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RESEARCH

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The modified degenerate q-Bernoulli polynomials arising from p-adic invariant integral on $\mathbb{Z}_p$

Jeong Gon Lee[1] and Jongkyum Kwon[2*]

*Correspondence:
mathkjk26@gnu.ac.kr
[2]Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea
Full list of author information is available at the end of the article

Abstract

Dolgy et al. introduced the modified degenerate Bernoulli polynomials, which are different from Carlitz's degenerate Bernoulli polynomials (see Dolgy et al. in Adv. Stud. Contemp. Math. (Kyungshang) 26(1):1-9, 2016). In this paper, we study some explicit identities and properties for the modified degenerate q-Bernoulli polynomials arising from the p-adic invariant integral on $\mathbb{Z}_p$.

MSC: 11B68; 11S40; 11S80

Keywords: degenerate Bernoulli polynomials; modified degenerate q-Bernoulli polynomials

1 Introduction

For a fixed prime number p, $\mathbb{Z}_p$ refers to the ring of p-adic integers, $\mathbb{Q}_p$ to the field of p-adic rational numbers, and $\mathbb{C}_p$ to the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let q be in $\mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp(x \log q)$ for $|x|_p < 1$. Then the q-analogue of x is defined to be $[x]_q = \frac{1-q^x}{1-q}$.

The Bernoulli polynomials are given by the generating function

$$\left(\frac{t}{e^t-1}\right)e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (\text{see [1–25]}). \tag{1.1}$$

When $x = 0$, $B_n = B_n(0)$ are called Bernoulli numbers.

Carlitz [4, 5, 8] defined the degenerate Bernoulli polynomials as follows:

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda)\frac{t^n}{n!}. \tag{1.2}$$

When $x = 0$, $\beta_n(0|\lambda) = \beta_n(\lambda)$ are called Carlitz's degenerate Bernoulli numbers.

From (1.2) we note that

$$\sum_{n=0}^{\infty} \lim_{\lambda\to 0} \beta_n(x|\lambda)\frac{t^n}{n!} = \lim_{\lambda\to 0} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$= \left(\frac{t}{e^t - 1}\right) e^{xt}$$
$$= \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.3}$$

Using the derivation given in (1.3), we have

$$\lim_{\lambda \to 0} \beta_n(x|\lambda) = B_n(x) \quad (n \geq 0). \tag{1.4}$$

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic invariant integral on $\mathbb{Z}_p$ (also called the Volkenborn integral on $\mathbb{Z}_p$) is defined by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{n=0}^{p^N - 1} f(x) \quad \bigl(\text{see } [1, 9, 10, 15, 17]\bigr). \tag{1.5}$$

By using the formula defined in (1.1) we note that

$$\int_{\mathbb{Z}_p} f_1(x)\, du_0(x) - \int_{\mathbb{Z}_p} f(x)\, du_0(x) = f'(0) \tag{1.6}$$

and

$$\int_{\mathbb{Z}_p} f_n(x)\, du_0(x) - \int_{\mathbb{Z}_p} f(x)\, du_0(x) = \sum_{l=0}^{n-1} f'(l), \tag{1.7}$$

where $f_n(x) = f(x+n)$ $(n \in \mathbb{N})$; see [1, 9, 10, 15, 17].

Thus, by (1.6) we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t}\, du_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.8}$$

The modified degenerate Bernoulli polynomials are recently revisited by Dolgy et al., and they are formulated with the p-adic invariant integral on $\mathbb{Z}_p$ to be

$$\int_{\mathbb{Z}_p} (1+\lambda)^{(\frac{x+y}{\lambda})t}\, du_0(x) = \frac{t}{(1+\lambda)^{\frac{t}{\lambda}} - 1} \left(\frac{\log(1+\lambda)}{\lambda}\right) (1+\lambda)^{\frac{xt}{\lambda}}$$
$$= \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!} \quad \bigl(\text{see } [1]\bigr), \tag{1.9}$$

where $\lambda \in \mathbb{C}_p$ with $|\lambda|_p < p^{-\frac{1}{p-1}}$.

When $x = 0$, we call $\beta_{n,\lambda}(0) = \beta_{n,\lambda}$ the modified degenerate Bernoulli numbers.

Recently, Kim introduced p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_q(f) &= \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) \\ &= \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N - 1} f(x) q^x \quad (\text{see } [17]). \end{aligned} \tag{1.10}$$

The degenerate q-Bernoulli polynomials are also defined by Kim as follows.

$$\sum_{n=0}^{\infty} \beta_{n,q,\lambda}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}}\,d\mu_q(y) \quad \text{(see [20])}. \tag{1.11}$$

The generating functions of Stirling numbers are given by

$$\bigl(\log(1+t)\bigr)^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{t^l}{l!} \quad (n \geq 0) \tag{1.12}$$

and

$$\bigl(e^t-1\bigr)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \quad (n \geq 0), \tag{1.13}$$

where $S_1(l,n)$ are the Stirling numbers of the first kind, and $S_2(l,n)$ are the Stirling numbers of the second kind.

The following diagram illustrates the variations of several types of q-Bernoulli polynomials and numbers. The definitions of the q-Bernoulli polynomials and the degenerate q-Bernoulli polynomials applied in the given diagram are provided by Carlitz [4, 5, 8] and Kim [20], respectively. In this paper, we investigate some of the explicit identities to characterize the modified degenerate q-Bernoulli polynomials used in the diagram

$\int_{\mathbb{Z}_p} e^{[x+y]_q t}\,d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x)\frac{t^n}{n!}$ (q-Bernoulli polynomials)	$\int_{\mathbb{Z}_p} q^{-x}e^{[x+y]_q t}\,d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!}$ (modified q-Bernoulli polynomials)
$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}}\,d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q,\lambda}(x)\frac{t^n}{n!}$ (degenerate q-Bernoulli polynomials)	$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,q,\lambda}(x)\frac{t^n}{n!}$ (modified degenerate q-Bernoulli polynomials)

A few studies have identified some of the properties of the degenerate q-Bernoulli polynomials and numbers. This paper defines the modified q-Bernoulli polynomials and numbers arising from the p-adic invariant integral on $\mathbb{Z}_p$ and introduces additional characteristic properties of these polynomials and numbers, which are defined from the generating functions and p-adic invariant integral on $\mathbb{Z}_p$.

2 The modified degenerate *q*-Bernoulli polynomials and numbers

In the following discussions, we assume that $\lambda, t \in \mathbb{C}_p$ with $0 < |\lambda| \leq 1$ and $|t|_p < p^{-\frac{1}{p-1}}$. Then, as $|\lambda t|_p < p^{-\frac{1}{p-1}}$, $|\log(1+\lambda t)|_p = |\lambda t|_p$, and hence $|\frac{1}{\lambda}\log(1+\lambda t)|_p = |t|_p < p^{-\frac{1}{p-1}}$, it makes sense to take the limit as $\lambda \to 0$.

Following (1.3), we define the modified degenerate q-Bernoulli polynomials given by the generating function

$$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y) = \sum_{n=0}^{\infty} \widetilde{B}_{n,q,\lambda}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $\widetilde{B}_{n,q,\lambda}(0) = \widetilde{B}_{n,q,\lambda}$ are called the modified degenerate q-Bernoulli numbers.

Note that

$$\begin{aligned}
&\lim_{\lambda\to 0}\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y) \\
&\quad= \int_{\mathbb{Z}_p} q^{-y}e^{[x+y]_q t}\,d\mu_q(y) \\
&\quad= \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!},
\end{aligned} \tag{2.2}$$

where $B_{n,q}(x)$ are the modified Carlitz q-Bernoulli polynomials.

Now, we consider

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y) \\
&\quad= \int_{\mathbb{Z}_p} q^{-y}e^{\frac{[x+y]_q}{\lambda}t\log(1+\lambda)}\,d\mu_q(y) \\
&\quad= \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n \int_{\mathbb{Z}_p} q^{-y}[x+y]_q^n\,d\mu_q(y)\frac{t^n}{n!} \\
&\quad= \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n B_{n,q}(x)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

By the definitions provided in (2.1), (2.2), and (2.3) we are able to derive the following theorem.

Theorem 2.1 *For $n \geq 0$, $\widetilde{B}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}_{n,q,\lambda}(x) = \left(\frac{\log(1+\lambda)}{\lambda}\right)^n B_{n,q}(x). \tag{2.4}$$

Note that $(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l$ ($n \geq 0$), where S_1 are the Stirling numbers of the first kind.

Then, by using (2.1) we are able to state

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y) \\
&\quad= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} q^{-y}\binom{\frac{[x+y]_q}{\lambda}t}{n}\lambda^n\,d\mu_q(y) \\
&\quad= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} q^{-y}\lambda^n\sum_{l=0}^{n} S_1(n,l)\left(\frac{[x+y]_q}{\lambda}\right)^l\frac{t^l}{n!}\,d\mu_q(y)
\end{aligned}$$

$$= \sum_{l=0}^{\infty}\sum_{n=l}^{\infty} S_1(n,l)\lambda^{n-l}\frac{t^l}{n!}\int_{\mathbb{Z}_p} q^{-y}[x+y]_q^l\,d\mu_q(y)$$
$$= \sum_{l=0}^{\infty}\left(\sum_{n=l}^{\infty} S_1(n,l)\lambda^{n-l}\frac{l!}{n!}B_{l,q}(x)\right)\frac{t^l}{l!}. \tag{2.5}$$

Given the descriptions in (2.1) and (2.5), we have another theorem.

Theorem 2.2 *For $n \geq 0$, $\widetilde{B}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}_{n,q,\lambda}(x) = \sum_{n=l}^{\infty} S_1(n,l)\lambda^{n-l}\frac{l!}{n!}B_{l,q}(x). \tag{2.6}$$

We observe that

$$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y)$$
$$= \int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x]_q}{\lambda}t}(1+\lambda)^{\frac{[y]_q}{\lambda}q^x t}\,d\mu_q(y)$$
$$= (1+\lambda)^{\frac{[x]_q}{\lambda}t}\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[y]_q}{\lambda}q^x t}\,d\mu_q(y)$$
$$= \left(\sum_{l=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l [x]_q^l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\widetilde{B}_{m,q,\lambda}\frac{q^{mx}t^m}{m!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{m,q,\lambda}[x]_q^{n-m}q^{mx}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-m}\right)\frac{t^n}{n!}. \tag{2.7}$$

The third theorem is obtained by (2.1) and (2.7) as follows.

Theorem 2.3 *For $n \geq 0$, $\widetilde{B}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}_{n,q,\lambda}(x) = \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{m,q,\lambda}[x]_q^{n-m}q^{mx}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-m}. \tag{2.8}$$

Remark 2.4

$$\lim_{\lambda\to 0}\widetilde{B}_{m,q,\lambda}(x) = \lim_{\lambda\to 0}\sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{m,q,\lambda}[x]_q^{n-m}q^{mx}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-m}$$
$$= \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{m,q}q^{mx}$$
$$= B_{m,q}(x). \tag{2.9}$$

Note that

$$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y)$$
$$= \lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{y=0}^{dp^N-1}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}$$

$$= \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} (1+\lambda)^{\frac{[x+a+dy]_q}{\lambda} t}$$

$$= \lim_{N\to\infty} \frac{1}{[d]_q [p^N]_{q^d}} \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{x+a}{d}+y]_{q^d} t}$$

$$= \frac{1}{[d]_q} \sum_{a=0}^{d-1} \lim_{N\to\infty} \frac{1}{[p^N]_{q^d}} \sum_{y=0}^{p^N-1} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{x+a}{d}+y]_{q^d} t} q^{-dy} q^{dy}$$

$$= \frac{1}{[d]_q} \sum_{a=0}^{d-1} \left(\int_{\mathbb{Z}_p} q^{-dy} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{x+a}{d}+y]_{q^d} t} \, du_{q^d}(y) \right)$$

$$= \frac{1}{[d]_q} \sum_{a=0}^{d-1} \sum_{n=0}^{\infty} \widetilde{B}_{n,q^d,\lambda}\left(\frac{x+a}{d}\right) \frac{[d]_q^n t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left([d]_q^{n-1} \sum_{a=0}^{d-1} \widetilde{B}_{n,q^d,\lambda}\left(\frac{x+a}{d}\right)\right) \frac{t^n}{n!}, \tag{2.10}$$

where $d \in \mathbb{N}$.

The following theorem is obtained from (2.10).

Theorem 2.5 *For* $n \geq 0$ *and* $d \in \mathbb{N}$, $\widetilde{B}_{n,q,\lambda}(x)$ *can be written as*

$$\widetilde{B}_{n,q,\lambda}(x) = [d]_q^{n-1} \sum_{a=0}^{d-1} \widetilde{B}_{n,q^d,\lambda}\left(\frac{x+a}{d}\right). \tag{2.11}$$

Now, we observe that

$$\int_{\mathbb{Z}_p} q^{-y} e^{[x+y]_q t} \, du_q(y) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}. \tag{2.12}$$

We obtain Theorem 2.1 as follows by substituting t by $\log(1+\lambda)^{\frac{t}{\lambda}}$ in (2.12):

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{-y} e^{[x+y]_q \log(1+\lambda)^{\frac{t}{\lambda}}} \, du_q(y) &= \int_{\mathbb{Z}_p} q^{-y} (1+\lambda)^{\frac{[x+y]_q}{\lambda} t} \, du_q(y) \\ &= \sum_{n=0}^{\infty} B_{n,q}(x) \frac{1}{n!} \left(\log(1+\lambda)^{\frac{t}{\lambda}}\right)^n \\ &= \sum_{n=0}^{\infty} B_{n,q}(x) \left(\frac{\log(1+\lambda)}{\lambda}\right)^n \frac{t^n}{n!}. \end{aligned} \tag{2.13}$$

For $r \in \mathbb{N}$, we define the *modified degenerate q-Bernoulli polynomials of order r* as follows:

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)} (1+\lambda)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda} t} \, du_q(x_1) \, du_q(x_2) \cdots du_q(x_r) \\ &= \sum_{n=0}^{\infty} \widetilde{B}_{n,q,\lambda}^{(r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.14}$$

When $x = 0$, $\widetilde{B}^{(r)}_{n,q,\lambda}(0) = \widetilde{B}^{(r)}_{n,q,\lambda}$ are called the *modified degenerate q-Bernoulli numbers of order r*.

We observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}(1+\lambda)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda}t}\,d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\\
&\quad=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}\sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n\\
&\qquad\times[x_1+x_2+\cdots+x_r+x]_q^n\frac{t^n}{n!}\,d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\\
&\quad=\sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}[x_1+x_2+\cdots+x_r+x]_q^n\\
&\qquad\times d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\frac{t^n}{n!}\\
&\quad=\sum_{n=0}^{\infty}\left(\left(\frac{\log(1+\lambda)}{\lambda}\right)^n B^{(r)}_{n,q}(x)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.15}$$

Therefore, we are able to derive the following theorem.

Theorem 2.6 *For $n \geq 0$, $\widetilde{B}^{(r)}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}^{(r)}_{n,q,\lambda}(x) = \left(\frac{\log(1+\lambda)}{\lambda}\right)^n B^{(r)}_{n,q}(x).\tag{2.16}$$

Now, we consider

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}(1+\lambda)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda}t}\,d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\\
&\quad=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}\sum_{l=0}^{\infty}\binom{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda}t}{l}\lambda^l\,d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\\
&\quad=\sum_{l=0}^{\infty}\sum_{n=0}^{l}\frac{S_1(l,n)}{l!}\lambda^{l-n}t^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}\\
&\qquad\times[x_1+x_2+\cdots+x_r+x]_q^n\,d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\\
&\quad=\sum_{l=0}^{\infty}\sum_{n=0}^{l}\frac{S_1(l,n)}{l!}\lambda^{l-n}t^n B^{(r)}_{n,q}(x)\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{l=n}^{\infty}\frac{S_1(l,n)}{l!}\lambda^{l-n}n!B^{(r)}_{n,q}(x)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.17}$$

Now, (2.17) yields the following theorem.

Theorem 2.7 *For $n \geq 0$, $\widetilde{B}^{(r)}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}^{(r)}_{n,q,\lambda}(x) = \sum_{l=n}^{\infty}\frac{S_1(l,n)}{l!}\lambda^{l-n}n!B^{(r)}_{n,q}(x).\tag{2.18}$$

Now, we observe that, for $d \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)} (1+\lambda)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda} t}\, d\mu_q(x_1)\, d\mu_q(x_2) \cdots d\mu_q(x_r) \\
&= \lim_{N\to\infty} \frac{1}{[dp^N]_q^r} \sum_{x_1=0}^{dp^N-1} \cdots \sum_{x_r=0}^{dp^N-1} (1+\lambda)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda} t} \\
&= \lim_{N\to\infty} \frac{1}{[dp^N]_q^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \sum_{x_1=0}^{dp^N-1} \cdots \sum_{x_r=0}^{dp^N-1} (1+\lambda)^{\frac{[a_1+\cdots+a_r+x+dx_1+dx_2+\cdots+dx_r]_q}{\lambda} t} \\
&= \lim_{N\to\infty} \frac{1}{[d]_q^r [p^N]_{q^d}^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \sum_{x_1=0}^{p^N-1} \cdots \sum_{x_r=0}^{p^N-1} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{a_1+\cdots+a_r+x}{d}+x_1+x_2+\cdots+x_r]_{q^d} t} \\
&= \frac{1}{[d]_q^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \lim_{N\to\infty} \frac{1}{[p^N]_{q^d}^r} \sum_{x_1=0}^{p^N-1} \cdots \sum_{x_r=0}^{p^N-1} (1+\lambda)^{\frac{1}{\lambda}[\frac{a_1+\cdots+a_r+x}{d}+x_1+x_2+\cdots+x_r]_{q^d} [d]_q t} \\
&= \frac{1}{[d]_q^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-d(x_1+x_2+\cdots+x_r)} \\
&\quad \times (1+\lambda)^{\frac{1}{\lambda}[\frac{a_1+\cdots+a_r+x}{d}+x_1+x_2+\cdots+x_r]_{q^d} [d]_q t}\, d\mu_{q^d}(x_1)\, d\mu_{q^d}(x_2) \cdots d\mu_{q^d}(x_r) \\
&= \sum_{n=0}^{\infty} \left([d]_q^{n-r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \widetilde{B}_{n,q^d,\lambda}^{(r)} \left(\frac{a_1+\cdots+a_r+x}{d} \right) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.19}$$

Finally, by comparing the coefficients on both sides of (2.19) we get the following theorem.

Theorem 2.8 *For $n \geq 0$ and $d \in \mathbb{N}$, $\widetilde{B}_{n,q}^{(r)}(x)$ can be written as*

$$\widetilde{B}_{n,q,\lambda}^{(r)}(x) = [d]_q^{n-r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \widetilde{B}_{n,q^d,\lambda}^{(r)} \left(\frac{a_1+\cdots+a_r+x}{d} \right). \tag{2.20}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to this work. Both authors read and approved the final manuscript.

Author details

[1] Division of Mathematics and informational Statistics, Nanoscale Science and Technology Institute, Wonkwang University, Iksan, 54538, Republic of Korea. [2] Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea.

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