1) = 0, \end{aligned} \tag{1.4}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $p \in PC^1([0,1])$, $f \in C([0,1] \times R, R)$, $I_j \in C(R,R)$, $j = 1, 2, \ldots, m$, $\Delta p(t_j)u'(t_j) = p(t_j^+)u'(t_j^+) - p(t_j^-)u'(t_j^-)$, $p(t_j^+)u'(t_j^+)$ and $p(t_j^-)u'(t_j^-)$ denote the right and the left limits, respectively, $\lambda \in [0, +\infty)$ is a real parameter.

2 Preliminaries

Let $p_0 = \min_{t\in[0,1]} p(t) > 0$, $M_0 = \max\{\frac{1}{p_0}, 1\}$, $X = W^{1,2}[0,1]$ with the norm

$$\|u\| = \left(\int_0^1 \big(p(t)\big|u'(t)\big|^2 + \big|u(t)\big|^2\big)\,dt\right)^{\frac{1}{2}}.$$

Define the norm in $C([0,1])$ by $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$.

Lemma 2.1 *For any $u \in X$, we have $\|u\|_\infty \leq \sqrt{2M_0}\|u\|$.*

Proof For $u \in X$ by the mean-value theorem, there exists $\tau \in (0,1)$ such that $\int_0^1 u(s)\,ds = u(\tau)$. Hence, for $t \in [0,1]$, we have

$$\begin{aligned} \big|u(t)\big| &= \left|u(\tau) + \int_\tau^t u'(s)\,ds\right| \leq \big|u(\tau)\big| + \int_0^1 \big|u'(s)\big|\,ds \\ &\leq \int_0^1 \big|u(s)\big|\,ds + \int_0^1 \big|u'(s)\big|\,ds \\ &\leq \left(\int_0^1 \big|u(s)\big|^2\,ds\right)^{\frac{1}{2}} + \sqrt{1/p_0}\left(\int_0^1 p(t)\big|u'(t)\big|^2\,dt\right)^{\frac{1}{2}} \\ &\leq \sqrt{2M_0}\|u\|. \end{aligned}$$

For every $u \in X$, we define the functional $\varphi(u) : X \to R$ by

$$\varphi(u) = \Phi(u) - \lambda\Psi(u);$$

here

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\,ds$$

and

$$\Psi(u) = \int_0^1 F(t,u)\,dt,$$

where $F(t,u) = \int_0^{u(t)} f(t,s)\,ds$.

We easily show that φ is differentiable at any $u \in X$ and

$$\varphi'(u)v = \int_0^1 \big(p(t)u'(t)v'(t) + u(t)v(t)\big)\,dt + \sum_{j=1}^m I_j\big(u(t_j)\big)v(t_j) - \lambda \int_0^1 f\big(t,u(t)\big)v(t)\,dt.$$

Obviously, Φ is a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and Ψ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. □

Lemma 2.2 ([1]) *If $u \in X$ is a critical point of the functional φ, then u is a classical solution of problem* (1.4).

Suppose that $E \subset X$. We denote $\overline{E}^{\omega}$ as the weak closure of E, that is, $u \in \overline{E}^{\omega}$ if there exists a sequence $\{u_n\} \subset E$ such that $g(u_n) \to g(u)$ for every $g \in X^*$. Our main tool is the following three critical points theorem obtained in [6].

Lemma 2.3 ([6], Theorem 2.1) *Let X be separable and reflexive real Banach space. $\Phi : X \to R$ a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*. $J : X \to R$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$ and that*

(i) $\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda J(x)) = +\infty$ *for all* $\lambda \in [0,+\infty)$.

Further, assume that there are $r > 0$, $x_1 \in X$ such that

(ii) $r < \Phi(x_1)$.

(iii) $\sup_{x\in\overline{\Phi^{-1}((-\infty,r))}^{\omega}} J(x) < \frac{r}{r+\Phi(x_1)}J(x_1)$.

Then, for each

$$\lambda \in \Lambda_1 = \left(\frac{\Phi(x_1)}{J(x_1) - \sup_{x\in\overline{\Phi^{-1}((-\infty,r))}^{\omega}} J(x)}, \frac{r}{\sup_{x\in\overline{\Phi^{-1}((-\infty,r))}^{\omega}} J(x)}\right),$$

the equation

$$\Phi'(x) - \lambda J'(x) = 0 \tag{2.1}$$

has at least three solutions in X and, moreover, for each $h > 1$, there exist an open interval

$$\Lambda_2 \subseteq \left[0, \frac{hr}{r(J(x_1)/\Phi(x_1)) - \sup_{x\in\overline{\Phi^{-1}((-\infty,r))}^{\omega}} J(x)}\right)$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, (1.2) *has at least three solutions in X whose norms are less than σ.*

3 Main results

Theorem 3.1 *The following conditions are given.*

(H_1) $\sum_{j=1}^m \int_0^{u(t_j)} I_j(t)\,dt \geq 0$.

(H_2) *Let* $a_i > 0$ $(i = 1, 2)$, $M > 0$, *and* $0 < \mu < 2$ *such that*

$$F(t,u) \leq a_1|u|^\mu - a_2, \quad \text{for } |u| \geq M, t \in [0,1].$$

(H_3) *There exist two positive constants* c, c_1 *with* $c_1 > \frac{c}{\sqrt{2M_0}}$, *such that*

$$4M_0 \int_0^1 \max_{|u|\leq c} F(t,u)\,dt < c^2 \left(\frac{c^2}{4M_0} + \frac{c_1^2}{2} + \sum_{j=1}^m \int_0^{c_1} I_j(t)\,dt \right)^{-1} \int_0^1 F(t,c_1)\,dt.$$

Furthermore, put

$$\begin{aligned} \lambda_1 &= \frac{4M_0 \int_0^1 \max_{|u|\leq c} F(t,u)\,dt}{c^2}, \\ \lambda_2 &= \frac{\int_0^1 F(t,c_1)\,dt - \int_0^1 \max_{|u|\leq c} F(t,u)\,dt}{\frac{c_1^2}{2} + \sum_{j=1}^m \int_0^{c_1} I_j(s)\,ds}. \end{aligned} \tag{3.1}$$

Then, for each $\lambda \in (\frac{1}{\lambda_2}, \frac{1}{\lambda_1})$, *problem* (1.4) *has at least three solutions in* X.

Proof Now we show the conditions (i)-(iii) of Lemma 2.3 are satisfied.

For any $u \in X$, $|u| \geq M$, and $\lambda \geq 0$, and the assumptions (H_1)-(H_2) we have

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\,ds - \lambda \int_0^1 F\big(t,u(t)\big)\,dt \\ &\geq \frac{1}{2}\|u\|^2 - \lambda\big[a_1|u|^\mu - a_2\big] \\ &\geq \frac{1}{2}\|u\|^2 - \lambda\big[a_1(2M_0)^{\mu/2}\|u\|^\mu - a_2\big], \end{aligned}$$

$0 < \mu < 2$ implies that

$$\lim_{\|u\|\to\infty} \big(\Phi(u) - \lambda J(u)\big) = +\infty,$$

which shows the condition (i) of Lemma 2.3 is satisfied.

Let $u_1 = c_1 \in X$ and $c_1 > \frac{c}{\sqrt{2M_0}}$. Then

$$\begin{aligned} \Phi(u_1) &= \frac{1}{2}\|u_1\|^2 + \sum_{j=1}^m \int_0^{u_1(t_j)} I_j(s)\,ds \\ &= \frac{1}{2}c_1^2 + \sum_{j=1}^m \int_0^{c_1} I_j(s)\,ds \geq \frac{1}{2}c_1^2 > \frac{c^2}{4M_0} = r, \end{aligned}$$

so the condition (ii) of Lemma 2.3 is obtained.

By Lemma 2.1, if $\Phi(u) \leq r$, then

$$\big|u(t)\big|^2 \leq 2M_0\|u\|^2 \leq 4M_0\Phi(u) \leq 4M_0 r = c^2, \quad \text{for } t \in [0,1],$$

which implies that

$$\Phi^{-1}(-\infty, r) \subseteq \{u \in X, |u(t)| \leq c, t \in [0,1]\}.$$

So for any $u \in X$, we have

$$\sup_{u \in \overline{\Phi^{-1}(-\infty,r)}^{\omega}} \Psi(u) = \sup_{u \in \Phi^{-1}(-\infty,r)} \Psi(u) \leq \int_0^1 \max_{|u| \leq c} F(t,u)\,dt.$$

On the other hand, we obtain

$$\frac{r}{r+\Phi(u_1)}\Psi(u_1) = c^2\left[4M_0\left(\frac{c^2}{4M_0} + \frac{c_1^2}{2} + \sum_{j=1}^{m}\int_0^{c_1} I_j(t)\,dt\right)\right]^{-1}\int_0^1 F(t,c_1)\,dt.$$

From the assumption (H_3) we have

$$\sup_{u \in \overline{\Phi^{-1}(-\infty,r)}^{\omega}} \Psi(u) < \frac{r}{r+\Phi(u_1)}\Psi(u_1),$$

which shows the condition (iii) of Lemma 2.3 is satisfied.

Note that

$$\frac{\Phi(u_1)}{\Psi(u_1) - \sup_{u \in \overline{\Phi^{-1}(-\infty,r)}^{\omega}} \Psi(u)} \leq \frac{\frac{1}{2}c_1^2 + \sum_{j=1}^m \int_0^{c_1} I_j(s)\,ds}{\int_0^1 F(t,c_1)\,dt - \int_0^1 \max_{|u|\leq c} F(t,u)\,dt} = \frac{1}{\lambda_2},$$
$$\frac{r}{\sup_{u \in \overline{\Phi^{-1}(-\infty,r)}^{\omega}} \Psi(u)} \geq \frac{c^2}{4M_0 \int_0^1 \max_{|u|\leq c} F(t,u)} = \frac{1}{\lambda_1}.$$

The condition (H_3) implies $\lambda_2 > \lambda_1$. In the light of Lemma 2.3, the problem (1.4) has at least three solutions in X for each $\lambda \in (1/\lambda_2, 1/\lambda_1)$.

The proof is complete. □

4 Examples

Consider the following problem:

$$\begin{aligned}
&-\left(e^t u'(t)\right)' + u(t) = \lambda f(t,u), \quad t \in [0,1], t \neq t_1,\\
&\Delta\left(e^{t_1} u'(t_1)\right) = u(t_1), \quad t_1 = \frac{1}{2},\\
&u'(0) = u'(1) = 0,
\end{aligned} \tag{4.1}$$

where

$$f(t,u) = \begin{cases} e^{2u}, & u \leq 4,\\ u^{1/2} + e^8 - 4, & u > 4,\end{cases}$$

then

$$F(t,u) = \begin{cases} \frac{1}{2}(e^{2u} - 1), & u \leq 4,\\ \frac{2}{3}u^{3/2} + (e^8 - 4)u + \frac{61}{6} - \frac{7}{2}e^8, & u > 4.\end{cases}$$

Clearly $M_0 = 1$. Let $c = 1$, $c_1 = 4$, it follows that

$$\begin{aligned} &4M_0 \int_0^1 \max_{|u|\le c} F(t,u)\,dt \\ &\quad = 2\bigl(e^2-1\bigr) < c^2 \left(\frac{c^2}{4M_0} + \frac{c_1^2}{2} + \sum_{j=1}^{m} \int_0^{c_1} I_j(s)\,ds \right)^{-1} \int_0^1 F(t,c_1)\,dt = \frac{2(e^8-1)}{65}, \\ &\frac{1}{\lambda_1} = \frac{1}{2(e^2-1)}, \qquad \frac{1}{\lambda_2} = \frac{32}{e^8-e^2}, \end{aligned}$$

which shows that all conditions of Theorem 3.1 are satisfied, so the problem (4.1) admits at least three solutions for $\lambda \in (\frac{32}{e^8-e^2}, \frac{1}{2(e^2-1)})$.

Competing interests
The author declares that she has no competing interests.

Received: 4 August 2016 Accepted: 6 January 2017 Published online: 18 January 2017

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