Abbas et al. *Advances in Difference Equations* (2017) 2017:75
DOI 10.1186/s13662-017-1110-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Stability results for partial fractional differential equations with noninstantaneous impulses

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Abstract

In this article, we investigate some uniqueness and Ulam's type stability concepts for the Darboux problem of partial functional differential equations with noninstantaneous impulses and delay in Banach spaces. The main techniques rely on fractional calculus, integral equations and inequalities. Two examples are also provided to illustrate our results.

MSC: 26A33; 34K37; 34K45

Keywords: fractional differential equation; Riemann-Liouville integral; Caputo derivative; Darboux problem; noninstantaneous impulses; Ulam-Hyers-Rassias stability

1 Introduction

The fractional calculus deals with extensions of derivatives and integrals to noninteger orders. It represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology, and bioengineering. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [1], Kilbas *et al.* [2], Miller and Ross [3], Zhou [4, 5], the papers [6–25], and the references therein.

In [9], Abbas *et al.* studied some existence, uniqueness and stability results for functional partial impulsive differential equations. In [26], Wang *et al.* studied the stability of first-order impulsive evolution equations.

In pharmacotherapy, the above instantaneous impulses cannot describe the certain dynamics of evolution processes. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. From the viewpoint of general theories, Hernández and O'Regan [27] initially offered to study a new class of abstract semilinear impulsive differential equations with noninstantaneous impulses in a PC-normed Banach space. Meanwhile, in [27, 28] the authors continue to study other new classes of differential equations with noninstantaneous impulses.

However, Ulam-Hyers-Rassias stability of fractional differential equations with this kind of impulses has not been studied. Motivated by recent work [29, 30], we investigate the uniqueness and Ulam-Hyers-Rassias stability of the following partial fractional differential equations with noninstantaneous impulses and finite delay:

$$\begin{cases} {}^cD^r_{\theta_k}u(t,x) = f(t,x,u_{(t,x)}); & \text{if } (t,x) \in I_k, k = 0,\ldots,m, \\ u(t,x) = g_k(t,x,u(t,x)); & \text{if } (t,x) \in J_k, k = 1,\ldots,m, \\ u(t,x) = \phi(t,x); & \text{if } (t,x) \in \tilde{J} := [-\alpha,a] \times [-\beta,b] \setminus (0,a] \times (0,b], \\ u(t,0) = \varphi(t); & t \in [0,a], \\ u(0,x) = \psi(x); & x \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{1}$$

where $I_0 = [0,t_1] \times [0,b]$, $I_k := (s_k,t_{k+1}] \times [0,b]$, $J_k := (t_k,s_k] \times [0,b]$; $k = 1,\ldots,m$, $a,b,\alpha,\beta > 0$, $\theta_k = (s_k,0)$; $k = 0,\ldots,m$, ${}^cD^r_{\theta_k}$ is the fractional Caputo derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1]$, $0 = s_0 < t_1 \le s_1 \le t_2 \le \cdots \le s_{m-1} \le t_m \le s_m \le t_{m+1} = a$, $f : I_k \times \mathcal{C} \to E$; $k = 0,\ldots,m$, $g_k : J_k \times E \to E$; $k = 1,\ldots,m$, $\phi : \tilde{J} \to E$ are given continuous functions, $\varphi : [0,a] \to E$ and $\psi : [0,b] \to E$ are given absolutely continuous functions with $\varphi(t) = \Phi(t,0)$, $\psi(x) = \Phi(0,x)$ for each $(t,x) \in J := [0,a] \times [0,b]$, E is a complete Banach space, and $\mathcal{C}$ is the Banach space defined by

$$\begin{aligned} \mathcal{C} &= C_{(\alpha,\beta)} \\ &= \bigl\{u : [-\alpha,0] \times [-\beta,0] \to E : \text{continuous} \\ &\quad \text{and there exist } \tau_k \in (-\alpha,0) \text{ with } u\bigl(\tau_k^-,\tilde{x}\bigr) \text{ and } u\bigl(\tau_k^+,\tilde{x}\bigr); k = 1,\ldots,m, \\ &\quad \text{exist for any } \tilde{x} \in [-\beta,0] \text{ with } u\bigl(\tau_k^-,\tilde{x}\bigr) = u(\tau_k,\tilde{x})\bigr\}, \end{aligned}$$

with the norm

$$\|u\|_{\mathcal{C}} = \sup_{(t,x)\in[-\alpha,0]\times[-\beta,0]} \bigl\|u(t,x)\bigr\|_E.$$

We denote by $u_{(t,x)}$ the element of $\mathcal{C}$ defined by

$$u_{(t,x)}(\tau,\xi) = u(t+\tau,x+\xi); \quad (\tau,\xi) \in [-\alpha,0] \times [-\beta,0],$$

here $u_{(t,x)}(\cdot,\cdot)$ represents the history of the state from time $t-\alpha$ up to the present time t and from time $x-\beta$ up to the present time x.

Next, we consider the following partial fractional differential equations with noninstantaneous impulses and infinite delay:

$$\begin{cases} {}^cD^r_{\theta_k}u(t,x) = f(t,x,u_{(t,x)}); & \text{if } (t,x) \in I_k, k = 0,\ldots,m, \\ u(t,x) = g_k(t,x,u(t,x)); & \text{if } (t,x) \in J_k, k = 1,\ldots,m, \\ u(t,x) = \phi(t,x) & \text{if } (t,x) \in \tilde{J}' := (-\infty,a] \times (-\infty,b] \setminus (0,a] \times (0,b], \\ u(t,0) = \varphi(t); & t \in [0,a], \\ u(0,x) = \psi(x); & x \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{2}$$

where J, φ, ψ are as in problem (1), $f: I_k \times \mathcal{B} \to E$; $k = 0, \dots, m$, $g_k : J_k \times E \to E$; $k = 1, \dots, m$, $\phi : \tilde{J}' \to E$ are given continuous functions and $\mathcal{B}$ is called a phase space that will be specified in Section 4.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Denote $L^1(J)$ the space of Bochner-integrable functions $u : J \to E$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(t,x)\|_E \, dx \, dt,$$

where $\|\cdot\|_E$ denotes a suitable complete norm on E. As usual, by $\mathrm{AC}(J)$ we denote the space of absolutely continuous functions from J into E, and $C(J)$ is the Banach space of all continuous functions from J into E with the norm $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(t,x)\in J} \|u(t,x)\|_E.$$

Let $\theta = (0,0)$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J)$, the expression

$$\left(I_\theta^r u\right)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-\xi)^{r_2-1} u(\tau,\xi) \, d\xi \, d\tau,$$

is called the left-sided mixed Riemann-Liouville integral of order r, where $\Gamma(\cdot)$ is the (Euler) Gamma function defined by $\Gamma(\varsigma) = \int_0^\infty t^{\varsigma-1} e^{-t} \, dt$; $\varsigma > 0$.

In particular,

$$\left(I_\theta^\theta u\right)(t,x) = u(t,x), \qquad \left(I_\theta^\sigma u\right)(t,x) = \int_0^t \int_0^x u(\tau,\xi) \, d\xi \, d\tau; \quad \text{for almost all } (t,x) \in J,$$

where $\sigma = (1,1)$. For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_\theta^r u) \in C(J)$, moreover,

$$\left(I_\theta^r u\right)(t,0) = \left(I_\theta^r u\right)(0,x) = 0; \quad t \in [0,a], x \in [0,b].$$

Example 2.1 Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$, $r = (r_1, r_2)$, $r_1, r_2 \in (0,\infty)$ and $h(t,x) = t^\lambda x^\omega$; $(t,x) \in J$. We have $h \in L^1(J)$, and we get

$$\left(I_\theta^r h\right)(t,x) = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}; \quad \text{for almost all } (t,x) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \, \partial x}$ the mixed second-order partial derivative.

Definition 2.2 ([17]) Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J)$. The Caputo fractional-order derivative of order r of u is defined by the expression

$${}^cD_\theta^r u(t,x) = \left(I_\theta^{1-r} D_{tx}^2 u\right)(t,x) = \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^t \int_0^x \frac{D_{\tau\xi}^2 u(\tau,\xi)}{(t-\tau)^{r_1}(x-\xi)^{r_2}} \, d\xi \, d\tau.$$

The case $\sigma = (1,1)$ is included and we have

$$\left({}^cD^\sigma_\theta u\right)(t,x) = \left(D^2_{tx}u\right)(t,x); \quad \text{for almost all } (t,x) \in J.$$

Example 2.3 Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$ and $r = (r_1,r_2) \in (0,1] \times (0,1]$, then

$${}^cD^r_\theta t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1}x^{\omega-r_2}; \quad \text{for almost all } (t,x) \in J.$$

Let $a_1 \in [0,a]$, $z^+ = (a_1,0) \in J$, $J_z = (a_1,a] \times [0,b]$, $r_1, r_2 > 0$ and $r = (r_1,r_2)$. For $u \in L^1(J_z)$, the expression

$$\left(I^r_{z^+}u\right)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^t \int_0^x (t-\tau)^{r_1-1}(x-\xi)^{r_2-1}u(\tau,\xi)\,d\xi\,d\tau,$$

is called the left-sided mixed Riemann-Liouville integral of order r of u.

Definition 2.4 ([17]) For $u \in L^1(J_z)$ where $D^2_{tx}u$ is Bochner integrable on J_z, the Caputo fractional-order derivative of order r of u is defined by the expression

$$\left({}^cD^r_{z^+}u\right)(t,x) = \left(I^{1-r}_{z^+}D^2_{tx}u\right)(t,x).$$

Now, we consider the Ulam stability for our problems. Let $\epsilon > 0$, $\Psi \geq 0$ and $\Phi : J \to [0,\infty)$ be a continuous function. We consider the following inequalities:

$$\begin{cases} \|{}^cD^r_{\theta_k}u(t,x) - f(t,x,u_{(t,x)})\|_E \leq \epsilon; & \text{if } (t,x) \in I_k, k = 0,\ldots,m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \epsilon; & \text{if } (t,x) \in J_k, k = 1,\ldots,m, \end{cases} \tag{3}$$

$$\begin{cases} \|{}^cD^r_{\theta_k}u(t,x) - f(t,x,u_{(t,x)})\|_E \leq \Phi(t,x); & \text{if } (t,x) \in I_k, k = 0,\ldots,m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \Psi; & \text{if } (t,x) \in J_k, k = 1,\ldots,m, \end{cases} \tag{4}$$

$$\begin{cases} \|{}^cD^r_{\theta_k}u(t,x) - f(t,x,u_{(t,x)})\|_E \leq \epsilon\Phi(t,x); & \text{if } (t,x) \in I_k, k = 0,\ldots,m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \epsilon\Psi; & \text{if } (t,x) \in J_k, k = 1,\ldots,m. \end{cases} \tag{5}$$

Definition 2.5 ([9, 29]) Problem (1) is Ulam-Hyers stable if there exists a real number $c_{f,g_k} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (3) there exists a solution $v \in PC$ of problem (1) with

$$\left\|u(t,x) - v(t,x)\right\|_E \leq \epsilon c_{f,g_k}; \quad (t,x) \in J.$$

Definition 2.6 ([9, 29]) Problem (1) is generalized Ulam-Hyers stable if there exists $c_{f,g_k} : C([0,\infty),[0,\infty))$ with $c_{f,g_k}(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (3) there exists a solution $v \in PC$ of problem (1) with

$$\left\|u(t,x) - v(t,x)\right\|_E \leq c_{f,g_k}(\epsilon); \quad (t,x) \in J.$$

Definition 2.7 ([9, 29]) Problem (1) is Ulam-Hyers-Rassias stable with respect to (Φ,Ψ) if there exists a real number $c_{f,g_k,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$

of the inequality (5) there exists a solution $v \in PC$ of problem (1) with

$$\left\|u(t,x)-v(t,x)\right\|_E \le \epsilon c_{f,g_k,\Phi}\big(\Psi+\Phi(t,x)\big); \quad (t,x)\in J.$$

Definition 2.8 ([9, 29]) Problem (1) is generalized Ulam-Hyers-Rassias stable with respect to (Φ,Ψ) if there exists a real number $c_{f,g_k,\Phi}>0$ such that for each solution $u\in PC$ of the inequality (4) there exists a solution $v\in PC$ of problem (1) with $\|u(t,x)-v(t,x)\|_E \le c_{f,g_k,\Phi}(\Psi+\Phi(t,x))$; $(t,x)\in J$.

Remark 2.9 It is clear that: (i) Definition 2.5 $\Rightarrow$ Definition 2.6, (ii) Definition 2.7 $\Rightarrow$ Definition 2.8, (iii) Definition 2.7 for $\Phi(\cdot,\cdot)=\Psi=1 \Rightarrow$ Definition 2.5.

Remark 2.10 A function $u\in PC$ is a solution of the inequality (3) if and only if there exist a function $G\in PC$ and a sequence G_k; $k=1,\dots,m$ in E (which depend on u) such that

(i) $\|G(t,x)\|_E\le\epsilon$ and $\|G_k\|_E\le\epsilon$; $k=1,\dots,m$,
(ii) ${}^cD^r_{\theta_k}u(t,x)=f(t,x,u(t,x))+G(t,x)$; if $(t,x)\in I_k$, $k=0,\dots,m$,
(iii) $u(t,x)=g_k(t,x,u(t,x))+G_k$; if $(t,x)\in J_k$, $k=1,\dots,m$,

One can have similar remarks for the inequalities (4) and (5). So, the Ulam stabilities of the impulsive fractional differential equations are some special types of data dependence of the solutions of impulsive fractional differential equations.

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 2.11 (Gronwall lemma [31]) *Let $\upsilon: J\to[0,\infty)$ be a real function and $\omega(\cdot,\cdot)$ be a nonnegative, locally integrable function on J. If there are constants $c>0$ and $0<r_1,r_2<1$ such that*

$$\upsilon(t,x)\le\omega(t,x)+c\int_0^t\int_0^x\frac{\upsilon(\tau,\xi)}{(t-\tau)^{r_1}(x-\xi)^{r_2}}\,d\xi\,d\tau,$$

then there exists a constant $\delta=\delta(r_1,r_2)$ such that

$$\upsilon(t,x)\le\omega(t,x)+\delta c\int_0^t\int_0^x\frac{\omega(\tau,\xi)}{(t-\tau)^{r_1}(x-\xi)^{r_2}}\,d\xi\,d\tau,$$

for every $(t,x)\in J$.

3 Uniqueness and Ulam stabilities results for finite delay

In this section, we present conditions for the uniqueness and Ulam stability of problem (1). Consider the Banach space

$$\begin{aligned} PC=\big\{u:[-\alpha,a]\times[-\beta,b]\to E: u\in C\big((t_k,t_{k+1}]\times[0,b]\big); k=0,1,\dots,m,\\ \text{and there exist } u\big(t_k^-,x\big) \text{ and } u\big(t_k^+,x\big); k=1,\dots,m,\\ \text{with } u\big(t_k^-,x\big)=u(t_k,x) \text{ for each } x\in[0,b]\big\},\end{aligned}$$

with the norm

$$\|u\|_{PC} = \sup_{(t,x)\in[-\alpha,a]\times[-\beta,b]} \|u(t,x)\|_E.$$

By Lemma 2.14 in [1], we conclude to the following lemma.

Lemma 3.1 *Let $r_1, r_2 \in (0,1]$, $\mu(t,x) = \varphi(t) + \psi(x) - \varphi(0)$. A function $u \in PC$ is called a solution of the problem* (1), *if u satisfies*

$$\begin{cases} u(t,x) = \mu(t,x) \\ \qquad + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x) \in I_0, \\ u(t,x) = \varphi(t) + g_k(s_k,x,u(s_k,x)) - g_k(s_k,0,u(s_k,0)) \\ \qquad + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x) \in I_k, k = 1,\ldots,m, \\ u(t,x) = g_k(t,x,u(t,x)); \quad \text{if } (t,x) \in J_k, k = 1,\ldots,m, \\ u(t,x) = \phi(t,x); \quad \text{if } (t,x) \in \tilde{J}, \\ u(t,0) = \varphi(t); \quad t \in [0,a], \qquad u(0,x) = \psi(x); \quad x \in [0,b] \text{ and } \varphi(0) = \psi(0). \end{cases} \tag{6}$$

Lemma 3.2 *If $u \in PC$ is a solution of the inequality* (3) *then u is a solution of the following integral inequality*:

$$\begin{cases} \|u(t,x) - \mu(t,x) - \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad \le \frac{\epsilon a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}; \quad \text{if } (t,x) \in I_0, \\ \|u(t,x) - \varphi(t) - g_k(s_k,x,u(s_k,x)) + g_k(s_k,0,u(s_k,0)) \\ \qquad - \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad \le \frac{\epsilon a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}; \quad \text{if } (t,x) \in I_k, k = 1,\ldots,m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \le \epsilon; \quad \text{if } (t,x) \in J_k, k = 1,\ldots,m. \end{cases} \tag{7}$$

Proof By Remark 2.10 we have

$$\begin{cases} {}^cD^r_{\theta_k} u(t,x) = f(t,x,u_{(t,x)}) + G(t,x); \quad \text{if } (t,x) \in I_k, k = 0,\ldots,m, \\ u(t,x) = g_k(t,x,u(t,x)) + G_k; \quad \text{if } (t,x) \in J_k, k = 1,\ldots,m. \end{cases}$$

Then

$$\begin{cases} u(t,x) = \mu(t,x) \\ \qquad + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} (f(\tau,\xi,u_{(\tau,\xi)}) + G(\tau,\xi))\,d\xi\,d\tau; \quad \text{if } (t,x) \in I_0, \\ u(t,x) = \varphi(t) + g_k(s_k,x,u(s_k,x)) - g_k(s_k,0,u(s_k,0)) \\ \qquad + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} (f(\tau,\xi,u_{(\tau,\xi)}) + G(\tau,\xi))\,d\xi\,d\tau; \\ \qquad \text{if } (t,x) \in I_k, k = 1,\ldots,m, \\ u(t,x) = g_k(t,x,u(t,x)) + G_k; \quad \text{if } (t,x) \in J_k, k = 1,\ldots,m. \end{cases}$$

Thus, it follows that

$$\begin{cases} \|u(t,x)-\mu(t,x)-\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad = \|\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}G(\tau,\xi)\,d\xi\,d\tau\|_E; \quad \text{if } (t,x)\in I_0, \\ \|u(t,x)-\varphi(t)-g_k(s_k,x,u(s_k,x))+g_k(s_k,0,u(s_k,0)) \\ \qquad -\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad = \|\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}G(\tau,\xi)\,d\xi\,d\tau\|_E; \quad \text{if } (t,x)\in I_k, k=1,\dots,m, \\ \|u(t,x)-g_k(t,x,u(t,x))\|_E = \|G_k\|_E; \quad \text{if } (t,x)\in J_k, k=1,\dots,m. \end{cases}$$

Hence, we obtain (7). □

Remark 3.3 We have similar results for the solutions of the inequalities (4) and (5).

Theorem 3.4 *Assume that the following hypotheses hold*:

(H_1) *There exists a constant* $l_f > 0$ *such that*

$$\|f(t,x,u)-f(t,x,\overline{u})\|_E \le l_f\|u-\overline{u}\|_{\mathcal{C}},$$

for each $(t,x)\in J$, *and each* $u,\overline{u}\in\mathcal{C}$.

(H_2) *There exist constants* $l_{g_k} > 0$; $k=1,\dots,m$, *such that*

$$\|g_k(t,x,u)-g_k(t,x,\overline{u})\|_E \le l_{g_k}\|u-\overline{u}\|_E,$$

for each $(t,x)\in J_k$, *and each* $u,\overline{u}\in E$, $k=1,\dots,m$.

If

$$\ell := 2l_g + \frac{l_f a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{8}$$

where $l_g = \max_{k=1,\dots,m} l_{g_k}$, *then the problem* (1) *has a unique solution on* J.

Furthermore, if the following hypothesis holds:

(H_3) *There exists* $\lambda_\Phi > 0$ *such that, for each* $(t,x)\in J$, *we have*

$$I_{\theta_k}^r \Phi(t,x) \le \lambda_\Phi \Phi(t,x); \quad k=0,\dots,m,$$

then the problem (1) *is generalized Ulam-Hyers-Rassias stable*.

Proof Consider the operator $N: PC \to PC$ defined by

$$\begin{cases} (Nu)(t,x) = \mu(t,x) \\ \qquad + \int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x)\in I_0, \\ (Nu)(t,x) = \varphi(t)+g_k(s_k,x,u(s_k,x))-g_k(s_k,0,u(s_k,0)) \\ \qquad + \int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x)\in I_k, k=1,\dots,m, \\ (Nu)(t,x) = g_k(t,x,u(t,x)); \quad \text{if } (t,x)\in J_k, k=1,\dots,m, \\ (Nu)(t,x) = \phi(t,x); \quad \text{if } (t,x)\in\tilde{J}. \end{cases}$$

Clearly, the fixed points of the operator N are solution of the problem (1). We shall use the Banach contraction principle to prove that N has a fixed point. *N is a contraction*. Let $u, v \in PC$, then, for each $(t,x) \in J$, we have

$$\begin{cases} \|(Nu)(t,x) - (Nv)(t,x)\|_E \\ \quad \leq \| \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} [f(\tau,\xi,u_{(\tau,\xi)}) - f(\tau,\xi,v_{(\tau,\xi)})]\, d\xi\, d\tau \|_E; \\ \qquad \text{if } (t,x) \in I_0, \\ \|(Nu)(t,x) - (Nv)(t,x)\|_E \\ \quad \leq \|g_k(s_k,x,u(s_k,x)) - g_k(s_k,x,v(s_k,x))\|_E \\ \qquad + \|g_k(s_k,0,u(s_k,0)) - g_k(s_k,0,v(s_k,0))\|_E \\ \qquad + \| \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} [f(\tau,\xi,u_{(\tau,\xi)}) - f(\tau,\xi,v_{(\tau,\xi)})]\, d\xi\, d\tau \|_E; \\ \qquad \text{if } (t,x) \in I_k, k = 1,\dots,m, \\ \|(Nu)(t,x) - (Nv)(t,x)\|_E \\ \quad = \|g_k(t,x,u(t,x)) - g_k(t,x,v(t,x))\|_E; \quad \text{if } (t,x) \in J_k, k = 1,\dots,m. \end{cases}$$

Thus, we get

$$\begin{cases} \|(Nu)(t,x) - (Nv)(t,x)\|_E \\ \quad \leq \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} l_f \|u-v\|_C\, d\xi\, d\tau \\ \quad \leq \frac{l_f a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|u-v\|_{PC}; \quad \text{if } (t,x) \in I_0, \\ \|(Nu)(t,x) - (Nv)(t,x)\|_E \\ \quad \leq 2l_g \|u-v\|_{PC} + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} l_f \|u-v\|_C\, d\xi\, d\tau \\ \quad \leq (2l_g + \frac{l_f a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}) \|u-v\|_{PC}; \quad \text{if } (t,x) \in I_k, k = 1,\dots,m, \\ \|(Nu)(t,x) - (Nv)(t,x)\|_E \leq l_g \|u-v\|_{PC}; \quad \text{if } (t,x) \in J_k, k = 1,\dots,m. \end{cases}$$

Hence

$$\left\| N(u) - N(v) \right\|_{PC} \leq \ell \|u-v\|_{PC}.$$

By the condition (8), we conclude that N is a contraction. As a consequence of Banach fixed point theorem, we deduce that N has a unique fixed point v which is a solution of the problem (1). Then we have

$$\begin{cases} v(t,x) = \mu(t,x) \\ \qquad + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,v_{(\tau,\xi)})\, d\xi\, d\tau; \quad \text{if } (t,x) \in I_0, \\ v(t,x) = \varphi(t) + g_k(s_k,x,v(s_k,x)) - g_k(s_k,0,v(s_k,0)) \\ \qquad + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,v_{(\tau,\xi)})\, d\xi\, d\tau; \quad \text{if } (t,x) \in I_k, k = 1,\dots,m, \\ v(t,x) = g_k(t,x,v(t,x)); \quad \text{if } (t,x) \in J_k, k = 1,\dots,m. \end{cases}$$

Let $u \in PC$ be a solution of the inequality (4). By the differential of this inequality, for each $(t,x) \in J$, we have

$$\begin{cases} \|u(t,x) - \mu(t,x) - \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad \leq \| \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \Phi(\tau,\xi)\,d\xi\,d\tau\|_E; \quad \text{if } (t,x) \in I_0, \\ \|u(t,x) - \varphi(t) - g_k(s_k,x,u(s_k,x)) + g_k(s_k,0,u(s_k,0)) \\ \qquad - \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad \leq \| \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \Phi(\tau,\xi)\,d\xi\,d\tau\|_E; \quad \text{if } (t,x) \in I_k, k = 1,\dots,m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \Psi; \quad \text{if } (t,x) \in J_k, k = 1,\dots,m. \end{cases}$$

Thus, by (H_3) for each $(t,x) \in J$, we get

$$\begin{cases} \|u(t,x) - \mu(t,x) - \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad \leq \lambda_\Phi \Phi(t,x); \quad \text{if } (t,x) \in I_0, \\ \|u(t,x) - \varphi(t) - g_k(s_k,x,u(s_k,x)) + g_k(s_k,0,u(s_k,0)) \\ \qquad - \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad \leq \lambda_\Phi \Phi(t,x); \quad \text{if } (t,x) \in I_k, k = 1,\dots,m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \Psi; \quad \text{if } (t,x) \in J_k, k = 1,\dots,m. \end{cases}$$

Hence

$$\begin{cases} \|u(t,x) - v(t,x)\|_E \\ \quad \leq \lambda_\Phi \Phi(t,x) \\ \qquad + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(\tau,\xi,u_{(\tau,\xi)}) - f(\tau,\xi,v_{(\tau,\xi)})\|_E\,d\xi\,d\tau; \\ \qquad \text{if } (t,x) \in I_0, \\ \|u(t,x) - v(t,x)\|_E \\ \quad \leq \lambda_\Phi \Phi(t,x) + 2l_g\|u(t,x) - v(t,x)\|_E \\ \qquad + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(\tau,\xi,u_{(\tau,\xi)}) - f(\tau,\xi,v_{(\tau,\xi)})\|_E\,d\xi\,d\tau; \\ \qquad \text{if } (t,x) \in I_k, k = 1,\dots,m, \\ \|u(t,x) - v(t,x)\|_E \\ \quad \leq \Psi + \|g_k(t,x,u(t,x)) - g_k(t,x,v(t,x))\|_E \\ \quad \leq \Psi + l_g\|u(t,x) - v(t,x)\|_E; \quad \text{if } (t,x) \in J_k, k = 1,\dots,m. \end{cases}$$

For each $(t,x) \in I_0$, we have

$$\|u(t,x) - v(t,x)\|_E \leq \lambda_\Phi \Phi(t,x) + l_f \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|u_{(\tau,\xi)} - v_{(\tau,\xi)}\|_C\,d\xi\,d\tau.$$

We consider the function γ defined by

$$\gamma(t,x) = \sup\{\|u(\tau,\xi) - v(\tau,\xi)\| : -\alpha \leq \tau \leq t, -\beta \leq \xi \leq x\}; \quad (t,x) \in J.$$

Let $(t^*,x^*) \in [-\alpha,x] \times [-\beta,y]$ be such that $\gamma(t,x) = \|u(t^*,x^*) - v(t^*,x^*)\|_E$. If $(t^*,x^*) \in \tilde{J}$, then $\gamma(t,x) = 0$. Now, if $(t^*,x^*) \in J$, then by the previous inequality, we have, for $(t,x) \in J$,

$$\gamma(t,x) \le \lambda_\Phi \Phi(t,x) + l_f \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \gamma(t,x)\, d\xi\, d\tau.$$

From Lemma 2.11, there exists a constant $\delta_1 := \delta_1(r_1,r_2)$ such that

$$\begin{aligned}\gamma(t,x) &\le \lambda_\Phi\bigl(\Phi(t,x) + l_f\delta_1 I_\theta^r \Phi(t,x)\bigr)\\ &\le \lambda_\Phi(1 + l_f\delta_1\lambda_\Phi)\Phi(t,x)\\ &:= c_{1,f,g_k,\Phi}\Phi(t,x).\end{aligned}$$

Since for every $(t,x) \in I_0$, $\|u_{(t,x)}\|_C \le \gamma(t,x)$, we get

$$\bigl\|u(t,x) - v(t,x)\bigr\|_E \le c_{1,f,g_k,\Phi}\bigl(\Psi + \Phi(t,x)\bigr).$$

Now, for each $(t,x) \in I_k$, $k = 1,\ldots,m$, we have

$$\begin{aligned}&\bigl\|u(t,x) - v(t,x)\bigr\|_E\\ &\quad\le \lambda_\Phi\Phi(t,x) + 2l_g\bigl\|u(t,x) - v(t,x)\bigr\|_E\\ &\qquad + l_f\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\bigl\|u(\tau,\xi) - v(\tau,\xi)\bigr\|_E\, d\xi\, d\tau.\end{aligned}$$

Then we obtain

$$\begin{aligned}&\bigl\|u(t,x) - v(t,x)\bigr\|_E\\ &\quad\le \frac{\lambda_\Phi}{1-2l_g}\Phi(t,x) + \frac{l_f}{1-2l_g}\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\bigl\|u(\tau,\xi) - v(\tau,\xi)\bigr\|_E\, d\xi\, d\tau.\end{aligned}$$

Again, from Lemma 2.11, there exists a constant $\delta_2 := \delta_2(r_1,r_2)$ such that

$$\begin{aligned}\bigl\|u(t,x) - v(t,x)\bigr\|_E &\le \frac{\lambda_\Phi}{1-2l_g}\left(\Phi(t,x) + \frac{l_f\delta_2}{1-2l_g} I_{\theta_k}^r \Phi(t,x)\right)\\ &\le \frac{\lambda_\Phi}{1-2l_g}\left(1 + \frac{l_f\delta_2\lambda_\Phi}{1-2l_g}\right)\Phi(t,x)\\ &:= c_{2,f,g_k,\Phi}\Phi(t,x).\end{aligned}$$

Hence, for each $(t,x) \in I_k$, $k = 1,\ldots,m$, we get

$$\bigl\|u(t,x) - v(t,x)\bigr\|_E \le c_{2,f,g_k,\Phi}\bigl(\Psi + \Phi(t,x)\bigr).$$

Now, for each $(t,x) \in J_k$, $k = 1,\ldots,m$, we have

$$\bigl\|u(t,x) - v(t,x)\bigr\|_E \le \Psi + l_g\bigl\|u(t,x) - v(t,x)\bigr\|_E.$$

This gives

$$\left\|u(t,x)-v(t,x)\right\|_E \leq \frac{\Psi}{1-l_g} := c_{3,f,g_k,\Phi}\Psi.$$

Thus, for each $(t,x) \in J_k$, $k = 1,\dots,m$, we get

$$\left\|u(t,x)-v(t,x)\right\|_E \leq c_{3,f,g_k,\Phi}\bigl(\Psi+\Phi(t,x)\bigr).$$

Set $c_{f,g_k,\Phi} := \max_{i\in\{1,2,3\}} c_{i,f,g_k,\Phi}$. Hence, for each $(t,x) \in J$, we obtain

$$\left\|u(t,x)-v(t,x)\right\|_E \leq c_{f,g_k,\Phi}\bigl(\Psi+\Phi(t,x)\bigr).$$

Consequently, problem (1) is generalized Ulam-Hyers-Rassias stable. □

4 The phase space $\mathcal{B}$

The notation of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [32]. For any $(t,x) \in J$ denote $\mathcal{E}_{(t,x)} := [0,t] \times \{0\} \cup \{0\} \times [0,x]$, furthermore in the case $t = a$, $x = b$ we write simply $\mathcal{E}$. Consider the space $(\mathcal{B}, \|(\cdot,\cdot)\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty,0] \times (-\infty,0]$ into E, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $z : (-\infty,a] \times (-\infty,b] \to E$ continuous on J and $z_{(t,x)} \in \mathcal{B}$, for all $(t,x) \in \mathcal{E}$, then there are constants $H,K,M > 0$ such that for any $(t,x) \in J$ the following conditions hold:
  (i) $z_{(t,x)}$ is in $\mathcal{B}$,
  (ii) $\|z(t,x)\| \leq H\|z_{(t,x)}\|_{\mathcal{B}}$,
  (iii) $\|z_{(t,x)}\|_{\mathcal{B}} \leq K\sup_{(\tau,\xi)\in[0,t]\times[0,x]}\|z(\tau,\xi)\| + M\sup_{(\tau,\xi)\in E_{(t,x)}}\|z_{(\tau,\xi)}\|_{\mathcal{B}}$,

(A_2) for the function $z(\cdot,\cdot)$ in (A_1), $z_{(t,x)}$ is a $\mathcal{B}$-valued continuous function on J,

(A_3) the space $\mathcal{B}$ is complete.

Now, we present some examples of phase spaces [33, 34].

Example 4.1 Let $\mathcal{B}$ be the set of all functions $\phi : (-\infty,0] \times (-\infty,0] \to E$ which are continuous on $[-\alpha,0] \times [-\beta,0]$, $\alpha,\beta \geq 0$, with the seminorm

$$\|\phi\|_{\mathcal{B}} = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]}\left\|\phi(s,t)\right\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\|\cdot\|_{\mathcal{B}}$ is isometric to the space $\mathcal{C}$, this means that partial differential functional equations with finite delay are included in our axiomatic model.

Example 4.2 Let γ be a real constant and Let C_γ be the set of all continuous functions $\phi : (-\infty,0] \times (-\infty,0] \to E$ for which a limit $\lim_{\|(s,t)\|\to\infty} e^{\gamma(s+t)}\phi(s,t)$ exists, with the norm

$$\|\phi\|_{C_\gamma} = \sup_{(s,t)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+t)}\left\|\phi(s,t)\right\|.$$

Then we have $H = 1$ and $K = M = \max\{e^{-\gamma(a+b)},1\}$.

Example 4.3 Let $\alpha, \beta, \gamma \geq 0$ and let

$$\|\phi\|_{CL_\gamma} = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,t)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+t)}\|\phi(s,t)\|\,dt\,ds$$

be the seminorm for the space CL_γ of all functions $\phi : (-\infty,0] \times (-\infty,0] \to E$ which are continuous on $[-\alpha,0] \times [-\beta,0]$ measurable on $(-\infty,-\alpha] \times (-\infty,0] \cup (-\infty,0] \times (-\infty,-\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1, \qquad K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+t)}\,dt\,ds, \qquad M = 2.$$

5 Uniqueness and Ulam stabilities results for infinite delay

In this section, we present conditions for the Ulam stability of problem (2). Consider the space

$$\Omega := \{u : (-\infty,a] \times (-\infty,b] \to E : u_{(t,x)} \in \mathcal{B} \text{ for } (t,x) \in \mathcal{E} \text{ and } u|_J \in PC\}.$$

Theorem 5.1 *Assume that the following hypotheses hold*:

(H_1') *There exists a constant* $l_f' > 0$ *such that*

$$\|f(t,x,u) - f(t,x,\overline{u})\|_E \leq l_f\|u - \overline{u}\|_{\mathcal{B}},$$

for each $(t,x) \in J$, *and each* $u, \overline{u} \in \mathcal{B}$.

(H_2') *There exist constants* $l_{g_k}' > 0$; $k = 1, \ldots, m$, *such that*

$$\|g_k(t,x,u) - g_k(t,x,\overline{u})\|_E \leq l_{g_k}'\|u - \overline{u}\|_E,$$

for each $(t,x) \in J_k$, *and each* $u, \overline{u} \in E$, $k = 1, \ldots, m$.

If

$$\ell' := 2l_g' + \frac{Kl_f' a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{9}$$

where $l_g' = \max_{k=1,\ldots,m} l_{g_k}'$, *then the problem* (2) *has a unique solution on* $(-\infty,a] \times (-\infty,b]$.

Furthermore, if the hypothesis (H_3) *holds, then the problem* (2) *is generalized Ulam-Hyers-Rassias stable.*

Proof Consider the operator $N' : \Omega \to \Omega$ defined by

$$\begin{cases} (N'u)(t,x) = \mu(t,x) \\ \qquad + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x) \in I_0, \\ (N'u)(t,x) = \varphi(t) + g_k(s_k,x,u(s_k,x)) - g_k(s_k,0,u(s_k,0)) \\ \qquad + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x) \in I_k, k = 1,\ldots,m, \\ (N'u)(t,x) = g_k(t,x,u(t,x)); \quad \text{if } (t,x) \in J_k, k = 1,\ldots,m, \\ (N'u)(t,x) = \phi(t,x); \quad \text{if } (t,x) \in \tilde{J}'. \end{cases}$$

Let $v(\cdot,\cdot):(-\infty,a]\times(-\infty,b]\to E$ be a function defined by

$$\begin{cases} v(t,x)=\mu(t,x); & \text{if } (t,x)\in J, \\ v(t,x)=\phi(t,x); & \text{if } (t,x)\in\tilde{J}'. \end{cases}$$

Then $v_{(t,x)}=\phi$ for all $(t,x)\in\mathcal{E}$. For each $w\in C(J)$ with $w(t,x)=0$; $(t,x)\in\mathcal{E}$ we denote by $\overline{w}$ the function defined by

$$\begin{cases} \overline{w}(t,x)=w(t,x); & \text{if } (t,x)\in J, \\ \overline{w}(t,x)=0; & \text{if } (t,x)\in\tilde{J}'. \end{cases}$$

If $u(\cdot,\cdot)$ satisfies

$$\begin{cases} u(t,x)=\mu(t,x) \\ \qquad +\int_0^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x)\in I_0, \\ u(t,x)=\varphi(t)+g_k(s_k,x,u(s_k,x))-g_k(s_k,0,u(s_k,0)) \\ \qquad +\int_{s_k}^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x)\in I_k, k=1,\ldots,m, \\ u(t,x)=g_k(t,x,u(t,x)); \quad \text{if } (t,x)\in J_k, k=1,\ldots,m, \end{cases}$$

then we can decompose $u(\cdot,\cdot)$ as $u(t,x)=\overline{w}(t,x)+v(t,x)$; $(t,x)\in J$, which implies $u_{(t,x)}=\overline{w}_{(t,x)}+v_{(t,x)}$, for every $(t,x)\in J$, and the function $w(\cdot,\cdot)$ satisfies

$$\begin{cases} w(t,x)=\int_0^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi,\overline{w}_{(\tau,\xi)}+v_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x)\in I_0, \\ w(t,x)=\varphi(t)+g_k(s_k,x,\overline{w}(s_k,x)+v(s_k,x))-g_k(s_k,0,\overline{w}(s_k,0)+v(s_k,0)) \\ \qquad +\int_{s_k}^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi,\overline{w}_{(\tau,\xi)}+v_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x)\in I_k, k=1,\ldots,m, \\ w(t,x)=g_k(t,x,\overline{w}(t,x)+v(t,x)); \quad \text{if } (t,x)\in J_k, k=1,\ldots,m. \end{cases}$$

Set

$$C_0=\left\{w\in PC: w(t,x)=0 \text{ for } (t,x)\in\mathcal{E}\right\},$$

and let $\|\cdot\|_{(a,b)}$ be the seminorm in C_0 defined by

$$\|w\|_{(a,b)}=\sup_{(t,x)\in E}\|w_{(t,x)}\|_{\mathcal{B}}+\sup_{(t,x)\in J}\left\|w(t,x)\right\|=\sup_{(t,x)\in J}\left\|w(t,x)\right\|; \quad w\in C_0.$$

C_0 is a Banach space with norm $\|\cdot\|_{(a,b)}$. Let the operator $P:C_0\to C_0$ be defined by

$$\begin{cases} (Pw)(t,x)=\int_0^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi,\overline{w}_{(\tau,\xi)}+v_{(\tau,\xi)})\,d\xi\,d\tau; \quad \text{if } (t,x)\in I_0, \\ (Pw)(t,x)=\varphi(t)+g_k(s_k,x,\overline{w}(s_k,x)+v(s_k,x))-g_k(s_k,0,\overline{w}(s_k,0)+v(s_k,0)) \\ \qquad +\int_{s_k}^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi,\overline{w}_{(\tau,\xi)}+v_{(\tau,\xi)})\,d\xi\,d\tau; \\ \qquad \text{if } (t,x)\in I_k, k=1,\ldots,m, \\ (Pw)(t,x)=g_k(t,x,\overline{w}(t,x)+v(t,x)); \quad \text{if } (t,x)\in J_k, k=1,\ldots,m. \end{cases}$$

Then the operator N' has a fixed point is equivalent to P has a fixed point. We shall show that $P : C_0 \to C_0$ is a contraction map. Indeed, consider $w, w^* \in C_0$. Then, for each $(t,x) \in J$, we have

$$\begin{cases} \|(Pw)(t,x) - (Pw^*)(t,x)\|_E \\ \quad \leq \| \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} [f(\tau,\xi,\overline{w}_{(\tau,\xi)} + v_{(\tau,\xi)}) - f(\tau,\xi,\overline{w}^*_{(\tau,\xi)} + v_{(\tau,\xi)})]\, d\xi\, d\tau \|_E; \\ \quad\quad \text{if } (t,x) \in I_0, \\ \|(Pw)(t,x) - (Pw^*)(t,x)\|_E \\ \quad \leq \|g_k(s_k,x,\overline{w}(s_k,x) + v(s_k,x)) - g_k(s_k,x,\overline{w}^*(s_k,x) + v(s_k,x))\|_E \\ \quad\quad + \|g_k(s_k,0,\overline{w}(s_k,0) + v(s_k,0)) - g_k(s_k,0,\overline{w}^*(s_k,0) + v(s_k,0))\|_E \\ \quad\quad + \| \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} [f(\tau,\xi,\overline{w}_{(\tau,\xi)} + v_{(\tau,\xi)}) - f(\tau,\xi,\overline{w}^*_{(\tau,\xi)} + v_{(\tau,\xi)})]\, d\xi\, d\tau \|_E; \\ \quad\quad \text{if } (t,x) \in I_k, k = 1,\ldots,m, \\ \|(Pw)(t,x) - (Pw^*)(t,x)\|_E \\ \quad = \|g_k(t,x,\overline{w}(t,x) + v(t,x)) - g_k(t,x,\overline{w}^*(t,x) + v(t,x))\|_E; \quad \text{if } (t,x) \in J_k, k = 1,\ldots,m. \end{cases}$$

Thus, we get

$$\begin{cases} \|(Pw)(t,x) - (Pw^*)(t,x)\|_E \\ \quad \leq \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} l'_f \|\overline{w}_{(\tau,\xi)} - \overline{w}^*_{(\tau,\xi)}\|\, d\xi\, d\tau \\ \quad \leq \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} l'_f \sup_{(\tau,\xi)\in[0,t]\times[0,x]} \|\overline{w}_{(\tau,\xi)} - \overline{w}^*_{(\tau,\xi)}\|\, d\xi\, d\tau \\ \quad \leq \frac{K l'_f a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|\overline{w} - \overline{w}^*\|_{(a,b)}; \quad \text{if } (t,x) \in I_0, \\ \|(Pw)(t,x) - (Pw^*)(t,x)\|_E \\ \quad \leq 2l'_g \|\overline{w}(\tau,\xi) - \overline{w}^*(\tau,\xi)\| \\ \quad\quad + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} l'_f \sup_{(\tau,\xi)\in[0,t]\times[0,x]} \|\overline{w}_{(\tau,\xi)} - \overline{w}^*_{(\tau,\xi)}\|\, d\xi\, d\tau \\ \quad \leq (2l'_g + \frac{K l'_f a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}) \|\overline{w} - \overline{w}^*\|_{(a,b)}; \quad \text{if } (t,x) \in I_k, k = 1,\ldots,m, \\ \|(Pw)(t,x) - (Pw^*)(t,x)\|_E \\ \quad \leq l'_g \|\overline{w}(\tau,\xi) - \overline{w}^*(\tau,\xi)\| \\ \quad \leq l'_g \|\overline{w} - \overline{w}^*\|_{(a,b)}; \quad \text{if } (t,x) \in J_k, k = 1,\ldots,m. \end{cases}$$

Therefore

$$\left\|P(w) - P\left(w^*\right)\right\|_{(a,b)} \leq \ell' \left\|\overline{w} - \overline{w}^*\right\|_{(a,b)}.$$

By the condition (9), we conclude that P is a contraction. As a consequence of Banach fixed point theorem, we deduce that P has a unique fixed point v. Then we have

$$\begin{cases} w^*(t,x) = \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,\overline{w}^*_{(\tau,\xi)} + v_{(\tau,\xi)})\, d\xi\, d\tau; \quad \text{if } (t,x) \in I_0, \\ (Pw)(t,x) = \varphi(t) + g_k(s_k,x,\overline{w}^*(s_k,x) + v(s_k,x)) - g_k(s_k,0,\overline{w}^*(s_k,0) + v(s_k,0)) \\ \quad\quad\quad\quad + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,\overline{w}^*_{(\tau,\xi)} + v_{(\tau,\xi)})\, d\xi\, d\tau; \\ \quad\quad\quad\quad \text{if } (t,x) \in I_k, k = 1,\ldots,m, \\ (Pw)(t,x) = g_k(t,x,\overline{w}^*(t,x) + v(t,x)); \quad \text{if } (t,x) \in J_k, k = 1,\ldots,m. \end{cases}$$

Let $w \in C_0$ be a solution of the inequality (4). By differential this inequality, for each $(t,x) \in J$, we have

$$\begin{cases} \|w(t,x) - \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad \leq \| \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \Phi(\tau,\xi)\,d\xi\,d\tau\|_E; \quad \text{if } (t,x) \in I_0, \\ \|u(t,x) - \varphi(t) - g_k(s_k,x,u(s_k,x)) + g_k(s_k,0,u(s_k,0)) \\ \qquad - \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,u_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad \leq \| \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \Phi(\tau,\xi)\,d\xi\,d\tau\|_E; \quad \text{if } (t,x) \in I_k, k = 1,\dots,m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \Psi; \quad \text{if } (t,x) \in J_k, k = 1,\dots,m. \end{cases}$$

Thus, by (H_3) for each $(t,x) \in J$, we get

$$\begin{cases} \|u(t,x) - \mu(t,x) - \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,\overline{w}_{(\tau,\xi)} + v_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad \leq \lambda_\Phi \Phi(t,x); \quad \text{if } (t,x) \in I_0, \\ \|u(t,x) - \varphi(t) - g_k(s_k,x,\overline{w}(s_k,x) + v(s_k,x)) + g_k(s_k,0,\overline{w}(s_k,0) + v(s_k,0)) \\ \qquad - \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi,\overline{w}_{(\tau,\xi)} + v_{(\tau,\xi)})\,d\xi\,d\tau\|_E \\ \quad \leq \lambda_\Phi \Phi(t,x); \quad \text{if } (t,x) \in I_k, k = 1,\dots,m, \\ \|u(t,x) - g_k(t,x,\overline{w}(t,x) + v(t,x))\|_E \leq \Psi; \quad \text{if } (t,x) \in J_k, k = 1,\dots,m. \end{cases}$$

Hence

$$\begin{cases} \|w(t,x) - w^*(t,x)\|_E \\ \quad \leq \lambda_\Phi \Phi(t,x) \\ \qquad + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(\tau,\xi,\overline{w}_{(\tau,\xi)} + v_{(\tau,\xi)}) - f(\tau,\xi,\overline{w}^*_{(\tau,\xi)} + v_{(\tau,\xi)})\|_E\,d\xi\,d\tau; \\ \qquad \text{if } (t,x) \in I_0, \\ \|w(t,x) - w^*(t,x)\|_E \\ \quad \leq \lambda_\Phi \Phi(t,x) + 2l_g \|\overline{w}(t,x) - \overline{w}^*(t,x)\|_E \\ \qquad + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(\tau,\xi,\overline{w}_{(\tau,\xi)} + v_{(\tau,\xi)}) - f(\tau,\xi,\overline{w}^*_{(\tau,\xi)} + v_{(\tau,\xi)})\|_E\,d\xi\,d\tau; \\ \qquad \text{if } (t,x) \in I_k, k = 1,\dots,m, \\ \|w(t,x) - w^*(t,x)\|_E \\ \quad \leq \Psi + \|g_k(t,x,\overline{w}(t,x) + v(t,x)) - g_k(t,x,\overline{w}^*(t,x) + v(t,x))\|_E \\ \quad \leq \Psi + l_g \|\overline{w}(t,x) - \overline{w}^*(t,x)\|_E; \quad \text{if } (t,x) \in J_k, k = 1,\dots,m. \end{cases}$$

For each $(t,x) \in I_0$, we have

$$\begin{aligned} &\big\|w(t,x) - w^*(t,x)\big\|_E \\ &\quad \leq \lambda_\Phi \Phi(t,x) \\ &\qquad + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \big\|f(\tau,\xi,\overline{w}_{(\tau,\xi)} + v_{(\tau,\xi)}) - f\big(\tau,\xi,\overline{w}^*_{(\tau,\xi)} + v_{(\tau,\xi)}\big)\big\|_E\,d\xi\,d\tau \\ &\quad \leq \lambda_\Phi \Phi(t,x) + l_f' \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \big\|\overline{w}_{(\tau,\xi)} - \overline{w}^*_{(\tau,\xi)}\big\|_{\mathcal{B}}\,d\xi\,d\tau. \end{aligned}$$

However,

$$\left\|\overline{w}_{(s,t)} - \overline{w}^*_{(s,t)}\right\|_{\mathcal{B}} \le K \sup_{(\tilde{s},\tilde{t})\in[0,s]\times[0,t]} w(\tilde{s},\tilde{t}).$$

If we name $z(s,t)$ the right hand side of this inequality, then we have

$$\left\|\overline{w}_{(s,t)} - \overline{w}^*_{(s,t)}\right\|_{\mathcal{B}} \le z(t,x),$$

and therefore, for each $(t,x)\in J$ we obtain

$$\left\|w(t,x) - w^*(t,x)\right\|_E \le \lambda_\Phi \Phi(t,x) + l'_f \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} z(\tau,\xi)\,d\xi\,d\tau.$$

Using the above inequality and the definition of z, for each $(t,x)\in J$ we have

$$z(t,x) \le K\lambda_\Phi \Phi(t,x) + Kl'_f \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} z(\tau,\xi)\,d\xi\,d\tau.$$

From Lemma 2.11, there exists a constant $\delta_1 := \delta_1(r_1,r_2)$ such that

$$\begin{aligned} z(t,x) &\le K\lambda_\Phi\big(\Phi(t,x) + Kl'_f\delta_1 I^r_\theta \Phi(t,x)\big) \\ &\le K\lambda_\Phi\big(1 + Kl'_f\delta_1\lambda_\Phi\big)\Phi(t,x) \\ &:= c'_{1,f,g_k,\Phi}\Phi(t,x). \end{aligned}$$

Thus, for each $(t,x)\in I_0$, we obtain

$$\begin{aligned} &\left\|w(t,x) - w^*(t,x)\right\|_E \\ &\quad\le \lambda_\Phi \Phi(t,x) + l'_f \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} c'_{1,f,g_k,\Phi}\Phi(\tau,\xi)\,d\xi\,d\tau \\ &\quad\le \big(\lambda_\Phi + l'_f c'_{1,f,g_k,\Phi}\lambda_\Phi\big)\Phi(t,x) \\ &\quad:= c_{1,f,g_k,\Phi}\Phi(t,x). \end{aligned}$$

Hence, for each $(t,x)\in I_0$, we get

$$\left\|u(t,x) - v(t,x)\right\|_E \le c_{1,f,g_k,\Phi}\big(\Psi + \Phi(t,x)\big).$$

Now, for each $(t,x)\in I_k$, $k=1,\dots,m$, we have

$$\begin{aligned} &\left\|w(t,x) - w^*(t,x)\right\|_E \\ &\quad\le \lambda_\Phi \Phi(t,x) + 2l'_g\left\|\overline{w}(t,x) - \overline{w}^*(t,x)\right\|_E \\ &\qquad + l_f \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \left\|\overline{w}_{(\tau,\xi)} - \overline{w}^*_{(\tau,\xi)}\right\|_{\mathcal{B}}\,d\xi\,d\tau. \end{aligned}$$

Then we obtain

$$\begin{aligned}&\|w(t,x)-w^*(t,x)\|_E\\&\quad\le\frac{\lambda_\Phi}{1-2l'_g}\Phi(t,x)+\frac{l'_f}{1-2l'_g}\int_{s_k}^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|\overline{w}_{(\tau,\xi)}-\overline{w}^*_{(\tau,\xi)}\|_{\mathcal{B}}\,d\xi\,d\tau.\end{aligned}$$

Again, from Lemma 2.11, there exists a constant $\delta_2 := \delta_2(r_1, r_2)$ such that

$$\begin{aligned}z(t,x)&\le\frac{K\lambda_\Phi}{1-2l'_g}\left(\Phi(t,x)+\frac{Kl'_f\delta_2}{1-2l'_g}I^r_{\theta_k}\Phi(t,x)\right)\\&\le\frac{K\lambda_\Phi}{1-2l_g}\left(1+\frac{Kl'_f\delta_2\lambda_\Phi}{1-2l_g}\right)\Phi(t,x)\\&:=c'_{2,f,g_k,\Phi}\Phi(t,x),\end{aligned}$$

and, then

$$\begin{aligned}&\|w(t,x)-w^*(t,x)\|_E\\&\quad\le\frac{\lambda_\Phi}{1-2l'_g}\Phi(t,x)+\frac{l'_f}{1-2l'_g}\int_{s_k}^t\int_0^x\frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}c'_{2,f,g_k,\Phi}\Phi(\tau,\xi)\,d\xi\,d\tau\\&\quad\le\frac{\lambda_\Phi+l'_f\lambda_\Phi c'_{2,f,g_k,\Phi}}{1-2l'_g}\Phi(t,x)\\&\quad:=c_{2,f,g_k,\Phi}\Phi(t,x).\end{aligned}$$

Hence, for each $(t,x)\in I_k$, $k=1,\dots,m$, we get

$$\|u(t,x)-v(t,x)\|_E\le c_{2,f,g_k,\Phi}\big(\Psi+\Phi(t,x)\big).$$

Now, for each $(t,x)\in J_k$, $k=1,\dots,m$, we have

$$\|w(t,x)-w^*(t,x)\|_E\le\Psi+l'_g\|\overline{w}(t,x)-\overline{w}^*(t,x)\|_E.$$

This gives

$$\|u(t,x)-v(t,x)\|_E\le\frac{\Psi}{1-l_g}:=c_{3,f,g_k,\Phi}\Psi.$$

Thus, for each $(t,x)\in J_k$, $k=1,\dots,m$, we get

$$\|u(t,x)-v(t,x)\|_E\le c_{3,f,g_k,\Phi}\big(\Psi+\Phi(t,x)\big).$$

Set $c_{f,g_k,\Phi}:=\max_{i\in\{1,2,3\}}c_{i,f,g_k,\Phi}$. Hence, for each $(t,x)\in J$, we obtain

$$\|u(t,x)-v(t,x)\|_E\le c_{f,g_k,\Phi}\big(\Psi+\Phi(t,x)\big).$$

Consequently, problem (2) is generalized Ulam-Hyers-Rassias stable. □

6 Examples

Example 6.1 Let $E = l^1 = \{w = (w_1, w_2, \ldots, w_n, \ldots) : \sum_{n=1}^{\infty} |w_n| < \infty\}$, be the Banach space with norm

$$\|w\|_E = \sum_{n=1}^{\infty} |w_n|.$$

Consider partial fractional differential equations with noninstantaneous impulses and finite delay of the form

$$\begin{cases} {}^cD^r_{\theta_k} u(t,x) = f(t,x,u(t-1,x-2)); & \text{if } (t,x) \in ([0,1] \cup (2,3]) \times [0,1], k \in \{0,1\}, \\ u(t,x) = g(t,x,u(t,x)); & \text{if } (t,x) \in (1,2] \times [0,1], \\ u(t,x) = 1 + x^2 + e^t; & (t,x) \in [-1,3] \times [-2,1] \setminus (0,3] \times (0,1], \\ u(t,0) = 1 + e^t; & t \in [0,3], \\ u(0,x) = 2 + x^2; & x \in [0,1], \end{cases} \tag{10}$$

where $r = (r_1, r_2) \in (0,1] \times (0,1]$, $\theta_0 = (0,0)$, $\theta_1 = (2,0)$, $\alpha = -1$, $\beta = -2$, $0 = s_0 < t_1 = 1 < s_1 = 2 < t_2 = 3$, $u = (u_1, u_2, \ldots, u_n, \ldots)$, $f = (f_1, f_2, \ldots, f_n, \ldots)$, $g = (g_1, g_2, \ldots, g_n, \ldots)$,

$${}^cD^r_\theta u = \left({}^cD^r_\theta u_1, {}^cD^r_\theta u_2, \ldots, {}^cD^r_\theta u_n, \ldots\right),$$

$$f_n(t,x,u_n) = \frac{1}{(1 + 110e^{t+x})(1 + \|u_n\|_{\mathcal{C}})}; \quad (t,x) \in ([0,1] \cup (2,3]) \times [0,1] \text{ and } n \in \mathbb{N},$$

$\mathcal{C} := C_{(1,2)}$ and

$$g_n(t,x,u_n) = \frac{1}{1 + 110e^{t+x}} \arctan\left(t^2 + x^2 + |u_n|\right); \quad (t,x) \in (1,2] \times [0,1] \text{ and } n \in \mathbb{N}.$$

Clearly, the functions f and g are continuous. For each $n \in \mathbb{N}$, $u, \overline{u} \in E$ and $(t,x) \in ([0,1] \cup (2,3]) \times [0,1]$, we have

$$\left|f_n(t,x,u_{n(t,x)}) - f_n(t,x,\overline{u}_{n(t,x)})\right| \leq \frac{1}{111} \|u_n - \overline{u}_n\|_{\mathcal{C}}.$$

Thus, for each $u, \overline{u} \in E$ and $(t,x) \in ([0,1] \cup (2,3]) \times [0,1]$ we get

$$\begin{aligned} \left\|f(t,x,u_{(t,x)}) - f(t,x,\overline{u}_{(t,x)})\right\|_E &= \sum_{n=1}^{\infty} \left|f_n(t,x,u_{(t,x)}) - f_n(t,x,\overline{u}_{(t,x)})\right| \\ &\leq \frac{1}{111} \sum_{n=1}^{\infty} \|u_n - \overline{u}_n\|_{\mathcal{C}} \\ &= \frac{1}{111} \|u_n - \overline{u}_n\|_{\mathcal{C}}. \end{aligned}$$

Also, for each $n \in \mathbb{N}$, $u, \overline{u} \in E$ and $(t,x) \in (1,2] \times [0,1]$, we get

$$\left\|g\left(t,x,u(t,x)\right) - g\left(t,x,\overline{u}(t,x)\right)\right\|_E \leq \frac{1}{111} \|u - \overline{u}\|_E.$$

Hence the conditions (H_1) and (H_2) are satisfied with $l_f = l_g = \frac{1}{111}$. We shall show that condition (8) holds with $a = 3$ and $b = 1$. Indeed, for each $(r_1, r_2) \in (0,1] \times (0,1]$ we get

$$\begin{aligned}\ell &= 2l_g + \frac{l_f a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \frac{2}{111} + \frac{3^{r_1}}{111\Gamma(1+r_1)\Gamma(1+r_2)} \\ &< \frac{14}{111} < 1.\end{aligned}$$

By Theorem 3.4, the problem (10) has a unique solution defined on $[-1,3] \times [-2,1]$. Finally, the hypothesis (H_3) is satisfied with $\Phi(t,x) = tx^2$ and $\lambda_\Phi = \frac{2\times 3^{r_1}}{\Gamma(2+r_1)\Gamma(3+r_2)}$. Indeed, for each $(t,x) \in [0,3] \times [0,1]$ we get

$$\left(I_\theta^r \Phi\right)(t,x) = \frac{\Gamma(2)\Gamma(3)t^{1+r_1}x^{2+r_2}}{\Gamma(2+r_1)\Gamma(3+r_2)} \le \frac{2\times 3^{r_1} tx^2}{\Gamma(2+r_1)\Gamma(3+r_2)} = \lambda_\Phi \Phi(t,x).$$

Consequently, Theorem 3.4 implies that the problem (10) is generalized Ulam-Hyersdz-Rassias stable.

Example 6.2 Consider now partial differential equations with noninstantaneous impulses and infinite delay of the form

$$\begin{cases} {}^cD_{\theta_k}^r u(t,x) = f(t,x,u_{(t,x)}); \quad \text{if } (t,x) \in ([0,1] \cup (2,3]) \times [0,1], k \in \{0,1\}, \\ u(t,x) = g(t,x,u(t,x)); \quad \text{if } (t,x) \in (1,2] \times [0,1], \\ u(t,x) = t + x^2; \quad \text{if } (t,x) \in \tilde{J} := [-\infty,3] \times [-\infty,1] \setminus (0,3] \times (0,1], \\ u(t,0) = t; \quad t \in [0,3], \\ u(0,x) = x^2; \quad x \in [0,1], \\ \varphi(0) = \psi(0), \end{cases} \tag{11}$$

where

$$r = (r_1,r_2) \in (0,1] \times (0,1], \qquad \theta_0 = (0,0), \qquad \theta_1 = (2,0),$$

$$0 = s_0 < t_1 = 1 < s_1 = 2 < t_2 = 3,$$

$$f(t,x,u_{(t,x)}) = \frac{ce^{t+x-\gamma(t+x)}\|u_{(t,x)}\|}{(e^{t+x}+e^{-t-x})(1+\|u_{(t,x)}\|)}; \quad \text{if } (t,x) \in ([0,1] \cup (2,3]) \times [0,1],$$

$c = \frac{11\times 3^{r_1}}{\Gamma(1+r_1)\Gamma(1+r_2)}$, $\gamma > 0$ and

$$g(t,x,u) = \frac{1}{1+10e^{t+x}} \arctan\left(t^2 + x^2 + |u|\right); \quad (t,x) \in (1,2] \times [0,1].$$

Let

$$B_\gamma = \left\{u \in C\left((-\infty,0] \times (-\infty,0], \mathbb{R}\right) : \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u(\theta,\eta) \text{ exists in } \mathbb{R}\right\}.$$

The norm of B_γ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(\theta+\eta)}\left|u(\theta,\eta)\right|.$$

Let

$$\mathcal{E} := [0,1]\times\{0\}\cup\{0\}\times[0,1],$$

and $u:(-\infty,3]\times(-\infty,1]\to\mathbb{R}$ such that $u_{(t,x)}\in B_\gamma$ for $(t,x)\in\mathcal{E}$, then

$$\begin{aligned}\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u_{(t,x)}(\theta,\eta) &= \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta-x+\eta-y)}u(\theta,\eta)\\ &= e^{\gamma(x+y)}\lim_{\|(\theta,\eta)\|\to\infty} u(\theta,\eta)<\infty.\end{aligned}$$

Hence $u_{(t,x)}\in B_\gamma$. Finally we prove that

$$\|u_{(t,x)}\|_\gamma = K\sup\left\{\left|u(\tau,\xi)\right|:(\tau,\xi)\in[0,t]\times[0,x]\right\} + M\sup\left\{\|u_{(\tau,\xi)}\|_\gamma:(\tau,\xi)\in\mathcal{E}_{(t,x)}\right\},$$

where $K=M=1$ and $H=1$. If $t+\theta\le 0$, $x+\eta\le 0$ we get

$$\|u_{(t,x)}\|_\gamma = \sup\left\{\left|u(\tau,\xi)\right|:(\tau,\xi)\in(-\infty,0]\times(-\infty,0]\right\},$$

and if $t+\theta\ge 0$, $x+\eta\ge 0$ then we have

$$\|u_{(t,x)}\|_\gamma = \sup_{(\tau,\xi)\in[0,t]\times[0,x]}\left|u(\tau,\xi)\right|.$$

Thus for all $(t+\theta,x+\eta)\in[0,3]\times[0,1]$, we get

$$\|u_{(t,x)}\|_\gamma = \sup_{(\tau,\xi)\in(-\infty,0]\times(-\infty,0]}\left|u(\tau,\xi)\right| + \sup_{(\tau,\xi)\in[0,t]\times[0,x]}\left|u(\tau,\xi)\right|.$$

Then

$$\|u_{(t,x)}\|_\gamma = \sup_{(s,t)\in E}\|u_{(s,t)}\|_\gamma + \sup_{(s,t)\in[0,t]\times[0,x]}\left|u(s,t)\right|.$$

$(B_\gamma,\|\cdot\|_\gamma)$ is a Banach space. We conclude that B_γ is a phase space.

For each $u,\overline{u}\in B_\gamma$ and $(t,x)\in([0,1]\cup(2,3])\times[0,1]$, we have

$$\begin{aligned}\left|f(t,x,u_{(t,x)})-f(t,x,\overline{u}_{(t,x)})\right| &\le \frac{e^{t+x}\|u-\overline{u}\|_B}{c(e^{t+x}+e^{-t-x})}\\ &\le \frac{1}{c}\|u-\overline{u}\|_B.\end{aligned}$$

Hence condition (H_1) is satisfied with $l_f=\frac{1}{c}$. Also, for each $u,\overline{u}\in\mathbb{R}$ and $(t,x)\in(1,2]\times[0,1]$, we have we get

$$\left|g\big(t,x,u(t,x)\big)-g\big(t,x,\overline{u}(t,x)\big)\right|\le\frac{1}{11}|u-\overline{u}|.$$

Hence the condition (H_2) is satisfied with $l_g = \frac{1}{11}$. We shall show that condition (9) holds with $a = 3$ and $b = K = 1$. Indeed,

$$\begin{aligned}\ell' &= 2l'_g + \frac{Kl'_f a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\\ &= \frac{2}{11} + \frac{3^{r_1}}{c\Gamma(1+r_1)\Gamma(1+r_2)}\\ &\le \frac{2}{11} + \frac{1}{11} = \frac{3}{11} < 1.\end{aligned}$$

By Theorem 5.1, the problem (11) has a unique solution defined on $(-\infty,3] \times (-\infty,1]$. Moreover, the hypothesis (H_3) is satisfied with $\Phi(t,x) = tx^2$ and $\lambda_\Phi = \frac{2\times 3^{r_1}}{\Gamma(2+r_1)\Gamma(3+r_2)}$. Indeed, for each $(t,x) \in [0,3] \times [0,1]$ we get

$$\left(I_\theta^r \Phi\right)(t,x) = \frac{\Gamma(2)\Gamma(3)t^{1+r_1}x^{2+r_2}}{\Gamma(2+r_1)\Gamma(3+r_2)} \le \frac{2\times 3^{r_1}tx^2}{\Gamma(2+r_1)\Gamma(3+r_2)} = \lambda_\Phi \Phi(t,x).$$

Consequently, Theorem 5.1 implies that the problem (11) is generalized Ulam-Hyers-Rassias stable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SA and MB contributed to Sections 1, 2, 3, and 5. AA and YZ contributed to Sections 4 and 6.

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Acknowledgements
The authors would like to thank all the anonymous reviewers and the editors for their helpful advice and hard work. The work was supported by the National Natural Science Foundation of China (No. 11671339).

Received: 20 November 2016 Accepted: 7 February 2017 Published online: 09 March 2017

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