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Topological properties of solution sets of fractional stochastic evolution inclusions

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Abstract

In this paper, we investigate the topological structure for the solution set of Caputo type neutral fractional stochastic evolution inclusions in Hilbert spaces. We introduce the concept of mild solutions for fractional neutral stochastic inclusions and show that the solution set is nonempty compact and R_δ-set, which means that the solution set may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point in the sense that it has the same homology group as one-point space. Finally, we illustrate the obtained theory with the aid of an example.

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Keywords: stochastic evolution inclusions; Brownian motion; multimaps; compact R_δ-set

1 Introduction

Stochastic differential inclusions play an important role in characterizing many social, physical, biological and engineering problems, see, for example, Gawarecki and Mandrekar [1], Kisielewicz [2], and Prato and Zabczyk [3]. Neutral stochastic differential equations and inclusions can be used to describe some systems such as aeroelasticity, the lossless transmission lines, stabilization of lumped control systems and theory of heat conduction in materials with fading memory when noise or stochastic perturbation is taken into account; for details, see Hernández and Henriquez [4], Luo [5], Mahmudov [6] and the references therein.

Fractional calculus tools are found to be quite effective in modelling anomalous diffusion processes as fractional-order operators can characterize the long memory processes. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas *et al.* [7], Diethelm [8], Zhou [9, 10], the recent papers [11–17], and the references therein. Therefore, it is reasonable and practical to import fractional-order operators into the investigation of stochastic differential systems. Recently, Toufik [18] obtained the existence of mild solutions for the fractional stochastic evolution inclusions. Zhou [10] derived topological properties of solution sets for fractional stochastic evolution inclusions. An important aspect of such structure is the R_δ-property, that is, an R_δ-set is acyclic (in particular, nonempty, compact and connected) and may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point in the sense that it has the same homology group as one point space. There has been a great interest in the study of topological structure of solution sets; for instance, see Andres and Pavlačková [19], Bothe [20], Bressan and Wang [21], Chen *et al.* [22], De Blasi and Myjak [23], Deimling [24], Gabor and Quincampoix [25], Górniewicz and Pruszko [26], Hu and Papageorgiou [27], Staicu [28], and Wang *et al.* [29, 30] and the references cited therein.

In this paper, we consider the following problem of fractional stochastic evolution inclusions in Hilbert spaces

$$\begin{cases} {}^CD_{0+}^q[x(t)-h(t,x_t)] \in Ax(t)+\Sigma(t,x_t)\frac{dW(t)}{dt}, & t\in[0,b],\\ x(t)=\phi(t), & t\in[-\tau,0], \end{cases} \tag{1.1}$$

where ${}^CD_{0+}^q$ is the Caputo fractional derivative of order $q\in(\frac{1}{2},1)$, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t):t\geq 0\}$ in a Hilbert space H with inner product $(\cdot,\cdot)$ and norm $|\cdot|$, $h:J\times C([-\tau,0],H)\to H$, $\Sigma:J\times C([-\tau,0],H)\multimap \mathcal{L}(K,H)$ is a nonempty, bounded, closed, and convex multimap, $\{W(t):t\geq 0\}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q\geq 0$. Here $C([-\tau,0],H)$ is the space of all continuous functions from $[-\tau,0]$ to H equipped with the norm $\|c\|_*^2=\sup_{\theta\in[-\tau,0]}E|c(\theta)|^2$, K is a Hilbert space with inner product $(\cdot,\cdot)_K$ and norm $|\cdot|_K$, $\mathcal{L}(K,H)$ denotes the Banach space of all bounded linear operators from K to H.

Let $q\in(0,1]$ and $x:[0,+\infty)\to X$. Then the fractional integral operator is defined by

$$I_{0+}^q x(t)=g_q(t)*x(t)=\int_0^t g_q(t-s)x(s)\,ds,\quad t>0,$$

where $*$ denotes the convolution and $g_q(t)=\frac{t^{q-1}}{\Gamma(q)}$. Similarly, the Riemann-Liouville fractional derivative operator is defined by

$${}^LD_{0+}^q x(t)=\bigl(g_{1-q}(t)*x(t)\bigr)'$$

and the Caputo fractional derivative operator can be defined by

$${}^CD_{0+}^q x(t)={}^LD_{0+}^q\bigl(x(t)-x(0)\bigr)$$

for all $t>0$. For more details, we refer the reader to [7–9].

The study of inclusions (1.1) constitutes an important area of research. However, this topic is relatively less developed and needs to be explored further [31]. To the best of our knowledge, the investigation of topological properties of the solution set for (1.1) is yet to be addressed. In this paper, our objective is to establish that the solution set for the inclusions (1.1) is nonempty compact R_δ-set.

The rest of the paper is organized as follows. Section 2 contains some preliminary material on differential inclusions and the notation to be followed in the sequel, while Section 3 presents the concept of mild solutions for fractional neutral stochastic inclusions. In Section 4, we show that the solution set for the inclusions problem (1.1) is nonempty compact and discuss its R_δ-structure. The paper concludes with an example demonstrating the application of the work established in this paper.

2 Preliminaries

Let H, K be real separable Hilbert spaces, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\mathcal{F}_t$, $t \in [0,b]$ satisfying the usual conditions (that is, right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). We consider a Q-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ with the linear bounded covariance operator Q satisfying the condition $\mathrm{Tr}(Q) < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}_{n\geq 1}$ on K, a bounded sequence of non-negative real numbers $\{\lambda_n\}$ such that $Qe_n = \lambda_n e_n$, $n = 1, 2, \dots$, and a sequence $\{W_n\}_{n\geq 1}$ of independent Brownian motion such that

$$\bigl(W(t), e\bigr)_K = \sum_{n=1}^{\infty} \sqrt{\lambda_n}(e_n, e)_K W_n(t), \quad e \in K, t \in [0,b],$$

and $\mathcal{F}_t = \mathcal{F}_t^{\omega}$, where $\mathcal{F}_t^{\omega}$ is the sigma algebra generated by $\{W(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Q^{\frac{1}{2}}K; H)$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}K$ to H with the inner product $(\Psi, \Upsilon)_{L_2^0} = \mathrm{tr}[\Psi Q \Upsilon^*]$.

Denote by $L^2(\Omega; H)$ the Banach space of all $\mathcal{F}_b$-measurable square integrable random variables with values in H with the norm $\|\cdot\|$. Let $\mathcal{C}[-\tau, b]$ be a subspace of all continuous H-valued stochastic processes $x \in C([-\tau, b]; L^2(\Omega; H))$ endowed with the norm

$$\|x\|_{C[-\tau,b]} = \Bigl(\sup_{t\in[-\tau,b]} E\bigl|x(t)\bigr|^2\Bigr)^{\frac{1}{2}}.$$

We assume that A is the infinitesimal generator of an analytic semigroup $\{T(t) : t \geq 0\}$ of uniformly bounded linear operators on H. Let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Under these conditions, it is possible to define the fractional power A^{β}, $0 < \beta \leq 1$, as a closed linear operator on its domain $D(A^{\beta})$. For an analytic semigroup $\{T(t) : t \geq 0\}$, the following properties will be used:

(i) there exists $M \geq 1$ such that $M := \sup_{t\geq 0} \|T(t)\| < \infty$;

(ii) for any $\beta \in (0,1]$, there exists a positive constant C_β such that

$$\bigl\|A^{\beta} T(t)\bigr\| \leq \frac{C_\beta}{t^{\beta}}, \quad 0 < t \leq b.$$

Let $P(H)$ stands for the collection of all nonempty subsets of H. As usual, we denote $P_{cp}(H) = \{D \in P(H) : \text{compact}\}$, $P_{cl,cv}(H) = \{D \in P(H) : \text{closed and convex}\}$, $P_{cp,cv}(H) = \{D \in P(H) : \text{compact and convex}\}$, $\mathrm{co}(D)$ (resp., $\overline{\mathrm{co}}(D)$) be the convex hull (resp., convex closed hull in D) of a subset D.

Definition 2.1 A subset D of a metric space is called an R_δ-set if there exists a decreasing sequence $\{D_n\}$ of compact and contractible sets such that

$$D = \bigcap_{n=1}^{\infty} D_n.$$

We need the following well-known results in the forthcoming analysis.

Lemma 2.1 ([32]) *Let H be a Hilbert spaces and $\varphi : H \to P(H)$ a closed quasicompact multimap with compact values. Then φ is u.s.c.*

Lemma 2.2 ([20]) *Let $\varphi : D \subset H \to P(H)$ be a multimap with weakly compact convex values. Then φ is weakly u.s.c. if and only if $\{x_n\} \subset D$ with $x_n \to x_0 \in D$ and $y_n \in \varphi(x_n)$ implies $y_n \rightharpoonup y_0 \in \varphi(x_0)$, up to a subsequence.*

Theorem 2.1 ([16]) *Let D be a bounded convex closed subset of Banach space X. Let $\varphi_1 : D \to X$ be a single-valued map and $\varphi_2 : D \to P_{cp,cv}(X)$ be a multimap such that $\varphi_1(x) + \varphi_2(x) \in P(D)$ for $x \in D$. In addition, it is assumed that*

(a) *φ_1 is a contraction with the contraction constant $k < \frac{1}{2}$, and*

(b) *φ_2 is u.s.c. and compact.*

Then the fixed point set $\mathrm{Fix}(\varphi_1 + \varphi_2) := \{x : x \in \varphi_1(x) + \varphi_2(x)\}$ *is a nonempty compact set.*

3 Statement of the problem

To study the fractional stochastic evolution inclusions (1.1), we assume that

(H_1) The function $h : [0,b] \times C([-\tau,0];H) \to H$ is continuous and there exist constants $\beta \in (\frac{1}{2q},1)$ and $d, d_1 > 0$ with $\sqrt{2d(\|A^{-\beta}\|^2 + \frac{M_\beta^2}{2q\beta-1}b^{2q\beta})} < \frac{1}{2}$ such that $h \in D(A^\beta)$ and for any $c_1, c_2 \in C([-\tau,0];H)$, the function $A^\beta h(t,\cdot)$ is strongly measurable and $A^\beta h(t,\cdot)$ satisfies the Lipschitz condition

$$E\left|A^\beta h(t,c_1) - A^\beta h(t,c_2)\right|^2 \le d\|c_1 - c_2\|_*^2$$

and the inequality

$$E\left|A^\beta h(t,c_1)\right|^2 \le d_1\left(1 + \|c_1\|_*^2\right) \quad \text{for every } t \in [0,b].$$

(H_2) The multimap $\Sigma : [0,b] \times C([-\tau,0];H) \to P(L_2^0)$ has closed bounded and convex values and satisfies the following conditions:

(i) $\Sigma(t,\cdot)$ is weakly u.s.c. for a.e. $t \in [0,b]$, and the multimap $\Sigma(\cdot,c)$ has a strongly measurable selection for every $c \in C([-\tau,0];H)$;

(ii) there exists $q_1 \in [\frac{1}{2},q)$ and a function $\alpha \in L^{\frac{1}{2q_1-1}}([0,b];\mathbb{R}^+)$ such that

$$E\left\|\Sigma(t,c)\right\|_{L_2^0}^2 \le \alpha(t)\left(1 + \|c\|_*^2\right) \quad \text{for a.e. } t \in [0,b] \text{ and } c \in C\left([-\tau,0];H\right),$$

where $E\|\Sigma(t,c)\|_{L_2^0}^2 = \sup\{E\|\sigma(t)\|_{L_2^0}^2 : \sigma \in \Sigma(t,c)\}$.

Given $x \in \mathcal{C}[-\tau,b]$, let us denote

$$\mathrm{Sel}_\Sigma(x) := \left\{\sigma \in L^2\left([0,b];L_2^0\right) : \sigma(t) \in \Sigma(t,x_t), \text{ for a.e. } t \in [0,b]\right\}.$$

Notice that the set $\mathrm{Sel}_\Sigma(x)$ is always nonempty by Lemma 3.1 stated below. Now we state some more well-known results [10] that we need later.

Lemma 3.1 *Let the condition* (H_2) *be satisfied. Then* $\mathrm{Sel}_\Sigma : \mathcal{C}[-\tau,b] \to P(L^2([0,b];L_2^0))$ *is weakly u.s.c. with nonempty, convex and weakly compact values.*

Lemma 3.2 *Assume that* (H_2) *is satisfied. Then there exists a sequence $\{\Sigma_n\}$ with $\Sigma_n : [0,b] \times C([-\tau,0];H) \to P_{cl,cv}(L_2^0)$ such that*

(i) $\Sigma(t,c) \subset \Sigma_{n+1}(t,c) \subset \Sigma_n(t,c) \subset \overline{\mathrm{co}}(\Sigma(t,B_{3^{1-n}}(c)))$, $n \geq 1$, *for each* $t \in [0,b]$ *and* $c \in C([-\tau,0];H)$;

(ii) $E\|\Sigma_n(t,c)\|^2_{L_2^0} \leq \alpha(t)(3+2\|c\|_*^2)$, $n \geq 1$, *for a.e.* $t \in [0,b]$ *and each* $c \in C([-\tau,0];H)$;

(iii) *there exists* $E \subset [0,b]$ *with* $\mathrm{mes}(E) = 0$ *such that for each* $x^* \in H$, $\epsilon > 0$ *and* $(t,c) \in [0,b] \setminus E \times C([-\tau,0];H)$, *there exists* $N > 0$ *such that for all* $n \geq N$,

$$\langle x^*, \Sigma_n(t,c)\rangle \subset \langle x^*, \Sigma(t,c)\rangle + (-\epsilon,\epsilon);$$

(iv) $\Sigma_n(t,\cdot): C([-\tau,0];H) \to P_{cl,cv}(L_2^0)$ *is continuous for a.e.* $t \in [0,b]$ *with respect to Hausdorff metric for each* $n \geq 1$;

(v) *for each* $n \geq 1$, *there exists a selection* $\tilde{\sigma}_n : [0,b] \times C([-\tau,0];H) \to L_2^0$ *of* Σ_n *such that* $\tilde{\sigma}_n(\cdot,c)$ *is measurable for each* $C([-\tau,0];H)$ *and for any compact subset* $\mathcal{D} \subset H$ *there exist constants* $C_V > 0$ *and* $\delta > 0$ *for which the estimate*

$$E\big\|\tilde{\sigma}_n(t,c_1) - \tilde{\sigma}_n(t,c_2)\big\|^2_{L_2^0} \leq C_V\alpha(t)\|c_1 - c_2\|_*^2$$

holds for a.e. $t \in [0,b]$ *and each* $c_1, c_2 \in V$ *with* $V := \mathcal{D} + B_\delta(0)$;

(vi) Σ_n *verifies the condition* (H_2)(i) *with* Σ_n *instead of* Σ *for each* $n \geq 1$.

Let us first introduce two families of operators on H:

$$S_q(t) = \int_0^\infty \xi_q(\theta)T(t^q\theta)\,d\theta, \quad \text{for } t \geq 0,$$
$$K_q(t) = \int_0^\infty q\theta\xi_q(\theta)T(t^q\theta)\,d\theta, \quad \text{for } t \geq 0,$$

where

$$\xi_q(\theta) = \frac{1}{\pi q}\sum_{n=1}^\infty (-\theta)^{n-1}\frac{\Gamma(1+qn)}{n!}\sin(n\pi q), \quad \theta \in (0,+\infty).$$

Definition 3.1 A stochastic process $x \in \mathcal{C}[-\tau,b]$ is said to be a mild solution of the problem (1.1) if $x(t) = \phi(t)$ for $t \in [-\tau,0]$ and there exists $\sigma \in L^2([0,b];L_2^0)$ such that $\sigma(t) \in \Sigma(t,x_t)$ for a.e. $t \in [0,b]$, and x satisfies the following integral equation

$$\begin{aligned} x(t) &= S_q(t)\big[\phi(0) - h(0,\phi)\big] + h(t,x_t) + \int_0^t (t-s)^{q-1}AK_q(t-s)h(s,x_s)\,ds \\ &\quad + \int_0^t (t-s)^{q-1}K_q(t-s)\sigma(s)\,dW(s), \quad t \in [0,b]. \end{aligned}$$

Lemma 3.3 ([9]) *The operators* $S_q(t)$ *and* $K_q(t)$ *have the following properties*:

(i) *for each fixed* $t \geq 0$, $S_q(t)$ *and* $K_q(t)$ *are linear and bounded operators, i.e., for any* $x \in H$,

$$\big|S_q(t)x\big| \leq M|x| \quad \textit{and} \quad \big|K_q(t)x\big| \leq \frac{M|x|}{\Gamma(q)};$$

(ii) $\{S_q(t): t \geq 0\}$ *and* $\{K_q(t): t \geq 0\}$ *are strongly continuous*;

(iii) $\{S_q(t) : t > 0\}$ *is compact if* $\{T(t) : t > 0\}$ *is compact;*
(iv) *for any* $x \in H$, $\beta \in (0,1)$, *we have* $AK_q(t)x = A^{1-\beta}K_q(t)A^{\beta}x$ *and*

$$\|A^{1-\beta}K_q(t)\| \leq \frac{M_\beta}{t^{q(1-\beta)}},$$

where $M_\beta = \frac{qC_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+q\beta)}$.

Remark 3.1 For any $x \in \mathcal{C}[-\tau,b]$, define a solution multioperator $\mathcal{F} : \mathcal{C}[-\tau,b] \to P(\mathcal{C}[-\tau,b])$ as follows:

$$\mathcal{F} = \mathcal{F}_1(x) + \mathcal{F}_2(x),$$

where

$$\mathcal{F}_1(x)(t) = \begin{cases} -S_q(t)h(0,\phi) + h(t,x_t) + \int_0^t (t-s)^{q-1}AK_q(t-s)h(s,x_s)\,ds, & t \in [0,b], \\ 0, & t \in [-\tau,0], \end{cases}$$

$$\mathcal{F}_2(x)(t) = \left\{ y \in \mathcal{C}[-\tau,b] : y(t) = \begin{cases} S(\sigma)(t), \sigma \in \mathrm{Sel}_\Sigma(x), & t \in [0,b], \\ \phi(t), & t \in [-\tau,0] \end{cases} \right\},$$

and the operator $S : L^2([0,b];L_2^0) \to \mathcal{C}[-\tau,b]$ is defined by

$$S(\sigma) = S_q(t)\phi(0) + \int_0^t (t-s)^{q-1}K_q(t-s)\sigma(s)\,dW(s).$$

Observe that the fixed points of the multioperator $\mathcal{F}$ are mild solutions of the problem (1.1).

Lemma 3.4 *Let D be a bounded set of $\mathcal{C}[-\tau,b]$. If (H_1) holds, then $\{\Phi(x)(t) : x \in D\}$ is equicontinuous on $[0,b]$, where*

$$\Phi(x)(t) = \int_0^t (t-s)^{q-1}AK_q(t-s)h(s,x_s)\,ds, \quad t \in [0,b].$$

Proof For each $x \in D$ with $0 \leq t_1 < t_2 \leq b$, we obtain

$$\begin{aligned} &E\left|\Phi(x)(t_2) - \Phi(x)(t_1)\right|^2 \\ &\quad\leq 3(t_2-t_1)\int_{t_1}^{t_2} (t_2-s)^{2(q-1)}E\left|A^{1-\beta}K_q(t_2-s)A^\beta h(s,x_s)\right|^2 ds \\ &\qquad + 3t_1\int_0^{t_1} \left((t_2-s)^{q-1} - (t_1-s)^{q-1}\right)^2 E\left|A^{1-\beta}K_q(t_2-s)A^\beta h(s,x_s)\right|^2 ds \\ &\qquad + 3t_1\int_0^{t_1} (t_1-s)^{2(q-1)}E\left|\left[A^{1-\beta}K_q(t_2-s) - A^{1-\beta}K_q(t_1-s)\right]A^\beta h(s,x_s)\right|^2 ds \\ &\quad=: J_1(t_1,t_2) + J_2(t_1,t_2) + J_3(t_1,t_2). \end{aligned}$$

After a fundamental calculation, one can estimate each term as follows:

$$\begin{aligned}
J_1(t_1,t_2) &\le 3d_1 M_\beta^2 (t_2-t_1)\int_{t_1}^{t_2}(t_2-s)^{2(q\beta-1)}\big(1+\|x_s\|_*^2\big)\,ds \\
&\le 3d_1 M_\beta^2\big(1+\|x\|_{\mathcal{C}[-\tau,b]}^2\big)\frac{(t_2-t_1)^{2q\beta}}{2q\beta-1}, \\
J_2(t_1,t_2) &\le 3d_1 M_\beta^2 t_1\int_0^{t_1}\big((t_2-s)^{q-1}-(t_1-s)^{q-1}\big)^2(t_2-s)^{2q(\beta-1)}\big(1+\|x_s\|_*^2\big)\,ds \\
&\le 3d_1 M_\beta^2 t_1\big(1+\|x\|_{\mathcal{C}[-\tau,b]}^2\big)\int_0^{t_1}\big((t_2-s)^{2(q-1)}-(t_1-s)^{2(q-1)}\big)(t_2-s)^{2q(\beta-1)}\,ds \\
J_3(t_1,t_2) &\le 3t_1\sup_{s\in[0,t_1-\delta]}\big\|A^{1-\beta}K_q(t_2-s)-A^{1-\beta}K_q(t_1-s)\big\|^2 \\
&\quad\times\int_0^{t_1-\delta}(t_1-s)^{2(q-1)}\big(1+\|x_s\|_*^2\big)\,ds \\
&\quad+6d_1 M_\beta^2 t_1\int_{t_1-\delta}^{t}(t_1-s)^{2(q-1)}\big[(t_2-s)^{2q(\beta-1)}+(t_1-s)^{2q(\beta-1)}\big]\big(1+\|x_s\|_*^2\big)\,ds \\
&\le 3t_1\big(1+\|x\|_{\mathcal{C}[-\tau,b]}^2\big)\frac{\delta^{2q-1}-(t_1-\delta)^{2q-1}}{2q-1} \\
&\quad\times\sup_{s\in[0,t_1-\delta]}\big\|A^{1-\beta}K_q(t_2-s)-A^{1-\beta}K_q(t_1-s)\big\|^2 \\
&\quad+12d_1 M_\beta^2 t_1\big(1+\|x\|_{\mathcal{C}[-\tau,b]}^2\big)\frac{\delta^{2q\beta-1}}{2q\beta-1}.
\end{aligned}$$

Therefore, for t_2-t_1 sufficiently small, $J_1(t_1,t_2)$ and $J_3(t_1,t_2)$ tend to zero by Lemma 3.3. For $J_2(t_1,t_2)$, notice that

$$\begin{aligned}
&\int_0^{t_1}\big((t_2-s)^{2(q-1)}-(t_1-s)^{2(q-1)}\big)(t_2-s)^{2q(\beta-1)}\,ds \\
&\quad\le\int_0^{t_1}(t_2-s)^{2(q\beta-1)}+(t_1-s)^{2(q-1)}(t_2-s)^{2q(\beta-1)}\,ds \\
&\quad\le\int_0^{t_1}(t_2-s)^{2(q\beta-1)}+(t_1-s)^{2(q\beta-1)}\,ds<\infty.
\end{aligned}$$

Using Lebesgue's dominated convergence theorem, one can deduce that $J_2(t_1,t_2)\to 0$ as $t_2-t_1\to 0$. Hence we obtain the result. □

4 Main results

In this section we study the topological properties of solution sets. For computational convenience, set

$$\begin{aligned}
\Lambda &= \frac{4M^2\operatorname{Tr}(Q)}{\Gamma^2(q)}\left(\frac{1-q_1}{q-q_1}\right)^{2-2q_1}, \\
\tilde{d} &= 4d_1\left(\big\|A^{-\beta}\big\|^2+\frac{M_\beta^2 b^{2q\beta}}{2q\beta-1}\right)+\Lambda b^{2(q-q_1)}\|\alpha\|_{\frac{1}{2q_1-1}}.
\end{aligned}$$

The following compactness characterization of the solution set to the problem (1.1) will be useful.

Lemma 4.1 *Suppose that $T(t)$ is compact for $t > 0$ and there exists $\gamma \in L^{\frac{1}{2q_1-1}}([0,b];\mathbb{R}^+)$ such that*

$$E\|\Sigma(t,c)\|^2_{L_2^0} \leq \gamma(t) \quad \text{for a.e. } t \in [0,b] \text{ and } c \in C([-\tau,0];H).$$

Then the multimap $\mathcal{F}_2$ is compact in $\mathcal{C}[-\tau,b]$.

Proof Let D be a bounded set of $\mathcal{C}[-\tau,b]$. For each $t \in [-\tau,b]$, it will be shown that $\Delta(t) = \{\mathcal{F}_2(x)(t) : x \in D\}$ is relatively compact in $L^2(\Omega;H)$.

Obviously, for $t \in [-\tau,0]$, $\Delta(t) = \{\phi(t)\}$ is relatively compact in $L^2(\Omega;H)$. Let $t \in [0,b]$ be fixed, for $x \in D$ and $y \in \Delta(t)$, there exists $\sigma \in \mathrm{Sel}_\Sigma(x)$ such that

$$y(t) = S_q(t)\big[\phi(0) - h(0,\phi)\big] + \int_0^t (t-s)^{q-1}K_q(t-s)\sigma(s)\,dW(s).$$

Let $t \in [0,b]$ be arbitrary and $\varepsilon > 0$ be small enough. Define the operator $\Psi_\varepsilon : \Delta(t) \to L^2(\Omega;H)$ by

$$\begin{aligned}\Psi_\varepsilon y(t) &= S_q(t)\big[\phi(0) - h(0,\phi)\big] \\ &\quad + T(\varepsilon^q\delta)\int_0^{t-\varepsilon}\int_\delta^\infty q\theta(t-s)^{q-1}\xi_q(\theta)T\big((t-s)^q\theta - \varepsilon^q\delta\big)\sigma(s)\,d\theta\,dW(s).\end{aligned}$$

Using the compactness of $T(t)$ for $t > 0$, we deduce that the set $\{\Psi_\varepsilon y(t) : y \in \Delta(t)\}$ is relatively compact in $L^2(\Omega;H)$ for every ε, $0 < \varepsilon < t$. Moreover, we have

$$\begin{aligned}E\big|\Psi_\varepsilon y(t) - y(t)\big|^2 &\leq 2E\left\|\int_0^t\int_0^\delta q\theta(t-s)^{q-1}\xi_q(\theta)T\big((t-s)^q\theta\big)\sigma(s)\,d\theta\,dW(s)\right\|^2_{L_2^0} \\ &\quad + 2E\left\|\int_{t-\varepsilon}^t\int_\delta^\infty q\theta(t-s)^{q-1}\xi_q(\theta)T\big((t-s)^q\theta\big)\sigma(s)\,d\theta\,dW(s)\right\|^2_{L_2^0} \\ &\leq 2\,\mathrm{Tr}(Q)\left(M\int_0^\delta q\theta\xi_q(\theta)\,d\theta\right)^2\int_0^t (t-s)^{2(q-1)}E\|\sigma(s)\|^2_{L_2^0}\,ds \\ &\quad + 2\,\mathrm{Tr}(Q)\left(M\int_\delta^\infty q\theta\xi_q(\theta)\,d\theta\right)^2\int_{t-\varepsilon}^t (t-s)^{2(q-1)}E\|\sigma(s)\|^2_{L_2^0}\,ds \\ &\leq 2\,\mathrm{Tr}(Q)\left(M\int_0^\delta q\theta\xi_q(\theta)\,d\theta\right)^2\int_0^t (t-s)^{2(q-1)}\gamma(s)\,ds \\ &\quad + \frac{2M^2\,\mathrm{Tr}(Q)}{\Gamma^2(q)}\int_{t-\varepsilon}^t (t-s)^{2(q-1)}\gamma(s)\,ds \\ &\leq 2\,\mathrm{Tr}(Q)\left(M\int_0^\delta q\theta\xi_q(\theta)\,d\theta\right)^2 b^{2(q-q_1)}\left(\frac{1-q_1}{q-q_1}\right)^{2-2q_1}\|\gamma\|_{\frac{1}{2q_1-1}} \\ &\quad + \frac{2M^2\,\mathrm{Tr}(Q)}{\Gamma^2(q)}\left(\frac{1-q_1}{q-q_1}\right)^{2-2q_1}\varepsilon^{2(q-q_1)}\left(\int_{t-\varepsilon}^t \gamma^{\frac{1}{2q_1-1}}(s)\,ds\right)^{2q_1-1} \\ &\to 0, \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Thus $\|\Psi_\varepsilon y(t) - y(t)\|_{\mathcal{C}} \to 0$, which shows that there is a relatively compact set arbitrarily close to the set $\Delta(t)$. Thus the set $\Delta(t)$ is also relatively compact in $L^2(\Omega; H)$ for each $t \in [0, b]$. Hence $\Delta(t) = \{\Gamma_2(x)(t) : x \in D\}$ is relatively compact in $L^2(\Omega; H)$ for each $t \in [-\tau, b]$.

Next we verify that the set $\{\mathcal{F}_2(x)(t) : x \in D\}$ is equicontinuous on $(0, b]$. For each $y \in \mathcal{F}_2(x)$ and $0 < t_1 < t_2 \leq b$, we obtain

$$\begin{aligned} E\bigl|y(t_2) - y(t_1)\bigr|^2 \leq{}& 4E\bigl|\bigl(S_q(t_2) - S_q(t_1)\bigr)\bigl[\phi(0) - h(0,\phi)\bigr]\bigr|^2 \\ &+ 4E\biggl\|\int_{t_1}^{t_2} (t_2 - s)^{q-1} K_q(t_2 - s)\sigma(s)\,dW(s)\biggr\|_{L_2^0}^2 \\ &+ 4E\biggl\|\int_0^{t_1} \bigl((t_2 - s)^{q-1} - (t_1 - s)^{q-1}\bigr) K_q(t_2 - s)\sigma(s)\,dW(s)\biggr\|_{L_2^0}^2 \\ &+ 4E\biggl\|\int_0^{t_1} (t_1 - s)^{q-1}\bigl[K_q(t_2 - s) - K_q(t_1 - s)\bigr]\sigma(s)\,dW(s)\biggr\|_{L_2^0}^2 \\ =:{}& I_1(t_1,t_2) + I_2(t_1,t_2) + I_3(t_1,t_2) + I_4(t_1,t_2). \end{aligned}$$

As before, one can obtain the following estimates:

$$\begin{aligned} I_1(t_1,t_2) \leq{}& 4\bigl\|S_q(t_2) - S_q(t_1)\bigr\|^2 E|x_0|^2, \\ I_2(t_1,t_2) \leq{}& \frac{4M^2\operatorname{Tr}(Q)}{\Gamma^2(q)} \int_{t_1}^{t_2} (t_2 - s)^{2(q-1)}\gamma(s)\,ds \\ \leq{}& \Lambda\|\gamma\|_{\frac{1}{2q_1-1}} (t_2 - t_1)^{2(q-q_1)}, \\ I_3(t_1,t_2) \leq{}& \frac{4M^2\operatorname{Tr}(Q)}{\Gamma^2(q)} \int_0^{t_1} \bigl[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\bigr]^2\gamma(s)\,ds \\ \leq{}& \Lambda\|\gamma\|_{\frac{1}{2q_1-1}} \bigl(t_1^{\frac{q-q_1}{1-q_1}} - t_2^{\frac{q-q_1}{1-q_1}} + (t_2 - t_1)^{\frac{q-q_1}{1-q_1}}\bigr)^{2(1-q_1)}, \\ I_4(t_1,t_2) \leq{}& 8\operatorname{Tr}(Q) \sup_{s\in[0,t_1-\delta]} \bigl\|K_q(t_2 - s) - K_q(t_1 - s)\bigr\|^2 \int_0^{t_1-\delta} (t_1 - s)^{2(q-1)}\gamma(s)\,ds \\ &+ \frac{8M^2\operatorname{Tr}(Q)}{\Gamma^2(q)} \int_{t_1-\delta}^{t} (t_1 - s)^{2(q-1)}\gamma(s)\,ds \\ \leq{}& 8\operatorname{Tr}(Q) \sup_{s\in[0,t_1-\delta]} \bigl\|K_q(t_2 - s) - K_q(t_1 - s)\bigr\|^2 \biggl(\frac{1-q_1}{q-q_1}\biggr)^{2-2q_1} \|\gamma\|_{\frac{1}{2q_1-1}} t_1^{2(q-q_1)} \\ &+ 4\Lambda\delta^{2(q-q_1)}\|\gamma\|_{\frac{1}{2q_1-1}}. \end{aligned}$$

Therefore, for $t_2 - t_1$ sufficiently small, the right side of each inequality tends to zero by Lemma 3.3. The equicontinuity of the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ is obvious.

Thus, an application of Arzela-Ascoli theorem justifies that $\{\mathcal{F}_2(x) : x \in D\}$ is relatively compact in $\mathcal{C}[-\tau, b]$. Hence $\mathcal{F}_2$ is compact in $\mathcal{C}[-\tau, b]$. This completes the proof. □

Let $a \in [0,b)$. We consider the singular integral equation of the form for

$$x(t) = \begin{cases} \varphi(t) + h(t,x_t) + \int_a^t (t-s)^{q-1} A K_q(t-s) h(s,x_s)\,ds \\ \quad + \int_a^t (t-s)^{q-1} K_q(t-s)\tilde{\sigma}(s,x_s)\,dW(s), & t \in [a,b], \\ \tilde{\varphi}(t), & t \in [-\tau,a], \end{cases} \tag{4.1}$$

where $\varphi \in C([a,b];H)$ and $\tilde{\varphi} \in C([-\tau,a];H)$ are such that $\varphi(a) = \tilde{\varphi}(a) - h(a,\tilde{\varphi})$. Similar to the proof of [30], Lemma 3.2, we can get the following lemma.

Lemma 4.2 *Let $q_1 \in [\frac{1}{2},q)$, $\tilde{\sigma}(\cdot,c)$ be $L^{\frac{1}{2q_1-1}}$-integrable for every $c \in C([a-\tau,a],H)$. Assume that $\{T(t): t > 0\}$ is compact. In addition, suppose that*

(i) *for any compact subset $K \subset H$, there exist $\delta > 0$ and $L_K \in L^1([a,b];\mathbb{R}^+)$ such that*

$$E\left\|\tilde{\sigma}(t,c_1) - \tilde{\sigma}(t,c_2)\right\|_{L_2^0}^2 \le L_K(t)\|c_1 - c_2\|_*^2,$$

for a.e. $t \in [a,b]$ and each $c_1, c_2 \in B_\delta(K)$;

(ii) *there exists $\gamma_1(t) \in L^{\frac{1}{2q_1-1}}([a,b];\mathbb{R}^+)$ such that $E\|\tilde{\sigma}(t,c)\|_{L_2^0}^2 \le \gamma_1(t)(c' + \|c\|_*^2)$ for a.e. $t \in [a,b]$ and every $c \in C([a-\tau,a],H)$, where c' is arbitrary, but fixed.*

If $4d_1\|A^{-\beta}\|^2 < 1$, then the integral equation (4.1) *admits a unique solution for every $\tilde{\varphi} \in C([-t,a];H)$. Moreover, the solution of* (4.1) *depends continuously on $\tilde{\varphi}$ and φ.*

Proof *Step 1. A priori* estimate. Assume that x is a solution of (4.1). Then

$$\begin{aligned} E\left|x(t)\right|^2 &\le 4E\left|A^{-\beta}A^{\beta}h(t,x_t)\right|^2 + 4(t-a)\int_a^t (t-s)^{2(q-1)} E\left|A^{1-\beta}K_q(t-s)A^{\beta}h(s,x_s)\right|^2 ds \\ &\quad + 4E\left|\varphi(t)\right|^2 + 4\operatorname{Tr}(Q)\int_a^t (t-s)^{2(q-1)} E\left\|K_q(t-s)\tilde{\sigma}(s,x_s)\right\|_{L_2^0}^2 ds \\ &\le 4d_1\left\|A^{-\beta}\right\|^2\left(1 + \|x_t\|_*^2\right) + 4d_1M_\beta^2(t-a)\int_a^t (t-s)^{2(q\beta-1)}\left(1 + \|x_s\|_*^2\right) ds \\ &\quad + 4\max_{[a,b]}\left|\varphi(t)\right| + \frac{4M^2\operatorname{Tr}(Q)}{\Gamma^2(q)}\int_a^t (t-s)^{2(q-1)}\gamma_1(s)\left(c' + \|x_s\|_*^2\right) ds \\ &\le 4d_1\left\|A^{-\beta}\right\|^2\left(1 + \|x\|_{C[a-\tau,t]}^2\right) \\ &\quad + 4(b-a)d_1M_\beta^2\int_a^t (t-s)^{2(q\beta-1)}\left(1 + \|x\|_{C[a-\tau,s]}^2\right) ds \\ &\quad + 4\max_{[a,b]} E\left|\varphi(t)\right|^2 + \frac{4M^2\operatorname{Tr}(Q)}{\Gamma^2(q)}\int_a^t (t-s)^{2(q-1)}\gamma_1(s)\left(c' + \|x\|_{C[a-\tau,s]}^2\right) ds \end{aligned}$$

for $t \in [a,b]$, and notice that $|x(t)| = |\tilde{\varphi}(t)|$ for $t \in [-\tau,a]$. Let $t^* \in [a-\tau,t]$ be such that $\|x(t^*)\| = \|x\|_{C[a-\tau,t]}$. Furthermore, we have

$$\begin{aligned} &\|x\|_{C[a-\tau,t]}^2 = \|x\|_{C[a-\tau,t^*]}^2 \\ &\quad \le \frac{1}{1 - 4d_1\|A^{-\beta}\|^2}\left[4d_1\left\|A^{-\beta}\right\|^2 + \frac{4d_1M_\beta^2(b-a)^{2q\beta}}{2q\beta - 1} + \widetilde{\Lambda} + 4\max_{[a,b]} E\left|\varphi(t)\right|^2\right. \end{aligned}$$

$$+\int_a^{t^*}\left(4(b-a)d_1M_\beta^2\left(t^*-s\right)^{2(q\beta-1)}+\frac{4M^2\operatorname{Tr}(Q)}{\Gamma^2(q)}\left(t^*-s\right)^{2(q-1)}\gamma_1(s)\right)$$
$$\times\|x\|_{\mathcal{C}[a-\tau,s]}\,ds\Bigg],$$

with $\widetilde{\Lambda}=c'\Lambda(b-a)^{2(q-q_1)}\|\gamma_1\|_{\frac{1}{2q_1-1}}$. By Gronwall's inequality, there exists $\bar{M}>0$ such that $\|x\|_{[-\tau,b]}\le\bar{M}$.

Step 2. Local existence. Let $\varphi\in C([a,b];H)$ and $\tilde{\varphi}\in C([-\tau,a];H)$ be fixed. From $4d_1\|A^{-\beta}\|^2<1$, we can find one ξ arbitrarily close to a such that

$$4d_1\left(\|A^{-\beta}\|^2+\frac{M_\beta^2(\xi-a)^{2q\beta}}{2q\beta-1}\right)+\Lambda(\xi-a)^{2(q-q_1)}\|\gamma_1\|_{L^{\frac{1}{2q_1-1}}[a,\xi]}<1.$$

Then, for such an ξ, we can choose ρ satisfying

$$\rho\ge\frac{4d_1(\|A^{-\beta}\|^2+\frac{M_\beta^2(\xi-a)^{2q\beta}}{2q\beta-1})+4\max_{[a,\xi]}E|\varphi(t)|^2+c'\Lambda(\xi-a)^{2(q-q_1)}\|\gamma_1\|_{L^{\frac{1}{2q_1-1}}[a,\xi]}}{1-4d_1(\|A^{-\beta}\|^2+\frac{M_\beta^2(\xi-a)^{2q\beta}}{2q\beta-1})-\Lambda(\xi-a)^{2(q-q_1)}\|\gamma_1\|_{L^{\frac{1}{2q_1-1}}[a,\xi]}},$$

that is,

$$4d_1(1+\rho)\left(\|A^{-\beta}\|^2+\frac{M_\beta^2(\xi-a)^{2q\beta}}{2q\beta-1}\right)+4\max_{[a,\xi]}E\left|\varphi(t)\right|^2$$
$$+\Lambda(\xi-a)^{2(q-q_1)}\left(c'+\rho\right)\|\gamma_1\|_{L^{\frac{1}{2q_1-1}}[a,\xi]}\le\rho.$$

Let us define

$$B_\rho(\xi)=\left\{x\in\mathcal{C}[-\tau,\xi]:\sup_{t\in[-\tau,\xi]}E\left|x(t)\right|^2\le\rho\text{ and }x(t)=\tilde{\varphi}(t)\text{ for }t\in[-\tau,a]\right\},$$

and introduce an operator Ξ on $B_\rho(\xi)$ as follows:

$$\Xi x(t)=\Xi_1x(t)+\Xi_2x(t),$$

where

$$\Xi_1x(t)=\begin{cases}h(t,x_t)+\int_a^t(t-s)^{q-1}AK_q(t-s)h(s,x_s)\,ds, & t\in[a,b],\\ 0, & t\in[-\tau,a],\end{cases}$$

and

$$\Xi_2x(t)=\begin{cases}\varphi(t)+\int_a^t(t-s)^{q-1}K_q(t-s)\tilde{\sigma}(s,x_s)\,dW(s), & t\in[a,b],\\ \tilde{\varphi}(t), & t\in[-\tau,a].\end{cases}$$

For $x \in B_\rho(\xi)$, we have

$$
\begin{aligned}
& E\left|\Xi_1 x(t) + \Xi_2 x(t)\right|^2 \\
& \quad \le 4\left\|A^{-\beta}\right\| d_1\left(1 + \|x\|^2_{\mathcal{C}[a-\tau,t]}\right) + 4(\xi - a) d_1 M_\beta^2 \int_a^t (t-s)^{2(q\beta-1)}\left(1 + \|x\|^2_{\mathcal{C}[a-\tau,s]}\right) ds \\
& \qquad + 4\max_{[a,\xi]} E\left|\varphi(t)\right|^2 + \frac{4M^2 \operatorname{Tr}(Q)}{\Gamma^2(q)} \int_a^t (t-s)^{2(q-1)} \gamma_1(s)\left(c' + \|x\|^2_{[a-\tau,s]}\right) ds \\
& \quad \le 4 d_1(1+\rho)\left(\left\|A^{-\beta}\right\|^2 + \frac{M_\beta^2(\xi - a)^{2q\beta}}{2q\beta - 1}\right) + 4\max_{[a,\xi]} E\left|\varphi(t)\right|^2 \\
& \qquad + \Lambda(\xi - a)^{2(q-q_1)}\left(c' + \rho\right)\|\gamma_1\|_{L^{\frac{1}{2q_1-1}}[a,\xi]} \\
& \quad \le \rho,
\end{aligned}
$$

for $t \in [a,\xi]$. Obviously, Ξ maps $B_\rho(\xi)$ into itself.

For any $x, y \in B_\rho(\xi)$ and $t \in [a,\xi]$, we have

$$
\begin{aligned}
& E\left|\Xi_1 x(t) - \Xi_1 y(t)\right|^2 \\
& \quad \le 2E\left|h(t,x_t) - h(t,y_t)\right|^2 + 2(t-a)\int_a^t (t-s)^{2(q-1)} E\left|AK_q(t-s)\left[h(s,x_s) - h(s,y_s)\right]\right|^2 ds \\
& \quad = 2E\left|A^{-\beta}A^\beta\left[h(t,x_t) - h(t,y_t)\right]\right|^2 \\
& \qquad + 2(t-a)M_\beta^2 \int_a^t (t-s)^{2(q\beta-1)} E\left|A^\beta\left[h(s,x_s) - h(s,y_s)\right]\right|^2 ds \\
& \quad \le 2d\left\|A^{-\beta}\right\|^2 \|x_t - y_t\|_*^2 + 2d(t-a)M_\beta^2 \int_a^t (t-s)^{2(q\beta-1)} \|x_s - y_s\|_*^2 ds \\
& \quad \le 2d\left\|A^{-\beta}\right\|^2 \|x - y\|^2_{\mathcal{C}[a-\tau,t]} + 2d(t-a)M_\beta^2 \int_a^t (t-s)^{2(q\beta-1)} \|x - y\|^2_{\mathcal{C}[a-\tau,s]} ds \\
& \quad \le 2d\left(\left\|A^{-\beta}\right\|^2 + \frac{M_\beta^2}{2q\beta - 1}(\xi - a)^{2q\beta}\right)\|x - y\|^2_{\mathcal{C}[a-\tau,\xi]}.
\end{aligned}
$$

Noting that $\Xi_1 x(t) = 0$ for $t \in [-\tau, a]$, we get

$$
\|\Xi_1 x - \Xi_1 y\|_{\mathcal{C}[a-\tau,\xi]} \le \sqrt{2d\left(\left\|A^{-\beta}\right\|^2 + \frac{M_\beta^2}{2q\beta - 1}\xi^{2q\beta}\right)}\|x - y\|_{\mathcal{C}[a-\tau,\xi]},
$$

which shows that Ξ_1 is a contraction.

Next, we will prove that Ξ_2 is continuous on $B_\rho(\xi)$. Let x^n, $x \in B_\rho(\xi)$ with $x^n \to x$ on $B_\rho(\xi)$. Noting that

$$
\begin{aligned}
\int_a^t (t-s)^{2(q-1)} L_K(s)\left\|x_s^n - x_s\right\|_*^2 ds &\le 2\int_a^t (t-s)^{2(q-1)} L_K(s)\left(\left\|x_s^n\right\|_*^2 + \|x_s\|_*^2\right) ds \\
&\le 4\rho \int_a^t (t-s)^{2(q-1)} L_K(s)\, ds < \infty,
\end{aligned}
$$

by (i) and the fact that $x_t^n \to x_t$ for $t \in [a,\xi]$, it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned} E\left|\Xi_2 x^n(t) - \Xi_2 x(t)\right|^2 &\le \frac{4M^2 \operatorname{Tr}(Q)}{\Gamma^2(q)} \int_a^t (t-s)^{2(q-1)} E\left\|\tilde{\sigma}\left(s,x_s^n\right) - \tilde{\sigma}(s,x_s)\right\|_{L_2^0}^2 \, ds \\ &\le \frac{4M^2 \operatorname{Tr}(Q)}{\Gamma^2(q)} \int_a^t (t-s)^{2(q-1)} L_K(s) \left\|x_s^n - x_s\right\|_*^2 \, ds \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Moreover, from the pf of Lemma 4.1 we see that Ξ_2 is a compact operator. Thus, Ξ_2 is a completely continuous operator. Hence, Krasnoselskii's fixed point theorem shows that there is a fixed point of Ξ, denoted by x, which is a local solution to equation (4.1).

Step 3. Uniqueness. In fact, let y be another local solution of equation (4.1). According to condition (i), we obtain

$$\begin{aligned} E\left|x(t) - y(t)\right|^2 \le{}& 3d\left\|A^{-\beta}\right\|^2 \|x-y\|_{\mathcal{C}[a-\tau,t]}^2 \\ &+ 3d(t-a)M_\beta^2 \int_a^t (t-s)^{2(q\beta-1)} \|x-y\|_{\mathcal{C}[a-\tau,s]}^2 \, ds \\ &+ \frac{3M^2 \operatorname{Tr}(Q)}{\Gamma^2(q)} \int_a^t (t-s)^{2(q-1)} L_K(s) \|x-y\|_{\mathcal{C}[a-\tau,s]}^2 \, ds \end{aligned}$$

for $t \in [a,\xi]$, and $E|x(t)-y(t)|^2 = 0$ for $t \in [-\tau,a]$. Let $\tilde{t} \in [a-\tau,t]$ be such that $\|x(\tilde{t})-y(\tilde{t})\| = \|x-y\|_{\mathcal{C}[a-\tau,t]}$. Thus we obtain

$$\begin{aligned} \|x-y\|_{\mathcal{C}[a-\tau,t]}^2 &= \|x-y\|_{\mathcal{C}[a-\tau,\tilde{t}]}^2 \\ &\le \frac{1}{1-3d\|A^{-\beta}\|^2} \int_a^{\tilde{t}} \biggl[3d(\xi-a)M_\beta^2 (\tilde{t}-s)^{2(q\beta-1)} \\ &\quad + \frac{3M^2 \operatorname{Tr}(Q)}{\Gamma^2(q)} (\tilde{t}-s)^{2(q-1)} L_K(s)\biggr] \|x-y\|_{\mathcal{C}[a-\tau,s]}^2 \, ds. \end{aligned}$$

Applying Gronwall's inequality, we get $\|x-y\|_{\mathcal{C}[a-\tau,t]}^2 = 0$, which implies $x(t) = y(t)$ for $t \in [-\tau,\xi]$.

Step 4. Continuation. In the sequel, the operator Ξ is treated as a mapping from $\mathcal{C}[a,b]$ to $\mathcal{C}[a,b]$. Define an operator

$$\Psi : [a,b] \times \mathcal{C}[a,b] \to \mathcal{C}[a,b], \qquad \Psi(t,y)(s) = \begin{cases} y(s), & s \in [a,t], \\ y(t), & s \in [t,b]. \end{cases}$$

Let $J = \{t \in [a,b] : y^t \in \mathcal{C}[a,b], y^t = \Psi(t,\Xi(y^t))\}$. Then it follows from $x^\xi = \Psi(\xi,\Xi(x^\xi))$ and $x^\xi = \Psi(t,x)$ that $J \neq \emptyset$ and $[a,t] \subset J$ for all $t \in J$. Letting $t_0 = \sup J$, there exists a sequence $\{t_n\} \subset J$ such that

$$t_n \le t_{n+1} \quad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to\infty} t_n = t_0.$$

By the continuity of φ and the assumption (H_1), and following the argument employed in Lemmas 3.4 and 4.1, we conclude that $E|x^{t_m}(t_0) - x^{t_n}(t_0)|^2 = E|x^{t_m}(t_m) - x^{t_n}(t_n)|^2 \to 0$ as $n \to \infty$. Accordingly, $\lim_{n\to\infty} x^{t_n}(t_0)$ exists.

Consider the function

$$x^{t_0}(s) = \begin{cases} x^{t_n}(s), & s \in [a, t_n], \\ \lim_{n\to\infty} x^{t_n}(t_0), & s \in [t_0, b]. \end{cases}$$

Clearly equicontinuity of the family $\{x^{t_n}\}$ implies that x^{t_0} is continuous. Using Lebesgue's dominated convergence theorem again, we have

$$\begin{aligned} x^{t_0}(t_0) &= \lim_{n\to\infty} x^{t_n}(t_n) \\ &= \lim_{n\to\infty} \biggl[\varphi(t_n) + h\bigl(t_n, x^{t_n}_{t_n}\bigr) + \int_a^{t_n} (t_n - s)^{q-1} A K_q(t_n - s) h\bigl(s, x^{t_n}_s\bigr)\, ds \\ &\quad + \int_a^{t_n} (t_n - s)^{q-1} K_q(t_n - s) \tilde{\sigma}\bigl(s, x^{t_n}_s\bigr)\, dW(s) \biggr] \\ &= \varphi(t_0) + h\bigl(t_0, x^{t_0}_{t_0}\bigr) + \int_a^{t} (t - s)^{q-1} A K_q(t - s) h\bigl(s, x^{t_0}_s\bigr)\, ds \\ &\quad + \int_a^{t} (t - s)^{q-1} K_q(t - s) \tilde{\sigma}\bigl(s, x^{t_0}_s\bigr)\, dW(s). \end{aligned}$$

Thus, we find that $x^{t_0} = \Psi(t_0, \Xi(x^{t_0}))$, which yields $t_0 \in J$.

Next, we show that $t_0 = b$. If this is not true, then $t_0 < b$. Let us set

$$\hat{\varphi}(t) = \varphi(t) + \int_a^{t_0} (t - s)^{q-1} A K_q(t - s) h\bigl(s, x^{t_0}_s\bigr)\, ds + \int_a^{t_0} (t - s)^{q-1} K_q(t - s) \tilde{\sigma}(s, x_s)\, dW(s),$$

with $\hat{\varphi} \in \mathcal{C}[t_0, b]$ and consider the following integral equation:

$$\begin{aligned} x(t) &= \hat{\varphi}(t) + h(t, x_t) + \int_{t_0}^{t} (t - s)^{q-1} A K_q(t - s) h(s, x_s)\, ds \\ &\quad + \int_{t_0}^{t} (t - s)^{q-1} K_q(t - s) \tilde{\sigma}\bigl(s, x^{t_0}_s\bigr)\, dW(s). \end{aligned}$$

Applying the earlier arguments, one can obtain that $z \in \mathcal{C}[t_0, t_0 + \xi']$. Let $x^{t_0+\xi'}(s)$ be equal to $x^{t_0}(s)$ for $s \in [a, t_0]$, equal to $z(s)$ for $s \in [t_0, t_0 + \xi']$ and equal to $z(t_0 + \xi')$ for $s \in [t_0 + \xi', b]$. Then it is clear that $x^{t_0+\xi'}(s) \in \mathcal{C}[a, b]$. Moreover,

$$\begin{aligned} x^{t_0+\xi'}(t) &= \varphi(t) + h\bigl(t, x^{t_0+\xi'}_t\bigr) + \int_a^{t} (t - s)^{q-1} A K_q(t - s) h\bigl(s, x^{t_0+\xi'}_s\bigr)\, ds \\ &\quad + \int_a^{t} (t - s)^{q-1} K_q(t - s) \tilde{\sigma}\bigl(s, x^{t_0+\xi'}_s\bigr)\, dW(s) \quad \text{for } t \in \bigl[a, t_0 + \xi'\bigr]. \end{aligned}$$

This shows $t_0 + \xi' \in J$, which is a contradiction.

Finally, let $\varphi^n \to \varphi^0$ in $C([a,b];H)$ and $\tilde{\varphi}^n \to \tilde{\varphi}^0$ in $C([-\tau,a];H)$ as $n \to \infty$, and x^n be the solution of (4.1) with the perturbation φ^n, *i.e.*,

$$\begin{aligned} x^n(t) &= \varphi^n(t) + h(t,x_t^n) + \int_a^t (t-s)^{q-1} A K_q(t-s) h(s,x_s^n)\,ds \\ &\quad + \int_a^t (t-s)^{q-1} K_q(t-s) \tilde{\sigma}(s,x_s^n)\,dW(s) \end{aligned} \tag{4.2}$$

for $t \in [a,b]$ and $x^n(t) = \tilde{\varphi}^n(t)$ for $t \in [-\tau,a]$. It is clear that $\lim_{n\to\infty} x^n$ exists in $\mathcal{C}[-\tau,a]$. From Lemma 4.1, it follows that the set

$$\left\{ \int_a^t (t-s)^{q-1} K_q(t-s) \tilde{\sigma}(s,x_s^n)\,dW(s) : n \geq 1 \right\}$$

is relatively compact in $\mathcal{C}[a,b]$. This implies that the family

$$\left\{ x^n(t) - h(t,x_t^n) - \int_a^t (t-s)^{q-1} A K_q(t-s) h(s,x_s^n)\,ds : n \geq 1 \right\}$$

is relatively compact in $\mathcal{C}[a,b]$. Next we only need to prove that $\lim_{n\to\infty} x^n$ exists in $\mathcal{C}[a,b]$. On the contrary, if $\lim_{n\to\infty} x^n$ does not exist in $\mathcal{C}[a,b]$, then for any $n \in \mathbb{N}$, we have n_1, n_2 with $n_1, n_2 > n$ such that $\|x^{n_1} - x^{n_2}\|_{\mathcal{C}[a,b]} > \varepsilon_0$ ($\varepsilon_0 > 0$ is a constant), that is, there exists t^* such that

$$E\left|x^{n_1}(t^*) - x^{n_2}(t^*)\right|^2 = \left\|x^{n_1} - x^{n_2}\right\|_{\mathcal{C}[a,b]}^2 > \varepsilon_0^2.$$

Let $u^n(t) = x^n(t) - h(t,x_t^n) - \int_a^t (t-s)^{q-1} A K_q(t-s) h(s,x_s^n)\,ds$. Using ($H_2$), we have

$$\begin{aligned} &3E\left|u^{n_1}(t^*) - u^{n_2}(t^*)\right|^2 \\ &\quad \geq E\left|x^{n_1}(t^*) - x^{n_2}(t^*)\right|^2 - 3E\left|h(t^*,x_{t^*}^{n_1}) - h(t^*,x_{t^*}^{n_2})\right|^2 \\ &\qquad - 3E\left|\int_a^{t^*} (t^*-s)^{q-1} A K_q(t^*-s)\left[h(s,x_s^{n_1}) - h(s,x_s^{n_2})\right]ds\right|^2 \\ &\quad \geq E\left|x^{n_1}(t^*) - x^{n_2}(t^*)\right|^2 - 3d\left\|A^{-\beta}\right\|\left\|x_{t^*}^{n_1} - x_{t^*}^{n_2}\right\|_*^2 \\ &\qquad - 3dM_\beta^2(t^*-a)\int_a^{t^*} (t^*-s)^{\beta-1}\left\|x_s^{n_1} - x_s^{n_2}\right\|_*^2\,ds \\ &\quad \geq E\left|x^{n_1}(t^*) - x^{n_2}(t^*)\right|^2 - 3d\left(\left\|A^{-\beta}\right\| + \frac{M_\beta^2(t^*-a)^\beta}{\beta}\right)\left\|x^{n_1} - x^{n_2}\right\|_{\mathcal{C}[a,b]}^2 \\ &\quad \geq \left[1 - 3d\left(\left\|A^{-\beta}\right\| + \frac{M_\beta^2 b^\beta}{\beta}\right)\right]\varepsilon_0^2, \end{aligned}$$

which contradicts the compactness of u^n in $\mathcal{C}[a,b]$. Hence $\{x^n\}$ converges in $\mathcal{C}[-\tau,b]$, the limit is denoted by x. Therefore, taking the limit in (4.2) as $n \to \infty$, one finds again by (H_2) and Lebesgue's dominated convergence theorem that x is the solution of (4.1) with the perturbation φ^0. This completes the proof. □

Next we present an approximation result. We do not provide the proof as it is similar to that of [33], Lemma 2.4.

Lemma 4.3 *Let $\{T(t) : t > 0\}$ be compact and* (H_1) *holds. Suppose that there exist two sequences $\{\sigma_n\} \subset L^2([0,b];L_2^0)$ and $\{x^n\} \subset \mathcal{C}[-\tau,b]$ such that $\lim_{n\to\infty}\sigma_n = \sigma$ weakly in $L^2([0,b];L_2^0)$ and $\lim_{n\to\infty} x^n = x$ in $\mathcal{C}[-\tau,b]$, where x^n is a mild solution of the stochastic problem*

$$\begin{cases} {}^C D_{0+}^q[x^n(t) - h(t,x_t^n)] = Ax^n(t) + \sigma_n(t)\frac{dW(t)}{dt}, & t \in [0,b], \\ x^n(t) = \phi(t), & t \in [-\tau,0]. \end{cases}$$

Then x is a mild solution of the limit problem

$$\begin{cases} {}^C D_{0+}^q[x(t) - h(t,x_t)] = Ax(t) + \sigma(t)\frac{dW(t)}{dt}, & t \in [0,b], \\ x(t) = \phi(t), & t \in [-\tau,0]. \end{cases}$$

Theorem 4.1 *Let the conditions* (H_1) *and* (H_2) *be satisfied. In addition, assume that $T(t)$ is compact for $t > 0$. If $\tilde{d} < 1$, then the solution set of the inclusion problem* (1.1) *is a nonempty compact subset of $\mathcal{C}[-\tau,b]$ for each $\phi \in C([-\tau,0],H)$.*

Proof Let us fix

$$R > \frac{\|\phi\|_*^2 + 8M^2[d_1\|A^{-\beta}\|^2(1 + \|\phi\|_*^2) + E|\phi(0)|^2] + \tilde{d}}{1 - \tilde{d}}$$

and consider

$$B_R(b) = \Bigl\{x \in \mathcal{C}[-\tau,b] : \sup_{t\in[-\tau,b]} E\bigl|x(t)\bigr|^2 \le R\Bigr\}.$$

Clearly $B_R(b)$ is a closed and convex subset of $\mathcal{C}[-\tau,b]$. We first show that $\mathcal{F}(B_R(b)) \subset B_R(b)$. Indeed, taking $x \in B_R(b)$ and $y \in \mathcal{F}(x)$, there exists $\sigma \in \mathrm{Sel}_\Sigma(x)$ such that

$$\begin{aligned} E\bigl|y(t)\bigr|^2 &\le 4E\bigl|A^{-\beta}A^\beta h(t,x_t)\bigr|^2 + 4t\int_0^t (t-s)^{2(q-1)}E\bigl|A^{1-\beta}K_q(t-s)A^\beta h(s,x_s)\bigr|^2\,ds \\ &\quad + 4E\bigl|S_q(t)\bigl[\phi(0) - h(0,\phi)\bigr]\bigr|^2 + 4\,\mathrm{Tr}(Q)\int_0^t (t-s)^{2(q-1)}E\bigl\|K_q(t-s)\sigma(s)\bigr\|_{L_2^0}^2\,ds \\ &\le 4d_1\bigl\|A^{-\beta}\bigr\|^2\bigl(1 + \|x\|_{\mathcal{C}[-\tau,t]}^2\bigr) + 4d_1M_\beta^2 t\int_0^t (t-s)^{2(q\beta-1)}\bigl(1 + \|x\|_{\mathcal{C}[-\tau,s]}^2\bigr)\,ds \\ &\quad + 8M^2\bigl[d_1\bigl\|A^{-\beta}\bigr\|^2\bigl(1 + \|\phi\|_*^2\bigr) + E\bigl|\phi(0)\bigr|^2\bigr] \\ &\quad + \frac{4M^2\,\mathrm{Tr}(Q)}{\Gamma^2(q)}\int_0^t (t-s)^{2(q-1)}\gamma_1(s)\bigl(1 + \|x\|_{\mathcal{C}[-\tau,s]}^2\bigr)\,ds \\ &\le 4d_1(1+R)\biggl(\bigl\|A^{-\beta}\bigr\|^2 + \frac{M_\beta^2 b^{2q\beta}}{2q\beta - 1}\biggr) + 8M^2\bigl[d_1\bigl\|A^{-\beta}\bigr\|^2\bigl(1 + \|\phi\|_*^2\bigr) + E\bigl|\phi(0)\bigr|^2\bigr] \\ &\quad + \Lambda b^{2(q-q_1)}(1+R)\|\alpha\|_{\frac{1}{2q_1-1}} \end{aligned}$$

for $t \in [0,b]$. With $y(t) = \phi(t)$ for $t \in [-\tau,0]$, we have

$$E\bigl|y(t)\bigr|^2 \le \|\phi\|_*^2 + 8M^2\bigl[d_1\bigl\|A^{-\beta}\bigr\|^2\bigl(1 + \|\phi\|_*^2\bigr) + E\bigl|\phi(0)\bigr|^2\bigr] + \tilde{d}(1+R) \le R$$

for $t \in [-\tau, b]$.

Letting $x, y \in \mathcal{C}[-\tau, b]$ and applying the argument employed in Lemma 4.2, we obtain

$$
\begin{aligned}
E\left|\mathcal{F}_1 x(t) - \mathcal{F}_1 y(t)\right|^2 &\leq 2E\left|A^{-\beta}A^{\beta}\left[h(t,x_t) - h(t,y_t)\right]\right|^2 \\
&\quad + 2tM_\beta^2 \int_0^t (t-s)^{2(q\beta-1)} E\left|A^\beta\left[h(s,x_s) - h(s,y_s)\right]\right|^2 ds \\
&\leq 2d\left(\left\|A^{-\beta}\right\|^2 + \frac{M_\beta^2}{2q\beta - 1} b^{2q\beta}\right) \|x - y\|_{\mathcal{C}[-\tau,b]}^2.
\end{aligned}
$$

As $\mathcal{F}_1 x(t) = 0$ for $t \in [-\tau, 0]$, we have

$$
\|\mathcal{F}_1 x - \mathcal{F}_1 y\|_{\mathcal{C}[-\tau,b]}^2 \leq 2d\left(\left\|A^{-\beta}\right\|^2 + \frac{M_\beta^2}{2q\beta - 1} b^{2q\beta}\right) \|x - y\|_{\mathcal{C}[-\tau,b]}^2.
$$

This shows that $\mathcal{F}_1$ is a contraction, since $\sqrt{2d(\|A^{-\beta}\|^2 + \frac{M_\beta^2}{2q\beta-1} b^{2q\beta})} < \frac{1}{2}$.

It follows from Lemma 3.1 that Sel_Σ is weakly u.s.c. with convex and weakly compact values. Moreover, using Lemmas 4.1, 4.3 and an argument similar to the one described in Zhou [10], we find that $\mathcal{F}_2 : \mathcal{C}[0,b] \to P(\mathcal{C}[0,b])$ is quasi-compact, closed (and therefore has closed values). This implies that $\mathcal{F}_2$ is u.s.c. due to Lemma 2.1, and therefore has compact values. Hence the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy all conditions of Theorem 2.1, and consequently the fixed points set of the operator $\mathcal{F}_1 + \mathcal{F}_2$ is a nonempty compact subset of $\mathcal{C}[-\tau, b]$. □

In the next result, we denote by $\Theta(\phi)$ the set of all mild solutions to the inclusion problem (1.1).

Theorem 4.2 *Let the hypotheses of Theorem* 4.1 *be satisfied. Then the solution set of the inclusion problem* (1.1) *is an* R_δ*-set.*

Proof Consider the fractional stochastic evolution inclusion

$$
\begin{cases}
{}^C D_{0+}^q [x(t) - h(t,x_t)] \in Ax(t) + \Sigma_n(t)\frac{dW(t)}{dt}, & t \in [0,b], \\
x(t) = \phi(t), & t \in [-\tau, 0],
\end{cases}
\tag{4.3}
$$

where $\Sigma_n : [0,b] \times C([-\tau,0], H) \to P_{cl,cv}(L_2^0)$ are the multivalued functions already established in Lemma 3.2. Let $\Theta_n(\phi)$ be the set of all mild solutions to the inclusion problem (4.3).

From Lemma 3.2(ii) and (vi), it follows that $\{\Sigma_n\}$ verifies the conditions (H_2) for each $n \geq 1$. Then, from Lemma 3.1, we find that Sel_{Σ_n} is nonempty weakly u.s.c. with convex and weakly compact values. As shown earlier, the solution set of the inclusion problem (4.3) is nonempty and compact in $\mathcal{C}[-\tau, b]$ for each $n \geq 1$.

Now we show that $\Theta_n(\phi)$ is contraction for all $n \geq 1$. To do this, let $x \in \Theta_n(\phi)$ and $\tilde{\sigma}_n$ be the selection of $\Sigma_n, n \geq 1$. For any $\lambda \in [0,1)$, we are concerned with the existence and uniqueness of solutions to the integral equation

$$\begin{aligned} y(t) &= S_q(t)\big[\phi(0) - h(0,\phi)\big] + h(t,y_t) + \int_0^{\lambda b} (t-s)^{q-1} A K_q(t-s) h(s,x_s)\, ds \\ &\quad + \int_0^{\lambda b} (t-s)^{q-1} K_q(t-s)\sigma^x(s)\, dW(s) + \int_{\lambda b}^t (t-s)^{q-1} A K_q(t-s) h(s,y_s)\, ds \\ &\quad + \int_{\lambda b}^t (t-s)^{q-1} K_q(t-s)\tilde{\sigma}_n(s)\, dW(s), \quad \text{for } t \in [\lambda b, b], \end{aligned} \tag{4.4}$$

where $\sigma^x \in \mathrm{Sel}_\Sigma(x)$ and $y(t) = x(t)$ for $t \in [\lambda b - \tau, \lambda b]$. Since the functions $\tilde{\sigma}_n$ satisfy the conditions of Lemma 4.2 due to Lemma 3.2(ii) and (v), it follows by by Lemma 4.2 that the problem (4.4) has a unique solution on $[\lambda b, b]$, denoted by $y(t, \lambda b, x(\lambda b))$. Moreover, $y(t, \lambda b, x(\lambda b))$ depends continuously on (λ, x).

Next we define a function $h_1 : [0,1] \times \Theta_n(\phi) \to \Theta_n(\phi)$ as

$$h_1(\lambda, x) = \begin{cases} x(t), & t \in [-\tau, \lambda b], \\ y(t, \lambda b, x), & t \in (\lambda b, b], \end{cases}$$

and observe that h_1 is well defined and continuous. Also, it is clear that

$$h_1(0,x) = y(t,0,\phi), \quad \text{and} \quad h_1(1,x) = x.$$

Thus $\Theta_n(\phi)$ is contraction for each $n \geq 1$.

Finally, by applying the arguments used in Zhou [10], we find that $\Theta(\phi) = \bigcap_{n\geq 1} \Theta_n(\phi)$. In consequence, we conclude that $\Theta(\phi)$ is a compact R_δ-set. The proof is completed. □

5 An example

Setting $H = L^2([0,\pi]; \mathbb{R}^+)$ and $K = \mathbb{R}$, we consider the fractional partial differential inclusions of neutral type given by

$$\begin{cases} \partial_t^q \big(z(t,\xi) - \int_0^\pi U(\xi,y) z_t(\theta,y)\, dy\big) & \\ \quad \in \frac{\partial^2 z(t,\xi)}{\partial \xi^2} + G(t, z_t(\theta,\xi)) \frac{dW(t)}{dt}, & t \in [0,1], \xi \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0, & t \in [0,1], \\ z(\theta,\xi) = \phi(\theta)(\xi), & \theta \in [-\tau,0], \xi \in [0,\pi], \end{cases} \tag{5.1}$$

where ∂_t^q is the Caputo fractional partial derivative of order $q \in (\frac{1}{2}, 1]$, $W(t)$ is a standard one-dimensional Wiener process defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, $\phi(\theta) \in H$ and $z_t(\theta,\xi) = z(t+\theta,\xi)$, $t \in [0,1]$, $\theta \in [-\tau, 0]$.

Define an operator $A : D(A) \subset H \to H$ by

$$\begin{aligned} D(A) &= \left\{ z \in H : z, \frac{\partial z}{\partial \xi} \text{ are absolutely continuous}, \frac{\partial^2 z}{\partial \xi^2} \in H, z(0) = z(\pi) = 0 \right\}, \\ Az &= \frac{\partial^2 z}{\partial \xi^2}. \end{aligned}$$

Then

$$Az = \sum_{n=1}^{\infty} n^2 (z, z_n) z_n,$$

where $z_n(t) = \sqrt{\frac{2}{\pi}} \sin(nt)$, $n = 1, 2, \ldots$, constitute the orthogonal basis of eigenvectors of A. It is well known that A generates a compact, analytic semigroup $\{T(t) : t \geq 0\}$ in H (see Pazy [34]). Then the system (5.1) can be reformulated as

$$\begin{cases} {}^C D_{0+}^q [x(t) - h(t, x_t)] \in Ax(t) + \Sigma(t, x_t) \frac{dW(t)}{dt}, & t \in [0,1], \\ x(t) = \phi(t), & t \in [-\tau, 0], \end{cases}$$

where $x(t)(\xi) = z(t, \xi)$, $x_t(\theta, \xi) = z_t(\theta, \xi)$, $\Sigma(t, x_t)(\xi) = G(t, z_t(\theta, \xi))$. Let the function $h(t, x_t) : [0,1] \times C([-\tau, 0]; H) \to H$ be defined by

$$h(t, x_t) = \int_0^{\pi} U(\xi, y) z_t(\theta, y)\, dy.$$

Moreover, we assume that the following conditions hold:

(h_1) the function $U(\xi, y)$ is measurable and

$$\int_0^{\pi} \int_0^{\pi} U^2(\xi, y)\, dy\, d\xi < \infty;$$

(h_2) the function $\partial_\xi U(\xi, y)$ is measurable, $U(0, y) = U(\pi, y) = 0$, and let

$$\overline{H} = \left(\int_0^{\pi} \int_0^{\pi} \left(\partial_\xi U(\xi, y) \right)^2 dy\, d\xi \right)^{\frac{1}{2}} < \infty.$$

Clearly, (H_1) is satisfied.

Let $\Sigma(t, c) = [f_1(t, c), f_2(t, c)]$. Now, we assume that $f_i : [0,1] \times C([-\tau, 0]; H) \to \mathbb{R}$, $i = 1, 2$, satisfy

(F_1) f_1 is l.s.c. and g_2 is u.s.c.;
(F_2) $f_1(t, c) \leq f_2(t, c)$ for each $(t, c) \in [0,1] \times C([-\tau, 0]; H)$;
(F_3) there exist $\alpha_1, \alpha_2 \in L^\infty([0,1]; \mathbb{R}^+)$ such that

$$\|f_i(t, c)\|_{L_2^0}^2 \leq \alpha_1(t) \|c\|_*^2 + \alpha_2(t), \quad i = 1, 2,$$

for each $(t, c) \in [0,1] \times C([-\tau, 0]; H)$.

In view of assumptions (F_1)-(F_3), it readily follows that the multivalued function $\Sigma(\cdot, \cdot) : [0,1] \times C([-\tau, 0]; H) \to P(L_2^0)$ satisfies (H_2). Thus, all the assumptions of Theorems 4.1 and 4.2 are satisfied and the conclusion of our result applies to the inclusion problem (5.1).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YZ and LP contributed to Sections 3 and 4. BA and AA contributed to Sections 1, 2, and 5.

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