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Global dynamics of a delayed chemostat model with harvest by impulsive flocculant input

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Abstract

A mathematical model describing continuous microbial culture and harvest in a chemostat, incorporating a control strategy and defined by impulsive differential equations, is presented and investigated. Theoretical results indicate that the model has a microbe-extinction periodic solution, which is globally attractive if the threshold R_1 is less than unity, and the model is permanent if the threshold R_2 is greater than unity. Further, we consider the control strategy under time delay and periodical impulsive effect. Analysis shows that continuous microbial culture and harvest process can be implemented by adjusting time delay, impulsive period or input amount of flocculant. Finally, we give an example with numerical simulations to illustrate the control strategy.

MSC: 34A37; 34K45; 92B05

Keywords: chemostat model; microbial flocculation; time delay; impulsive effect; global attractivity; permanence; control strategy

1 Introduction and model formulation

A chemostat is a classical bioreactor for microbes culture and has been widely applied in the field of microbiology and bioengineering [1, 2]. The chemostat model has attracted the attention of many scholars since it was introduced by Monod in 1942 [3]. These models include mathematical models [4–16] and experimental models [17–19]. A simple chemostat can be designed by using a pump or an overflow system (see Figure 1), by which the volume of the chemostat can be controlled either [20].

It was found that some microbes can produce flocculation under the action of flocculating agents. This phenomenon makes it possible to harvest microbes by flocculating agents [21–32]. Recently, based on a classical simple chemostat model in which a microbial species consumes a single growth limiting substrate [3, 8], Tai *et al.* [33] have proposed a delayed differential equations (DDEs) model to describe the process of microbial continuous culture and harvest as follows:

$$\begin{cases} \frac{dS(t)}{dt} = D(S_0 - S(t)) - h_1 x(t)S(t), \\ \frac{dx(t)}{dt} = hx(t-\tau)S(t-\tau) - Dx(t) - h_2 x(t)F(t), \\ \frac{dF(t)}{dt} = D(F_0 - F(t)) - h_3 x(t)F(t), \end{cases} \qquad (1)$$

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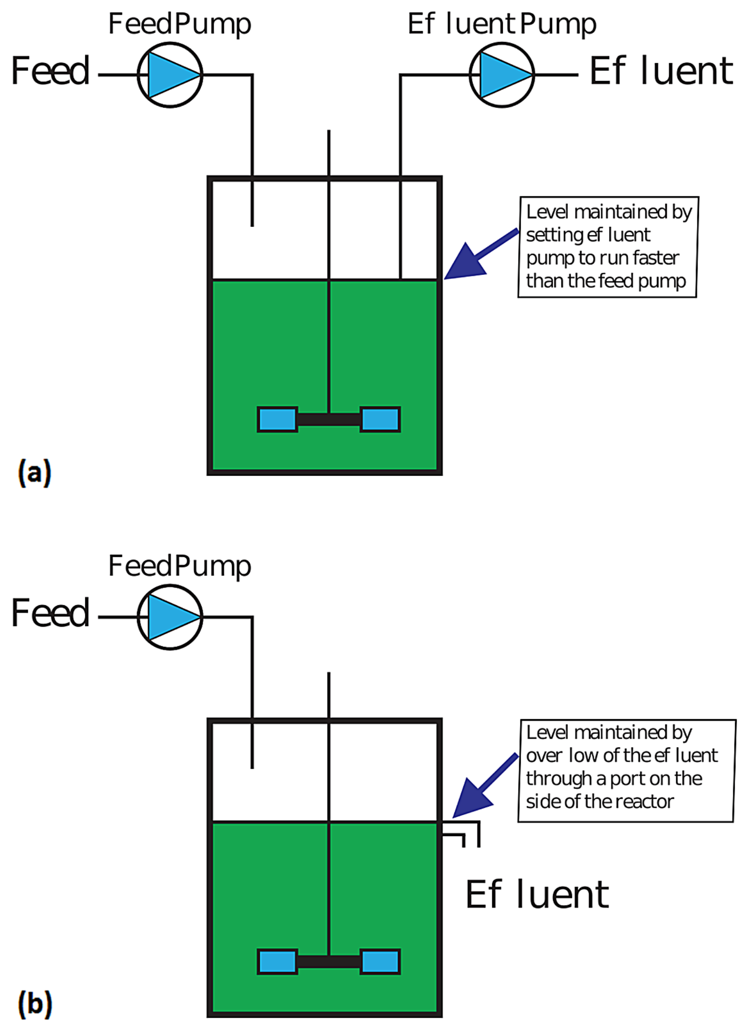

Figure 1 Two types of chemostat. (a) Type I chemostat. **(b)** Type II chemostat.

where $S(t)$, $x(t)$ and $F(t)$ represent the concentration of the substrate, microbia and flocculating agent at time t, respectively. D represents the velocity of medium, S_0 and F_0 represent the input concentration of substrate and flocculating agent, respectively. h_1 and h represent the consumption of medium and the yield of microbia, respectively. h_3 is the loss rate of flocculant. τ represents the time involved converting nutrient into the microbia [34–45]. The authors found that the model can produce backward bifurcation and complex dynamics. By establishing analytic thresholds for the existence of backward bifurcation, they analyzed the local stability of the equilibria.

In model (1), the flocculating agent is assumed to be added into the chemostat continuously. While in practice, by considering resource savings and the growth cycle of microorganisms, the flocculating agent can be periodically added into the chemostat at some fixed moment. Thus we perfect the chemostat system by adding input channel of flocculants and output channel of flocculation by two pumps, respectively (see Figure 2). It can be regulated through inputting flocculant to flocculate microbia according to the concentration of microbia. This process can be described by impulsive differential equations (IDEs). Impulsive differential equations, on the one hand, can fully reflect the actual control situ-

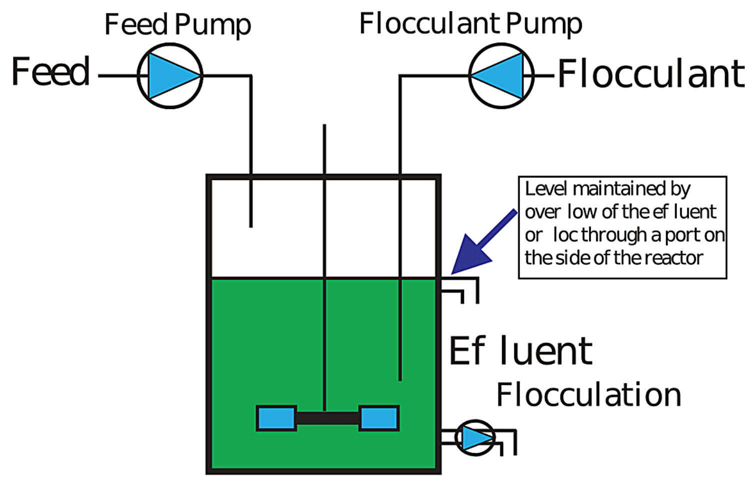

Figure 2 The modified chemostat.

ation; on the other hand, they can guide the operator to implement the impulsive control strategy conveniently and accurately [46–51]. Thus, we propose a new continuous culture chemostat model with time delay and impulsive harvest as follows:

$$\begin{cases} \left.\begin{aligned} \frac{dS(t)}{dt} &= D(S_0 - S(t)) - h_1 x(t)S(t), \\ \frac{dx(t)}{dt} &= e^{-D\tau} hx(t-\tau)S(t-\tau) - Dx(t) - h_2 x(t)F(t), \\ \frac{dF(t)}{dt} &= -DF(t) - h_3 x(t)F(t), \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} S(t^+) &= S(t), \\ x(t^+) &= x(t), \\ F(t^+) &= F(t) + \gamma F_0, \end{aligned}\right\} t = nT, \end{cases} \tag{2}$$

where T is the period of the impulsive effect, γF_0 is the input amount of flocculant at every impulsive period T. $F(nT^+) = \lim_{t\to nT^+} F(t)$, and $F(t)$ is left continuous at $t = nT$, *i.e.*, $F(nT) = \lim_{t\to nT^-} F(t)$, $S(t)$, $x(t)$ are continuous for all $t \geq 0$, the details can be seen in [52, 53].

Let $C_+ = C([-\tau, 0], R_+^3)$ be the Banach space, $\psi = (\psi_1(s), \psi_2(s), \psi_3(s))^T$, $\psi_i(\theta) \geq 0$ $(-\tau \leq \theta \leq 0, i = 1, 2, 3)$ the initial conditions are given as

$$\begin{aligned} &S(\theta) = \psi_1(\theta), \qquad x(\theta) = \psi_2(\theta), \qquad F(\theta) = \psi_3(\theta), \\ &\psi \in C_+, \qquad \psi_i(0) > 0 \quad (i = 1, 2, 3). \end{aligned} \tag{3}$$

The rest of the paper is organized as follows. In Section 2, we briefly introduce some concepts and fundamental results, which are necessary for future discussion. In Section 3, we focus our attention on the global property of system (2), including the existence, global attractivity of the microbe-extinction periodic solution and the permanence of system (2). In Section 4, we give the threshold of key parameters of system (2) and discuss the control strategy. We finally give a conclusion and numerical simulations in Section 5, from which it can be seen that all simulations agree with the theoretical results.

2 Preliminaries

In this section, we give some useful lemmas.

Let $f = (f_1, f_2, f_3)^T$ be the map defined by the right-hand side of the anterior three equations of system (2). Let $R_+ = [0,\infty)$, $R_+^3 = \{x \in R^3 : x \geq 0\}$, $\Omega = \operatorname{int} R_+^3$. Let $U : R_+ \times R_+^3 \to R_+$. If U satisfies the following conditions: (1) V is continuous in $((n-1)T, nT] \times R_+^3$, $n \in N$, and for each $x \in R_+^3$, $\lim_{(t,z)\to((n-1)T^+,x)} U(t,z) = U((n-1)T, x)$ and $\lim_{(t,z)\to(nT^+,x)} U(t,z) = U(nT^+, x)$ exists; (2) U is locally Lipschitzian in x. Then U is said to belong to class U_0.

Lemma 2.1 ([52, 53]) *Let $U : R_+ \times R_+^3 \to R_+$, $H : R_+ \times R_+ \to R$ and $U \in U_0$. Assume that*

$$\begin{cases} D^+ U(t, w(t)) \leq (\geq)\, H(t, U(t, w(t))), & t \neq n\omega, \\ U(t, w(t)^+) \leq (\geq)\, \Upsilon_n(U(t, w(t))), & t = n\omega, \end{cases} \tag{4}$$

here H is continuous in $(n\omega, (n+1)\omega] \times R_+$ and $\forall x \in R_+$, $n \in N$, $\lim_{(t,y)\to((n\omega)^+,x)} H(t,y) = H((n\omega)^+, x)$ exist; $\Upsilon_n : R_+ \to R_+$ is nondecreasing. Let $r(t) = r(t, 0, u_0)$ be the maximal (minimal) solution of the scalar impulsive differential equation

$$\begin{cases} u' = H(t, u), & t \neq n\omega, \\ u(t^+) = \Upsilon_n(u(t)), & t = n\omega, \end{cases} \tag{5}$$

existing on $[0,\infty)$. Then $U(0^+, w_0) \leq (\geq)\, u_0$ implies that $U(t, w(t)) \leq (\geq)\, r(t)$, $t \geq 0$, where $\omega(t) = \omega(t, 0, w_0)$ is any solution of (4) existing on $[0,\infty)$.

Lemma 2.2 ([54]) *Let q_1, q_2, τ be all positive constants and $z(t) > 0$ for $t \in [-\tau, 0]$. Consider the following delay differential equation:*

$$\frac{dz(t)}{dt} = q_1 z(t-\tau) - q_2 z(t),$$

then

(i) *if $q_1 < q_2$, then $\lim_{t\to\infty} z(t) = 0$;*
(ii) *if $q_1 > q_2$, then $\lim_{t\to\infty} z(t) = \infty$.*

Lemma 2.3 ([52, 53]) *Consider the following impulse differential inequalities:*

$$\begin{cases} u'(t) \leq (\geq)\, a(t)u(t) + c(t), & t \neq t_k, \\ u(t_k^+) \leq (\geq)\, b_k u(t_k) + d_k, & t = t_k, k \in N, \end{cases}$$

where $a(t), c(t) \in C(R_+, R)$, $b_k \geq 0$, and d_k are constants. Assume

(A_0) *the sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \cdots$, with $\lim_{t\to\infty} t_k = \infty$;*
(A_1) *$u \in PC'(R_+, R)$ and $u(t)$ is left-continuous at t_k, $k \in N$. Then*

$$\begin{aligned} u(t) \leq (\geq)\, u(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^t a(s)\,ds\right) + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^t a(s)\,ds\right)\right) d_k \\ + \int_{t_0}^t \prod_{s<t_k<t} b_k \exp\left(\int_s^t a(\theta)\,d\theta\right) c(s)\,ds, \quad t \geq t_0. \end{aligned}$$

Lemma 2.4 ([43]) *Consider the following impulsive differential system*:

$$\begin{cases} \frac{d\hbar(t)}{dt} = r_1 - r_2\hbar(t), & t \neq nT, \\ \hbar(t^+) = \hbar(t) + \mu, & t = nT, \end{cases} \tag{6}$$

for each solution $\hbar(t)$ *of* (6), $\hbar(t) \to \hbar^*(t)$ *as* $t \to \infty$, *where* $\hbar^*(t) = \frac{r_1}{r_2} + \frac{\mu e^{-r_2(t-nT)}}{1-e^{-r_2T}}$ *for* $t \in (nT, (n+1)T]$.

Lemma 2.5 *There exist constants* $M_1, M_2, M_3 > 0$ *such that* $S(t) \le M_1$, $x(t) \le M_2$, $F(t) \le M_3$ *for each solution of* (2) *with all t large enough*.

Proof Firstly, from the third and sixth equation of system (2), we have

$$\begin{cases} \frac{dF(t)}{dt} \le -FS(t), & t \neq nT, \\ F(t^+) = F(t) + \gamma F_0, & t = nT. \end{cases}$$

By Lemma 2.3, we have $F(t) \le \gamma F_0 \frac{e^{DT}}{e^{DT}-1} + \varepsilon_2$ for t large enough.

Let $V(t) = e^{-D\tau}\frac{h}{h_1}S(t-\tau) + x(t)$. It is clear that $V \in U_0$. Calculating the upper right derivative of $V(t)$ along a solution of system (2), one can get

$$\frac{dV(t)}{dt} \le e^{-D\tau}\frac{hDS_0}{h_1} - DV(t),$$

then by Lemma 2.3, we have $\limsup_{t\to\infty} V(t) \le e^{-D\tau}\frac{hS_0}{h_1}$, so $V(t)$ is ultimately bounded. Thus, $S(t)$ and $x(t)$ are ultimately bounded and $\limsup_{t\to\infty} S(t) \le S_0$, $\limsup_{t\to\infty} x(t) \le e^{-D\tau}\frac{hS_0}{h_1}$. Let $M_1 = S_0 + \varepsilon$, $M_2 = e^{-D\tau}\frac{hS_0}{h_1} + \varepsilon$, $M_3 = \gamma F_0\frac{e^{DT}}{e^{DT}-1} + \varepsilon$, we have $S(t) \le M_1$, $x(t) \le M_2$, $F(t) \le M_3$ for t large enough. The proof is completed. □

3 Global dynamical analysis for system (2)

In this section, we discuss the global dynamics of model (2), including the existence and global attractivity of the microbe-extinction periodic solution and the permanence.

3.1 Existence and global attractivity of the microbe-extinction periodic solution

Microbe-extinction solution describes that microbes are eventually absent from system (2), thus we let $x(t) = 0$ in system (2), then system (2) changes to the following system:

$$\begin{cases} \left.\begin{aligned} &\frac{dS(t)}{dt} = D(S_0 - S(t)), \\ &\frac{dF(t)}{dt} = -DF(t), \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} &S(t^+) = S(t), \\ &F(t^+) = F(t) + \gamma F_0, \end{aligned}\right\} t = nT. \end{cases} \tag{7}$$

Note that the variates $S(t)$ and $F(t)$ are independent of each other in system (7). Thus, by Lemma 2.4, we obtain that system (7) has a unique positive T-periodic solution

$(S^*(t), F^*(t))$ and for each solution $(S(t), F(t))$ of system (7), $S(t) \to S^*(t)$ and $F(t) \to F^*(t)$ as $t \to \infty$, where

$$\begin{cases} S^*(t) = S_0, \\ F^*(t) = \frac{\gamma F_0 e^{-D(t-nT)}}{1-e^{-DT}}. \end{cases} \tag{8}$$

Therefore, we have the existence theorem for system (2).

Theorem 3.1 *System* (2) *has a microbe-extinction periodic solution* $(S_0, 0, F^*(t))$.

Denote

$$R_1 = \frac{he^{-D\tau} S_0}{D + h_2 \frac{\gamma F_0 e^{-(D+h_3 M_2)T}}{1-e^{-(D+h_3 M_2)T}}}.$$

We have the following theorem about the attractivity of the microbe-extinction periodic solution of system (2).

Theorem 3.2 *If* $R_1 < 1$, *then the microbe-extinction periodic solution* $(S_0, 0, F^*(t))$ *of system* (2) *is globally attractive*.

Proof Let $(S(t), x(t), F(t))$ be any solution of system (2) satisfying initial condition (3). Since $R_1 < 1$, one can choose $\varepsilon_1, \varepsilon_2 > 0$ such that

$$he^{-D\tau}(S_0 + \varepsilon_1) < D + h_2 \left(\frac{\gamma F_0 e^{-(D+h_3 M_2)T}}{1 - e^{-(D+h_3 M_2)T}} - \varepsilon_2 \right). \tag{9}$$

By the first equation of system (2), we have

$$\frac{dS(t)}{dt} \le D\big(S_0 - S(t)\big).$$

According to Lemma 2.3, we have

$$\limsup_{t\to\infty} S(t) \le S_0.$$

Hence, there exists $n_1 \in N^+$ such that

$$S(t) \le S_0 + \varepsilon_1 \tag{10}$$

for all $t \ge n_1 T$, where ε_1 is an arbitrarily small positive constant.

By the third and sixth equations of system (2), we have

$$\begin{cases} \frac{dF(t)}{dt} \ge -(D + h_3 M_2) F(t), & t \ne nT, \\ F(t^+) = F(t) + \gamma F_0, & t = nT, \end{cases} \tag{11}$$

then consider the following impulsive differential system:

$$\begin{cases} \frac{dq(t)}{dt} = -(D + h_3 M_2)q(t), & t \neq nT, \\ q(t^+) = q(t) + \gamma F_0, & t = nT, \\ q(0^+) = F(0^+). \end{cases} \tag{12}$$

Then, by using Lemma 2.1, we have $F(t) \geq q(t)$ and $q(t) \to q^*(t)$ as $t \to \infty$, $q^*(t)$ is the periodic solution of (12), where $q^*(t) = \frac{\gamma F_0 e^{-(D+h_3M_2)(t-nT)}}{1-e^{-(D+h_3M_2)T}}$, $nT < t \leq (n+1)T$. By Lemma 2.4, we have that $q^*(t)$ is globally asymptotically stable. Hence there exists $n_2 \in N^+$ such that

$$F(t) \geq q(t) > q^*(t) - \varepsilon_2 > \frac{\gamma F_0 e^{-(D+h_3M_2)T}}{1 - e^{-(D+h_3M_2)T}} - \varepsilon_2 \tag{13}$$

for all $t \geq n_2 T$, where ε_2 is an arbitrarily small positive constant.

From the second equation, (10) and (13), there exists a positive integer $n_3 > \max\{n_1, n_2\}$, for $t > n_3 T + \tau$, we have

$$\frac{dx(t)}{dt} \leq he^{-D\tau}(S_0 + \varepsilon_1)x(t-\tau) - \left(D + h_2 \frac{\gamma F_0 e^{-(D+h_3M_2)T}}{1 - e^{-(D+h_3M_2)T}} - \varepsilon_2\right)x(t).$$

Consider the following delay differential equation:

$$\frac{dy(t)}{dt} = he^{-D\tau}(S_0 + \varepsilon_1)y(t-\tau) - \left(D + h_2\left(\frac{\gamma F_0 e^{-(D+h_3M_2)T}}{1 - e^{-(D+h_3M_2)T}} - \varepsilon_2\right)\right)y(t).$$

Since (9) holds, by Lemma 2.2, we get $\lim_{t\to\infty} y(t) = 0$. Notice that for all $\theta \in [-\tau, 0]$, $x(\theta) = y(\theta) = \psi_2(\theta) > 0$ holds. By the comparison theorem in differential equation and the positivity of solution (with $x(t) \geq 0$), we obtain $\lim_{t\to\infty} x(t) = 0$.

Next, we will prove $\lim_{t\to\infty} S(t) = S_0$ and $\lim_{t\to\infty} F(t) = F^*(t)$. We assume that $0 < x(t) < \varepsilon$ holds for all $t \geq 0$ in the following discussion without loss of generality. One the one hand, by the first equation of system (2), one gets

$$\frac{dS(t)}{dt} \geq DS_0 - (D + \varepsilon h_1)S(t).$$

Then, we have $\liminf_{t\to\infty} S(t) \geq S_0 \frac{D}{D+\varepsilon h_1}$. Thus, there exists $T > 0$ such that for any $\varepsilon_1 > 0$,

$$S(t) \geq S_0 \frac{D}{D + \varepsilon h_1} - \varepsilon_1 \tag{14}$$

for $t > T$. Let $\varepsilon \to 0$, from (10) and (14) we have

$$S_0 - \varepsilon_1 < S(t) < S_0 + \varepsilon_1$$

for t large enough, then we have $\lim_{t\to\infty} S(t) = S_0$.

On the other hand, from the third and sixth equations of system (2), we have

$$\begin{cases} \frac{dF(t)}{dt} \geq -(D + \varepsilon h_3)F(t), & t \neq nT, \\ F(t^+) = F(t) + \gamma F_0, & t = nT. \end{cases}$$

Then we get the following comparison system:

$$\begin{cases} \frac{dw_1(t)}{dt} = -(D+\varepsilon h_3)w_1(t), & t \neq nT, \\ w_1(t^+) = w_1(t) + \gamma F_0, & t = nT, \\ w_1(0^+) = F(0^+). \end{cases} \tag{15}$$

By Lemma 2.4, system (15) has a globally asymptotically stable positive periodic solution $w_1^*(t)$, where $w_1^*(t) = \frac{\gamma F_0 e^{-(D+\varepsilon h_3)(t-nT)}}{1-e^{-(D+\varepsilon h_3)T}}$. Thus, by Lemma 2.1, we have $F(t) \geq w_1(t)$ and $w_1(t) \to w_1^*(t)$ as $t \to \infty$. Therefore, there exists $T' > 0$ such that for any $\varepsilon_2 > 0$,

$$F(t) \geq w_1^*(t) - \varepsilon_2 \tag{16}$$

for $t > T'$. From the third and sixth equations of (2), one gets

$$\begin{cases} \frac{dF(t)}{dt} \leq -DF(t), & t \neq nT, \\ F(t^+) = F(t) + \gamma F_0, & t = nT. \end{cases}$$

Consider the following comparison system:

$$\begin{cases} \frac{dw_2(t)}{dt} = -Dw_2(t), & t \neq nT, \\ w_2(t^+) = w_2(t) + \gamma F_0, & t = nT, \\ w_2(0^+) = F(0^+), \end{cases} \tag{17}$$

then we have $F(t) \leq w_2(t)$ and $w_2(t) \to F^*(t)$. Then, for any $\varepsilon_2 > 0$, there exists $T'' > 0$ such that

$$F(t) \leq F^*(t) - \varepsilon_2 \tag{18}$$

for $t > T''$. Thus, by (16) and (18), for $t > \max\{T', T''\}$, we have

$$w_1^*(t) - \varepsilon_2 \leq F(t) \leq F^*(t) - \varepsilon_2.$$

Let $\varepsilon \to 0$, we have

$$F^*(t) - \varepsilon_2 < F(t) < F^*(t) + \varepsilon_2$$

for t large enough, thus we get $\lim_{t\to\infty} F(t) = F_0$. This completes the proof. □

3.2 Permanence

In this section, we prove that system (2) is persistent for $R_2 > 1$. Firstly, we give the following lemma supporting our main conclusion. Denote

$$R_2 = \frac{he^{-D\tau} S_0}{D + h_2 \frac{\gamma F_0}{1-e^{-DT}}}.$$

Lemma 3.1 *If $R_2 > 1$, then there exists a constant $m_4 > 0$ such that*

$$\liminf_{t\to\infty} x(t) \geq \min\left\{\frac{m_4}{2}, m_4 e^{-(D+h_2\varrho_2)\tau}\right\} = m_2.$$

Proof Let $X(t) = (S(t), x(t), F(t))$ be any positive solution of system (2) with initial condition (3). We rewrite the second equation of system (2) as follows:

$$\frac{dx(t)}{dt} = \bigl(e^{-D\tau} hS(t) - D - h_2F(t)\bigr)x(t) - e^{-D\tau} h \int_{t-\tau}^{t} x(\sigma)S(\sigma)\, d\sigma.$$

Define

$$G(t) = x(t) + e^{-D\tau} h \int_{t-\tau}^{t} x(\sigma)S(\sigma)\, d\sigma.$$

Calculating the derivative of $G(t)$ along the solution of (2) yields

$$\frac{dG(t)}{dt} = \bigl(e^{-D\tau} hS(t) - D - h_2F(t)\bigr)x(t). \tag{19}$$

Let $m_4 = \frac{(R_2-1)D}{(R_2+1)h_1}$. Since $R_2 > 1$, it is clear that $m_4 > 0$. For m_4, one can choose $\varepsilon_1, \varepsilon_2 > 0$ small enough such that

$$\frac{e^{-D\tau} h\varrho_1}{D + h_2\varrho_2} > 1, \tag{20}$$

where $\varrho_1 = \frac{DS_0}{D+h_1m_4} - \varepsilon_1$, $\varrho_2 = \frac{\gamma F_0}{1-e^{-DT}} + \varepsilon_2$. Then, for any positive constant t_0 and for all $t \geq t_0$, we claim that the inequality $x(t) < m_4$ cannot hold. Otherwise, there must exist a positive constant t_0 such that $x(t) < m_4$ for all $t \geq t_0$. The first equation of system (2) leads to

$$\frac{dS(t)}{dt} \geq DS_0 - (D + h_1m_4)S(t).$$

By Lemma 2.3, there exists such $T_1 > t_0 + \tau$ for $t > T_1$ that

$$S(t) > \frac{DS_0}{D + h_1m_4} - \varepsilon_1 = \varrho_1. \tag{21}$$

From (18), there exists such $T_2 > t_0 + \tau$ for $t \geq T_2$ that

$$F(t) < F^*(t) + \varepsilon_2 = \frac{\gamma F_0 e^{-D(t-nT)}}{1 - e^{-DT}} + \varepsilon_2 < \frac{\gamma F_0}{1 - e^{-DT}} + \varepsilon_2 = \varrho_2. \tag{22}$$

Thus by (21), (22) and (19), for $t > T_3 = \max\{T_1, T_2\}$, one gets

$$\frac{dG(t)}{dt} \geq (D + h_2\varrho)\left(\frac{e^{-D\tau} h\varrho_1}{D + h_2\varrho_2} - 1\right)x(t). \tag{23}$$

Let

$$x_l = \min_{t\in[T_1,T_1+\tau]} x(t).$$

We can prove that $x(t) \geq x_l$ for all $t \geq T_3$. Otherwise, there exists a constant $T_4 \geq 0$ such that $x(t) \geq x_l$ for $t \in [T_3, T_3 + \tau + T_4]$, $x(T_3 + \tau + T_4) = x_l$ and $\dot{x}(T_3 + \tau + T_4) \leq 0$. Then from the second equation of (2), (20) and (23), we have

$$\begin{aligned}\dot{x}(T_3 + \tau + T_4) &> \left(e^{-D\tau} h\varrho_1 - D - h_2\varrho_2\right)x_l \\ &= (D + h_2\varrho_2)\left(\frac{e^{-D\tau} h\varrho_1}{D + h_2\varrho_2} - 1\right)x_l \\ &> 0,\end{aligned}$$

which is a contradiction. Hence $x(t) \geq x_l > 0$ for all $t \geq T_3$. Inequality (23) implies

$$\frac{dG(t)}{dt} \geq (D + h_2\varrho)\left(\frac{e^{-D\tau} h\varrho_1}{D + h_2\varrho_2} - 1\right)x_l > 0, \tag{24}$$

which implies $G(t) \to +\infty$ as $t \to +\infty$. This is a contradiction to $G(t) \leq M(1 + M\tau e^{-DM})$ for t large enough. Therefore, for any positive constant t_0, the inequality $x(t) < m_4$ cannot hold for all $t \geq t_0$.

Step II: From Step I, we only need to consider:

(i) $x(t) > m_4$ for all t large enough;

(ii) $x(t)$ oscillates about m_4 for all large t.

However, case (i) is obviously the result of this theorem, so we only need to consider case (ii), in which we shall show that $x(t) > m_2$ for all large t, where

$$m_2 = \min\left\{\frac{m_4}{2}, m_4 e^{-(D+h_2\varrho_2)\tau}\right\}.$$

First, we notice that there would be two positive arbitrarily big constants $\bar{t}$, φ such that $x(t) < m_4$ for $\bar{t} < t < \bar{t} + \varphi$ and $x(\bar{t}) = x(\bar{t} + \varphi) = m_4$. Second, there exists a constant $0 < T_5 < \tau$ such that $x(t) > \frac{m_4}{2}$ for all $\bar{t} \leq t \leq \bar{t} + T_5$. Because $x(t)$ is not affected by impulses and, moreover, $x(t)$ is bound and continuous, then we conclude that T_5 is independent of the choice of $\bar{t}$. Next, according to the position of φ, T_5, τ, there will be three cases we should discuss.

Case ii(a): $\varphi \leq T_5$, obviously our aim is obtained.

Case ii(b): $T_5 < \varphi \leq \tau$. By (22), the second equation of (2) implies

$$\dot{x}(t) \geq -(D + h_2\varrho_2)x(t)$$

for $\bar{t} < t \leq \bar{t} + \varphi < \bar{t} + \tau$. Then we have

$$x(t) \geq x(\bar{t})e^{-(D+h_2\varrho_2)\tau}$$

for $\bar{t} < t \leq \bar{t} + \varphi \leq \bar{t} + \tau$, notice $x(\bar{t}) = m_4$, one can get

$$x(t) \geq m_4 e^{-(D+h_2\varrho_2)\tau}$$

for $\bar{t} < t \leq \bar{t} + \varphi \leq \bar{t} + \tau$. Thus we have $x(t) \geq m_2$ for $\bar{t} < t \leq \bar{t} + \varphi$.

Case ii(c): $\varphi \geq \tau$. We have proved $x(t) \geq m_2$ for $\bar{t} < t \leq \bar{t} + \tau$. For $\bar{t} + \tau \leq t \leq \bar{t} + \varphi$, we can analyze and prove $x(t) \geq m_2$ as the proof for the above claim. Because of the arbitrariness of interval $[\bar{t}, \bar{t} + \varphi]$ and because $\bar{t}$ is an arbitrarily big constant, we have that $x(t) \geq m_2$ holds for t large enough. Finally, notice that the choice of m_2 is independent of the positive solution of (2), which satisfies that $x(t) \geq m_2$ for t large enough. This completes the proof of Lemma 2.4. □

Theorem 3.3 *For* $R_2 > 1$, *then system* (2) *will be permanent.*

Proof Let $X(t) = (S(t), x(t), F(t))$ be any positive solution of system (2) with initial condition (3). From the first equation of system (2), we get

$$\frac{dS(t)}{dt} \geq DS_0 - (D + h_1 M)S(t),$$

then we have

$$\liminf_{t \to \infty} S(t) \geq \frac{DS_0}{D + h_1 M}.$$

Thus there exists a constant ε small enough such that

$$S(t) > \frac{DS_0}{D + h_1 M} - \varepsilon = m_1 > 0$$

for t large enough. And from (13) we have

$$F(t) \geq z(t) > z^*(t) - \varepsilon > \frac{\gamma F_0 e^{-(D+h_3 M)T}}{1 - e^{-(D+h_3 M)T}} = m_3 > 0 \tag{25}$$

for t large enough.

Set

$$D = \{(S, x, F) \in R_+^3 | m_1 \leq S \leq M_1, m_2 \leq x \leq M_2, m_3 \leq F \leq M_3\}.$$

Thus D is a bounded compact region and every solution of system (2) will eventually enter and remain in region D, then system (2) is permanent. The proof of Theorem 3.3 is completed. □

4 Control strategy of continuous microbial culture and harvest

In Section 3, we obtain the threshold values R_1 and R_2 associated with microbial extinction and existence. Next, we discuss the control strategy of continuous microbial culture and harvest by analyzing the key parameters of the threshold.

Denote

$$T^* = \frac{1}{D + h_3 M_2} \ln \frac{he^{-D\tau} S_0 + h_2 \gamma F_0 - D}{he^{-D\tau} S_0 - D},$$

$$F_0^* = \frac{he^{-D\tau} S_0 - D}{h_2 \frac{\gamma e^{-(D+h_3 M_2)T}}{1-e^{-(D+h_3 M_2)T}}},$$

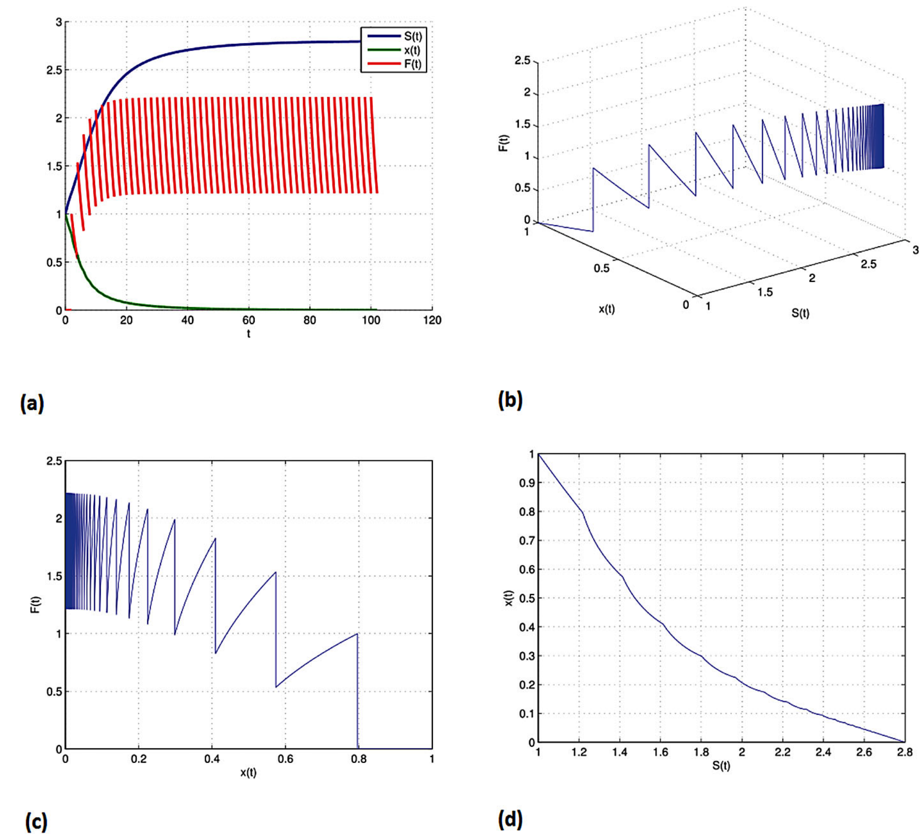

Figure 3 Dynamical behavior of solutions of system (26) with R_1 = 0.9910 < 1. (a) Time series of $S(t)$, $x(t)$, $F(t)$. **(b)** Three-dimensional phase diagrams of $S(t)$, $x(t)$, $F(t)$. **(c)** Phase diagrams of $x(t)$, $F(t)$. **(d)** Phase diagrams of $x(t)$, $S(t)$.

$$\tau^* = \frac{1}{D}\ln\frac{hS_0}{D + h_2\frac{\gamma F_0 e^{-(D+h_3M_2)T}}{1-e^{-(D+h_3M_2)T}}},$$

$$T_* = \frac{1}{D}\ln\frac{he^{-D\tau}S_0 + h_2\gamma F_0 - D}{he^{-D\tau}S_0 - D},$$

$$F_{0*} = \frac{(he^{-D\tau}S_0 - D)(1-e^{-DT})}{h_2\gamma},$$

$$\tau_* = \frac{1}{D}\ln\frac{hS_0}{D + h_2\frac{\gamma F_0}{1-e^{-DT}}}.$$

According to Theorem 3.2, we have that if $T < T^*$ or $F_0 > F_0^*$ or $\tau > \tau^*$, the microbe-extinction periodic solution $(S_0, 0, F^*(t))$ is globally attractive. That means the microbial continuous cultivation and harvest have failed. And from Theorem 3.3, we know that if $T > T_*$ or $F_0 < F_{0*}$ or $\tau < \tau_*$, then system (2) is permanent. That means we can achieve the process of microbial cultivation by increasing the time interval, or reducing the input amount of flocculant, or shortening the growth delay.

5 Discussion and numerical simulations

In this paper, to achieve the continuous microbial culture and harvest, we improve the classic chemostat model and propose a new chemostat model with time delay and peri-

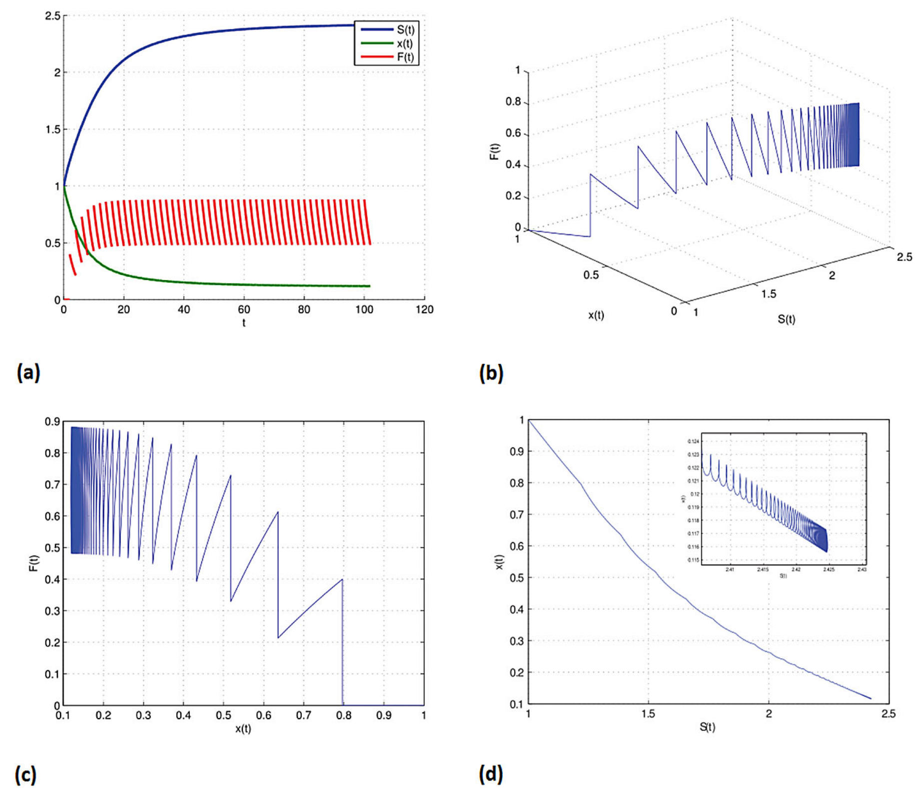

Figure 4 Dynamical behavior of solutions of system (26) with $R_2 = 1.0647 > 1$. (a) Time series of $S(t)$, $x(t)$, $F(t)$. **(b)** Three-dimensional phase diagrams of $S(t)$, $x(t)$, $F(t)$. **(c)** Phase diagrams of $x(t)$, $F(t)$. **(d)** Phase diagrams of $x(t)$, $S(t)$.

odical flocculant input. Our main aim is to investigate the control strategy of continuous microbial culture and harvest. By using the theory of impulsive delayed differential equations, global properties of the system are discussed. We prove that if $R_1 < 1$, then the microbe will be eventually extinct, and if $R_2 > 1$, the microbe species is permanent. Based on the threshold values associated with microbial extinction and existence, we consider the control strategy. Results show that we can culture microbia continuously and harvest microbia many times by adjusting the time interval (T) or the input amount of flocculant (γF_0), or the time delay (τ).

Next we will verify the effectiveness of control strategy by an example and some numerical simulations. Let $D = 0.3$, $S_0 = 2.8$, $h_1 = 0.4$, $h = 0.3$, $h_2 = 0.15$, $h_3 = 0.02$, $\gamma = 1$, and let the initial value be $(1, 1, 0)$. We get the following system:

$$\begin{cases} \left.\begin{aligned} \frac{dS(t)}{dt} &= 0.3(2.8 - S(t)) - 0.4x(t)S(t), \\ \frac{dx(t)}{dt} &= 0.3e^{-0.3\tau}x(t-\tau)S(t-\tau) - 0.3x(t) - 0.15x(t)F(t), \\ \frac{dF(t)}{dt} &= -0.3F(t) - 0.02x(t)F(t), \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} S(t^+) &= S(t), \\ x(t^+) &= x(t), \\ F(t^+) &= F(t) + \gamma F_0, \end{aligned}\right\} t = nT. \end{cases} \tag{26}$$

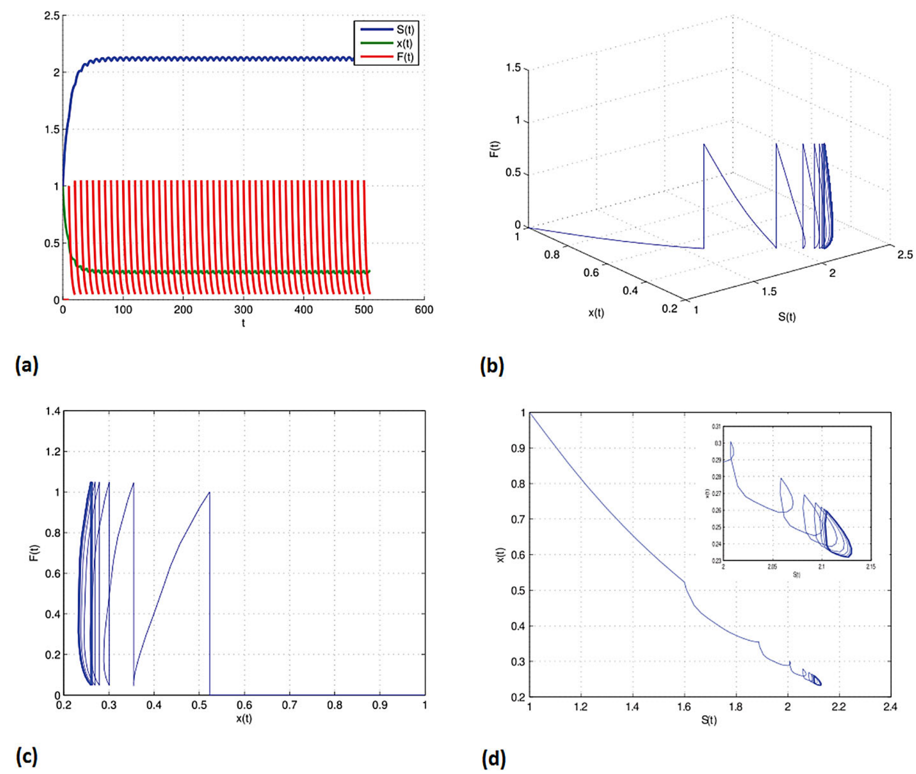

Figure 5 Dynamical behavior of solutions of system (26) with R_2 = 1.0069 > 1. (a) Time series of $S(t)$, $x(t)$, $F(t)$. **(b)** Three-dimensional phase diagrams of $S(t)$, $x(t)$, $F(t)$. **(c)** Phase diagrams of $x(t)$, $F(t)$. **(d)** Phase diagrams of $x(t)$, $S(t)$.

To investigate the effects of key parameters on the system, we assume $\tau = 2$, $T = 2$, $F_0 = 1$. By simple calculation, we have $R_1 = 0.9910 < 1$. Figure 3 shows that the microbe-eradication periodic solution $(S_0, 0, F^*(t))$ is globally attractive. That means microbial continuous cultivation and harvest have failed because the microbe will be eventually extinct. To achieve microbial continuous cultivation and harvest, we can take three kinds of control strategies.

(i) We can reduce the input of flocculant F_0 (from 1 to 0.4). By calculating, we have $R_2 = 1.0647 > 1$. According to Theorem 3.3, system (26) is permanent. A lower amount of flocculant can increase population microbia in the medium so that microbe can be cultured continuously in the chemostat system. Figure 4 shows that system (26) is permanent.

(ii) We can increase the time travail T (from 2 to 10). Longer time travail can decrease input mount of flocculant indirectly and increase population microbia in the medium, which makes microbia cultured and harvested continuously. By calculating, we have $R_2 = 1.0069 > 1$. According to Theorem 3.3, system (26) is permanent (see Figure 5). Figure 5 also shows that system (26) has an asymptotically stable periodic solution.

(iii) We can decrease time delay τ (from 2 to 0.5) by some biotechnology and biological engineering. Reduction of growth time delay makes the microbial growth cycle shorter, which is beneficial for microbial continuous cultivation. Figure 6 shows that system (26) is permanent. Moreover, system (26) has an asymptotically stable

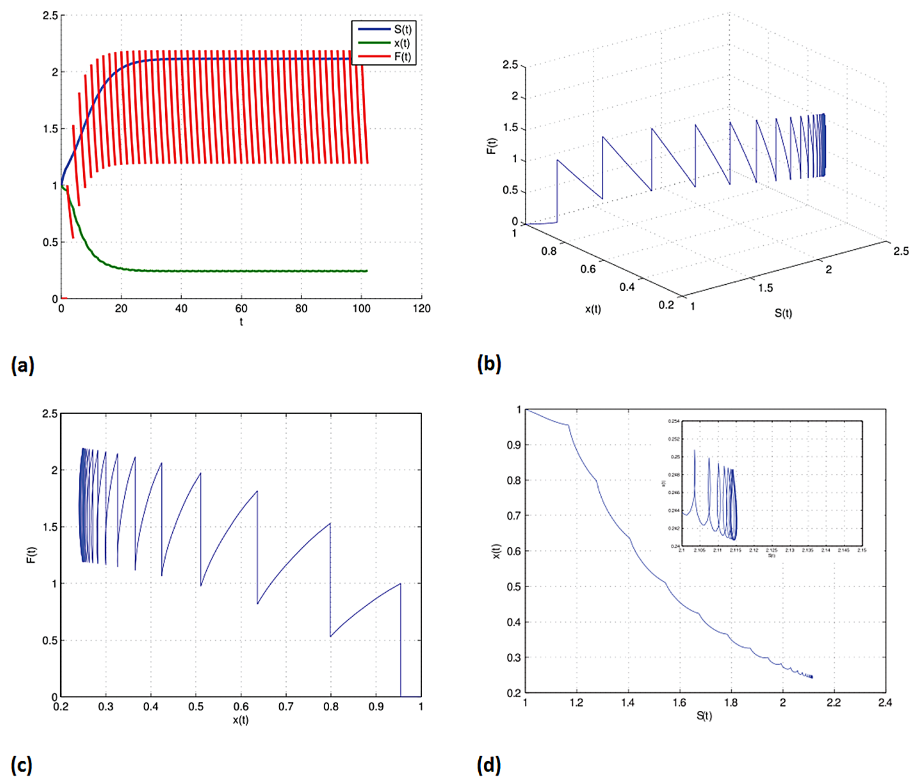

Figure 6 Dynamical behavior of solutions of system (26) with $R_2 = 1.1432 > 1$. (a) Time series of $S(t)$, $x(t)$, $F(t)$. **(b)** Three-dimensional phase diagrams of $S(t)$, $x(t)$, $F(t)$. **(c)** Phase diagrams of $x(t)$, $F(t)$. **(d)** Phase diagrams of $x(t)$, $S(t)$.

Table 1 Values of parameters, threshold and state of the system

τ	D	S_0	h_1	h	h_2	h_3	T	γ	F_0	R_i $(i = 1, 2)$	Microbe	Figure
2	0.3	2.8	0.4	0.3	0.15	0.02	2	1	1	$R_1 = 0.9910 < 1$	Eradication	Figure 3
2	0.3	2.8	0.4	0.3	0.15	0.02	2	1	0.4	$R_2 = 1.0647 > 1$	Permanence	Figure 4
2	0.3	2.8	0.4	0.3	0.15	0.02	10	1	1	$R_2 = 1.0069 > 1$	Permanence	Figure 5
0.5	0.3	2.8	0.4	0.3	0.15	0.02	2	1	1	$R_2 = 1.1432 > 1$	Permanence	Figure 6

periodic solution (where $R_2 = 1.1432 > 1$). Detailed parameter values, thresholds and states of system (26), please see Table 1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
TZ and WM designed the study and carried out the analysis. TZ, WM and XM contributed to writing the paper. TZ performed numerical simulations. All authors read and approved the final manuscript.

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