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# Reversed $S$-shaped connected component for a fourth-order boundary value problem

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**Abstract**

In this paper, we investigate the existence of a reversed $S$-shaped component in the positive solutions set of the fourth-order boundary value problem

$$\begin{cases} u''''(x) = \lambda h(x) f(u(x)), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $\lambda > 0$ is a parameter, $h \in C[0,1]$ and $f \in C[0,\infty)$, $f(0) = 0$, $f(s) > 0$ for all $s > 0$. By figuring the shape of unbounded continua of solutions, we show the existence and multiplicity of positive solutions with respect to parameter $\lambda$, and especially, we obtain the existence of three distinct positive solutions for $\lambda$ being in a certain interval.

**MSC:** 34B10; 34B18

**Keywords:** boundary value problem; positive solutions; principal eigenvalue; bifurcation

## 1 Introduction

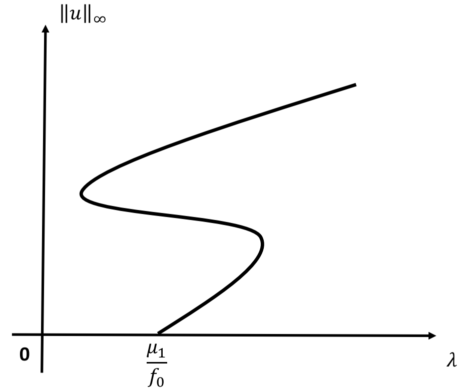
The fourth-order boundary value problem

$$\begin{cases} u''''(x) = \lambda f(x, u(x)), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{1.1}$$

describes the deformations of an elastic beam both of whose ends are simply supported at 0 and 1, see Gupta [1]. The existence and multiplicity of positive solutions for (1.1) with $\lambda \equiv 1$ have been extensively studied by many authors using topological degree theory, fixed point theorems, lower and upper solution methods, and critical point theory (see, for example, [2–17] and the references therein).

However, to the best of our knowledge, when parameter $\lambda$ varies in $\mathbb{R}^+$, there are few papers concerned with the global behavior of positive solutions of (1.1), see, for example, [18–20]. By using Rabinowitz's or Dancer's global bifurcation theorem, [18–20] investigated the global structure of the solutions set of (1.1), and accordingly, obtained the exis-

**Figure 1** *S*-shaped connected component of positive solutions set of (1.2).




tence and multiplicity of positive solutions and nodal solutions. Notice that these results give no information on direction turns of the connected component.

Very recently, Kim and Tanaka [21] considered the global structure of positive solutions set of the $p$-Laplacian problem

$$\begin{cases} (|y'|^{p-2}y')' = \lambda a(x)f(y), & x \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{1.2}$$

where the nonlinearity $f$ is asymptotic linear near 0, superlinear at some point, and sublinear near $\infty$. Based upon Rabinowitz's global bifurcation theorem, they proved that an unbounded subcontinuum of positive solutions of (1.2) bifurcates from the trivial solution, grows to the right from the initial point, to the left at some point, and to the right near $\lambda = \infty$. Roughly speaking, they concluded that there exists an $S$-shaped connected component in the positive solutions set of problem (1.2) (see Figure 1). Motivated by the above work, in a later paper [22], the present authors have established the existence result of an $S$-shaped connected component in the positive solutions set of the fourth-order boundary value problem

$$\begin{cases} u''''(x) = \lambda f(x, u(x), u''(x)), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

Now it is natural to ask whether we can get a reversed $S$-shaped connected component in the positive solutions set (see Figure 2) if the conditions on the nonlinearity are in contrast to these in [21]. In this paper, we will deal with this topic for problem (1.1) with $f(x,u) = h(x)f(u)$. More precisely, we will establish the existence result of a reversed $S$-shaped connected component in the positive solutions set of the problem

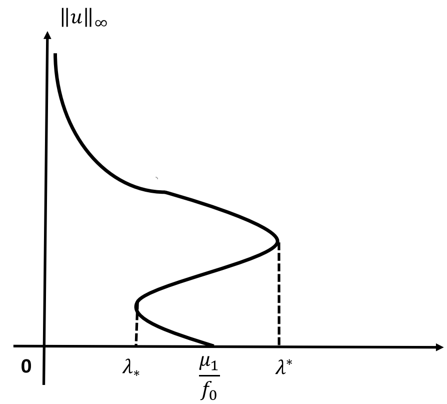
$$\begin{cases} u''''(x) = \lambda h(x)f(u(x)), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.3}$$

Throughout the paper, we assume that

(H1) $h(x) \geq 0$ in $[0,1]$ and $h \not\equiv 0$ in any subinterval of $[0,1]$;

(H2) there exist $\alpha > 0$, $f_0 > 0$, and $f_1 > 0$ such that $\lim_{s\to 0^+} \frac{f(s)-f_0 s}{s^{1+\alpha}} = f_1$;

**Figure 2 Reversed *S*-shaped connected of positive solutions set of (1.3).**




(H3) $f_\infty := \lim_{s\to\infty} \frac{f(s)}{s} = \infty$;

(H4) there exists $s_0 > 0$ such that $0 \le s \le s_0$ implies that

$$f(s) \le \frac{60f_0}{\mu_1 \widehat{h}} s_0,$$

where $\widehat{h} = \max_{x\in[0,1]} h(x)$ and $\mu_1 > 0$ is the first eigenvalue of the linear problem corresponding to (1.3) defined in Lemma 2.1.

It is easy to find that if (H2) holds, then

$$\lim_{s\to 0^+} \frac{f(s)}{s} = f_0, \tag{1.4}$$

that is, $f$ is asymptotic linear near 0. Contrary to the condition in [21], $f$ is superlinear near $\infty$ according to (H3), then we cannot find a constant $f^* > 0$ such that $f(s) \le f^* s$ for all $s \ge 0$. This will bring great difficulty to the study of the global structure of the positive solutions set of (1.3). On the other hand, the concavity and convexity of the solutions of second-order BVPs can be deduced directly from the nonlinearity in the equation, but for fourth-order BVPs, this becomes complicated, especially when the nonlinearity changes sign. So in this paper we only consider the case that $h \ge 0$ and $f(s) > 0$ for all $s > 0$.

Arguing the shape of a component in the positive solutions set of problem (1.3), we have the following result.

**Theorem 1.1** *Assume that* (H1), (H2), (H3), *and* (H4) *hold. Then there exist* $\lambda_* \in (0, \frac{\mu_1}{f_0})$ *and* $\lambda^* > \frac{\mu_1}{f_0}$ *such that*

(i) (1.3) *has at least one positive solution if* $0 < \lambda < \lambda_*$;

(ii) (1.3) *has at least two positive solutions if* $\lambda = \lambda_*$;

(iii) (1.3) *has at least three positive solutions if* $\lambda_* < \lambda < \frac{\mu_1}{f_0}$;

(iv) (1.3) *has at least two positive solutions if* $\frac{\mu_1}{f_0} \le \lambda < \lambda^*$;

(v) (1.3) *has at least one positive solution if* $\lambda = \lambda^*$;

(vi) (1.3) *has no positive solution if* $\lambda > \lambda^*$.

**Remark 1.1** Indeed, condition (H2) pushes the direction of bifurcation to the left near $u = 0$, while conditions (H4) and (H3) guarantee that the bifurcation curve grows to the right at some point and grows to the left near $\lambda = 0$, respectively.

**Remark 1.2** Let us consider the functions

$$h(x) \equiv 1, \quad x \in [0,1]$$

and

$$f(s) = s(s-1)^2 + \sqrt{s}\ln\left[1 + \left(\frac{e}{2} - 1\right)s\right], \quad s \in [0,\infty).$$

Obviously, $h$ satisfies (H1), $\widehat{h} = 1$ and the first eigenvalue of the linear problem corresponding to (1.3) is $\mu_1 = \pi^4$. It is easy to check that $f$ satisfies (H3) and (H2) with

$$\alpha = \frac{1}{2}, \qquad f_0 = 1, \qquad f_1 = \frac{e}{2} - 1.$$

Denote

$$w(s) = \frac{60}{\pi^4}s, \quad s \in [0,\infty).$$

By using *Mathematica 9.0*, we deduce that $f$ is increasing on $[0,\infty)$, and the equation

$$f(s) = w(s), \quad s \in [0,\infty)$$

has exactly three roots: $s_1 = 0$, $s_2 \doteq 0.356$, $s_3 \doteq 1.513$. Combining this with $f_0 = 1 > \frac{60}{\pi^4} = w'(0)$ and $f_\infty = \infty$, we conclude that for each fixed $s_0 \in [s_2, s_3]$,

$$f(s) \le \frac{60}{\pi^4}s_0, \quad \forall s \in [0, s_0],$$

that is, $f$ satisfies (H4).

Notice that the Conditions (H1)-(H4) are fulfilled, then Theorem 1.1 guarantees that there exist $\lambda_* \in (0,\pi^4)$ and $\lambda^* > \pi^4$ such that the conclusions (i)-(vi) are correct for problem

$$\begin{cases} u''''(x) = \lambda\{u(x)(u(x)-1)^2 + \sqrt{u(x)}\ln[1 + (\frac{e}{2} - 1)u(x)]\}, & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

The rest of this paper is arranged as follows. In Section 2, we show global bifurcation phenomena from the trivial branch with the leftward direction. Section 3 is devoted to showing that there are at least two direction turns of the component and to completing the proof of Theorem 1.1.

## 2 Leftward bifurcation

In this section, we state some preliminary results and show global bifurcation phenomena from the trivial branch with the leftward direction.

Let $g \in C[0,1]$, then the solution of the fourth-order linear boundary value problem

$$\begin{cases} v'''' = g(t), & t \in (0,1), \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases}$$

can be expressed by

$$v(t)=\int_0^1\int_0^1 G(t,s)G(s,\tau)g(\tau)\,d\tau\,ds,$$

where

$$G(t,s)=\begin{cases} s(1-t), & 0\le s\le t\le 1,\\ t(1-s), & 0\le t\le s\le 1.\end{cases}$$

Moreover, if $g(t)\ge 0$, $g\not\equiv 0$, then

$$v''(t)=-\int_0^1 G(t,s)g(s)\,ds\le 0,$$

and $v(t)\ge 0$ is concave.

Since the Green's function $G(t,s)$ has the properties:

(i) $0\le G(t,s)\le G(s,s)$, $\forall t,s\in[0,1]$;

(ii) $G(t,s)\ge \frac{1}{4}G(s,s)$, $\forall t\in[\frac{1}{4},\frac{3}{4}]$, $s\in[0,1]$,

then, for $t\in[\frac{1}{4},\frac{3}{4}]$, we have

$$\begin{aligned} v(t)&=\int_0^1 G(t,s)\left[\int_0^1 G(s,\tau)g(\tau)\,d\tau\right]ds\\ &\ge \frac{1}{4}\int_0^1 G(s,s)\left[\int_0^1 G(s,\tau)g(\tau)\,d\tau\right]ds\\ &\ge \frac{1}{4}\|v\|_\infty. \end{aligned} \tag{2.1}$$

Let us consider the linear eigenvalue problem

$$\begin{cases} u''''(x)=\lambda h(x)u(x), & x\in(0,1),\\ u(0)=u(1)=u''(0)=u''(1)=0. \end{cases} \tag{2.2}$$

**Lemma 2.1** (see [19], Theorem 2.1) *Assume that* (H1) *holds. Then the linear problem* (2.2) *has a positive simple principal eigenvalue*

$$\mu_1=\inf\left\{\frac{\int_0^1(u''(x))^2\,dx}{\int_0^1 h(x)u^2(x)\,dx}\;\middle|\; u\in C^4[0,1], u\not\equiv 0 \textit{ and } u(0)=u(1)=u''(0)=u''(1)=0\right\}.$$

*Moreover, the corresponding eigenfunction $\phi$ is positive in* $(0,1)$.

Extend $f$ to $\mathbb{R}$ with the oddity and rewrite (1.3) by

$$\begin{cases} u''''(x)=\lambda h(x)f_0u+\lambda h(x)(f(u)-f_0u), & x\in(0,1),\\ u(0)=u(1)=u''(0)=u''(1)=0. \end{cases} \tag{2.3}$$

Since condition (H2) implies (1.4), then following an argument similar to that in the proof of Theorem 1.1 in [19] or Theorem 2.2 in [20], we have the following.

**Lemma 2.2** *Assume that* (H1) *and* (H2) *hold, then from* $(\frac{\mu_1}{f_0}, 0)$ *there emanates an unbounded subcontinuum* $\mathcal{C}$ *of positive solutions of* (1.3) *in the set* $\mathbb{R} \times E$, *where* $E = \{u \in C^3[0,1] \mid u(0) = u(1) = u''(0) = u''(1) = 0\}$ *with the norm* $\|u\| = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty + \|u'''\|_\infty$.

**Lemma 2.3** *Assume that* (H1) *and* (H2) *hold. Let* $\{(\lambda_n, u_n)\}$ *be a sequence of positive solutions to* (1.3) *which satisfies* $\lambda_n \to \frac{\mu_1}{f_0}$ *and* $\|u_n\| \to 0$. *Let* $\phi$ *be the first eigenfunction of* (2.2) *which satisfies* $\|\phi\|_\infty = 1$. *Then there exists a subsequence of* $\{u_n\}$, *again denoted by* $\{u_n\}$, *such that* $\frac{u_n}{\|u_n\|_\infty}$ *converges uniformly to* $\phi$ *on* $[0,1]$.

*Proof* Set $v_n := \frac{u_n}{\|u_n\|_\infty}$. Then $\|v_n\|_\infty = 1$. For every $(\lambda_n, u_n)$, we have

$$u_n''(x) = -\lambda_n \int_0^1 G(x,s)h(s)f\bigl(u_n(s)\bigr)\,ds. \tag{2.4}$$

From the boundary condition $u_n(0) = u_n(1) = 0$, there exists $\widehat{x}_n \in (0,1)$ such that $u_n'(\widehat{x}_n) = 0$. Integrating (2.4) on $[x, \widehat{x}_n]$, we obtain

$$u_n'(x) = \lambda_n \int_x^{\widehat{x}_n} \int_0^1 G(t,s)h(s)f\bigl(u_n(s)\bigr)\,ds\,dt, \quad x \in [0,1]. \tag{2.5}$$

Dividing both sides of (2.5) by $\|u_n\|_\infty$, we get

$$v_n'(x) = \lambda_n \int_x^{\widehat{x}_n} \int_0^1 G(t,s)h(s)\frac{f(u_n(s))}{u_n(s)} v_n(s)\,ds\,dt, \quad x \in [0,1]. \tag{2.6}$$

Since $\|u_n\| \to 0$ implies $\|u_n\|_\infty \to 0$, then by (1.4) there exists a constant $m_1 > 0$ such that

$$\frac{f(u_n(s))}{u_n(s)} < m_1, \quad \forall n \in \mathbb{N}, s \in (0,1). \tag{2.7}$$

From $\lambda_n \to \frac{\mu_1}{f_0}$, it follows that there exists a constant $m_2 > 0$ such that

$$\lambda_n \le m_2, \quad \forall n \in \mathbb{N}. \tag{2.8}$$

Then, for $x \in [0,1]$, (2.6) implies that

$$\begin{aligned} v_n'(x) &= \lambda_n \int_x^{\widehat{x}_n} \int_0^1 G(t,s)h(s)\frac{f(u_n(s))}{u_n(s)} v_n(s)\,ds\,dt \\ &\le m_1 m_2 \|v_n\|_\infty \int_x^{\widehat{x}_n} \int_0^1 G(t,s)h(s)\,ds\,dt \\ &\le m_1 m_2 \|v_n\|_\infty \int_0^1 \int_0^1 G(t,s)h(s)\,ds\,dt \\ &= M\|v_n\|_\infty = M, \left(M = m_1 m_2 \int_0^1 \int_0^1 G(t,s)h(s)\,ds\,dt\right), \end{aligned} \tag{2.9}$$

that is,

$$\|v_n'\|_\infty \le M, \quad \forall n \in \mathbb{N}. \tag{2.10}$$

Since $\|v_n'\|_\infty$ is bounded, by the Ascoli-Arzela theorem, a subsequence of $\{v_n\}$ uniformly converges to a limit $v \in C[0,1]$ with $\|v\|_\infty = 1$, and we again denote by $\{v_n\}$ the subsequence.

For every $(\lambda_n, u_n)$, we have

$$u_n(x) = \lambda_n \int_0^1 G(x,s)\left[\int_0^1 G(s,\tau)h(\tau)f\bigl(u_n(\tau)\bigr)\,d\tau\right] ds. \tag{2.11}$$

Dividing both sides of (2.11) by $\|u_n\|_\infty$, we get

$$v_n(x) = \lambda_n \int_0^1 G(x,s)\left[\int_0^1 G(s,\tau)h(\tau)\frac{f(u_n(\tau))}{u_n(\tau)}v_n(\tau)\,d\tau\right] ds. \tag{2.12}$$

Since $\|u_n\|_\infty \to 0$, we conclude that $\frac{f(u_n(\tau))}{u_n(\tau)} \to f_0$ for each fixed $\tau \in [0,1]$. By recalling (2.7), Lebesgue's dominated convergence theorem shows that

$$\begin{aligned} v(x) &= \frac{\mu_1}{f_0}\int_0^1 G(x,s)\left[\int_0^1 G(s,\tau)h(\tau)f_0 v(\tau)\,d\tau\right] ds \\ &= \mu_1 \int_0^1 G(x,s)\left[\int_0^1 G(s,\tau)h(\tau)v(\tau)\,d\tau\right] ds, \end{aligned} \tag{2.13}$$

which means that $v$ is a nontrivial solution of (2.2) with $\lambda = \mu_1$, and hence $v \equiv \phi$. □

**Lemma 2.4** *Assume that* (H1) *and* (H2) *hold. Let* $\mathcal{C}$ *be as in Lemma* 2.2. *Then there exists* $\delta > 0$ *such that* $(\lambda, u) \in \mathcal{C}$ *and* $|\lambda - \frac{\mu_1}{f_0}| + \|u\| \le \delta$ *imply* $\lambda < \frac{\mu_1}{f_0}$.

*Proof* Assume to the contrary that there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ such that $\lambda_n \to \frac{\mu_1}{f_0}$, $\|u_n\| \to 0$ and $\lambda_n \ge \frac{\mu_1}{f_0}$. By Lemma 2.3, there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $\frac{u_n}{\|u_n\|_\infty}$ converges uniformly to $\phi$ on $[0,1]$. Multiplying the equation of (1.3) with $(\lambda, u) = (\lambda_n, u_n)$ by $\phi$ and integrating it over $[0,1]$, we have

$$\int_0^1 \phi(x)u_n''''(x)\,dx = \lambda_n \int_0^1 h(x)f\bigl(u_n(x)\bigr)\phi(x)\,dx. \tag{2.14}$$

By a simple computation, one has that

$$\int_0^1 \phi(x)u_n''''(x)\,dx = \int_0^1 \phi''''(x)u_n(x)\,dx = \mu_1 \int_0^1 h(x)\phi(x)u_n(x)\,dx. \tag{2.15}$$

Combining (2.14) with (2.15), we obtain

$$\int_0^1 h(x)f\bigl(u_n(x)\bigr)\phi(x)\,dx = \frac{\mu_1}{\lambda_n}\int_0^1 h(x)\phi(x)u_n(x)\,dx,$$

that is,

$$\frac{\int_0^1 h(x)\phi(x)[f(u_n(x)) - f_0 u_n(x)]\,dx}{\|u_n\|_\infty^{1+\alpha}} = \frac{\int_0^1 h(x)\phi(x)[\frac{\mu_1}{\lambda_n}u_n(x) - f_0 u_n(x)]\,dx}{\|u_n\|_\infty^{1+\alpha}}. \tag{2.16}$$

Since

$$\begin{aligned}
&\frac{\int_0^1 h(x)\phi(x)[f(u_n(x))-f_0u_n(x)]\,dx}{\|u_n\|_\infty^{1+\alpha}} \\
&\quad =\int_0^1 h(x)\phi(x)\frac{f(u_n(x))-f_0u_n(x)}{(u_n(x))^{1+\alpha}}\left[\frac{u_n(x)}{\|u_n\|_\infty}\right]^{1+\alpha}dx, \qquad (2.17)
\end{aligned}$$

then Lebesgue's dominated convergence theorem, Lemma 2.3, and condition (H2) imply that

$$\frac{\int_0^1 h(x)\phi(x)[f(u_n(x))-f_0u_n(x)]\,dx}{\|u_n\|_\infty^{1+\alpha}}\to f_1\int_0^1 h(x)\phi^{2+\alpha}(x)\,dx>0. \qquad (2.18)$$

Similarly,

$$\begin{aligned}
&\frac{\int_0^1 h(x)\phi(x)[\frac{\mu_1}{\lambda_n}u_n(x)-f_0u_n(x)]\,dx}{\|u_n\|_\infty^{1+\alpha}} \\
&\quad =\frac{\mu_1-f_0\lambda_n}{\lambda_n\|u_n\|_\infty^{\alpha}}\int_0^1 h(x)\phi(x)\frac{u_n(x)}{\|u_n\|_\infty}\,dx. \qquad (2.19)
\end{aligned}$$

By Lebesgue's dominated convergence theorem and Lemma 2.3 again, we conclude that

$$\int_0^1 h(x)\phi(x)\frac{u_n(x)}{\|u_n\|_\infty}\,dx\to\int_0^1 h(x)\phi^2(x)\,dx>0, \qquad (2.20)$$

this contradicts (2.16). □

## 3 Direction turns of the component and the proof of Theorem 1.1

In this section, we show that there are at least two direction turns of the component under conditions (H4) and (H3), that is, the component is reversed $S$-shaped, and accordingly we finish the proof of Theorem 1.1.

**Lemma 3.1** *Assume that* (H1) *and* (H4) *hold. Let* $u$ *be a solution of* (1.3) *with* $\|u\|_\infty=s_0$, *then* $\lambda>\frac{\mu_1}{f_0}$.

*Proof* Let $u$ be a solution of (1.3) with $\|u\|_\infty=s_0$, then by condition (H4) and the property of $G(x,s)$, we have

$$\begin{aligned}
s_0=\|u\|_\infty&=\max_{x\in[0,1]}\left\{\lambda\int_0^1 G(x,s)\left[\int_0^1 G(s,\tau)h(\tau)f\big(u(\tau)\big)\,d\tau\right]ds\right\} \\
&<\lambda\widehat{h}\frac{60f_0}{\mu_1\widehat{h}}s_0\int_0^1 G(s,s)\left[\int_0^1 G(s,\tau)\,d\tau\right]ds \\
&=\lambda\frac{f_0}{\mu_1}s_0, \qquad (3.1)
\end{aligned}$$

then $\lambda>\frac{\mu_1}{f_0}$. □

**Lemma 3.2** *Assume that* (H1), (H2), *and* (H3) *hold. Let* $\mathcal{C}$ *be as in Lemma* 2.2. *Then* $\sup\{\lambda\mid(\lambda,u)\in\mathcal{C}\}<\infty$.

*Proof* Note that if (H2) and (H3) hold, there exists a positive constant $k$ with $f_0 \geq k$ such that

$$f(s) \geq ks, \quad \forall s \in [0,\infty). \tag{3.2}$$

For any $(\lambda, u) \in \mathcal{C}$, by (3.2) and (2.1) and the property of $G(x,s)$, we have

$$\begin{aligned}
u(x) &= \lambda \int_0^1 G(x,s)\left[\int_0^1 G(s,\tau)h(\tau)f\big(u(\tau)\big)\,d\tau\right] ds \\
&\geq \lambda \int_0^1 G(x,s)\left[\int_0^1 G(s,\tau)h(\tau)ku(\tau)\,d\tau\right] ds \\
&\geq \lambda \frac{1}{4}\int_0^1 G(x,s)\left[\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)h(\tau)k\|u\|_\infty\,d\tau\right] ds \\
&\geq \lambda \frac{1}{4}k\|u\|_\infty \frac{1}{4}\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\left[\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)h(\tau)\,d\tau\right] ds \\
&= \lambda K\|u\|_\infty, \quad \forall x \in \left[\frac{1}{4},\frac{3}{4}\right],
\end{aligned} \tag{3.3}$$

where

$$K = \frac{1}{16}k\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\left[\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)h(\tau)\,d\tau\right] ds.$$

Then (3.3) implies that

$$\frac{1}{K} \geq \frac{\max_{x\in[\frac{1}{4},\frac{3}{4}]} u(x)}{K\|u\|_\infty} \geq \lambda.$$ □

**Lemma 3.3** *Assume that* (H1), (H2), *and* (H3) *hold. Let* $\{(\lambda_n, u_n)\}$ *be a sequence of positive solutions to* (1.3), *then* $\|u_n\| \to \infty$ *implies* $\|u_n\|_\infty \to \infty$.

*Proof* From Lemma 3.2, we conclude that $\{\lambda_n\}$ is bounded. Assume on the contrary that $\|u_n\|_\infty$ is bounded. By recalling (2.5) and (2.4), we have that $\|u_n'\|_\infty$ and $\|u_n''\|_\infty$ are bounded too.

From the boundary condition $u_n''(0) = u_n''(1) = 0$, there exists $x_n^* \in (0,1)$ such that $u_n'''(x_n^*) = 0$. Integrating the equation of (1.3) on $[x_n^*, x]$, we obtain

$$u_n'''(x) = \int_{x_n^*}^x u_n''''(s)\,ds = \lambda_n \int_{x_n^*}^x h(s)f\big(u_n(s)\big)\,ds, \quad x \in [0,1], \tag{3.4}$$

then $\|u_n'''\|_\infty$ is bounded. Finally, we conclude that $\|u_n\| = \|u_n\|_\infty + \|u_n'\|_\infty + \|u_n''\|_\infty + \|u_n'''\|_\infty$ is bounded, this deduces a contradiction. □

**Lemma 3.4** *Assume that* (H1), (H2), *and* (H3) *hold. Let* $\{(\lambda_n, u_n)\}$ *be a sequence of positive solutions to* (1.3), *then* $\|u_n\| \to \infty$ *implies* $\lambda_n \to 0$.

*Proof* For every $(\lambda_n, u_n)$, we have

$$u_n(x) = \lambda_n \int_0^1 G(x,s)\left[\int_0^1 G(s,\tau)h(\tau)f\big(u_n(\tau)\big)\,d\tau\right]ds. \tag{3.5}$$

Lemma 3.3 and Lemma 3.2 imply that $\|u_n\|_\infty \to \infty$. Dividing both sides of (3.5) by $\|u_n\|_\infty$, we get

$$\frac{u_n(x)}{\|u_n\|_\infty} = \lambda_n \int_0^1 G(x,s)\left[\int_0^1 G(s,\tau)h(\tau)\frac{f(u_n(\tau))}{u_n(\tau)}\frac{u_n(\tau)}{\|u_n\|_\infty}\,d\tau\right]ds, \quad x \in [0,1]. \tag{3.6}$$

Then by (H3) and the boundedness of $\frac{u_n(x)}{\|u_n\|_\infty}$, we conclude that $\lambda_n \to 0$. □

*Proof of Theorem* 1.1 Let $\mathcal{C}$ be as in Lemma 2.2. By Lemma 2.4, $\mathcal{C}$ is bifurcating from $(\frac{\mu_1}{f_0}, 0)$ and goes leftward. Since $\mathcal{C}$ is unbounded, there exists $\{(\lambda_n, u_n)\}$ such that $(\lambda_n, u_n) \in \mathcal{C}$ and $|\lambda_n| + \|u_n\| \to \infty$. By Lemmas 3.2 and 3.4, we have that $\|u_n\| \to \infty$ and $\lambda_n \to 0$. Lemma 3.3 implies that $\|u_n\|_\infty \to \infty$, then there exists $(\lambda_0, u_0) \in \mathcal{C}$ such that $\|u_0\|_\infty = s_0$, and Lemma 3.1 shows that $\lambda_0 > \frac{\mu_1}{f_0}$.

By Lemmas 2.2, 2.4, and 3.1, $\mathcal{C}$ passes through some points $(\frac{\mu_1}{f_0}, v_1)$ and $(\frac{\mu_1}{f_0}, v_2)$ with $\|v_1\|_\infty < s_0 < \|v_2\|_\infty$, and there exist $\underline{\lambda}$ and $\overline{\lambda}$ which satisfy $0 < \underline{\lambda} < \frac{\mu_1}{f_0} < \overline{\lambda}$ and both (i) and (ii):

(i) if $\lambda \in (\underline{\lambda}, \frac{\mu_1}{f_0})$, then there exist $u$ and $v$ such that $(\lambda, u), (\lambda, v) \in \mathcal{C}$ and $\|u\|_\infty < \|v\|_\infty < s_0$;

(ii) if $\lambda \in [\frac{\mu_1}{f_0}, \overline{\lambda})$, then there exist $u$ and $v$ such that $(\lambda, u), (\lambda, v) \in \mathcal{C}$ and $\|u\|_\infty < s_0 < \|v\|_\infty$.

Define $\lambda_* = \inf\{\underline{\lambda} : \underline{\lambda} \text{ satisfies (i)}\}$ and $\lambda^* = \sup\{\overline{\lambda} : \overline{\lambda} \text{ satisfies (ii)}\}$. Then (1.3) has a positive solution $u_{\lambda_*}$ at $\lambda = \lambda_*$ and $u_{\lambda^*}$ at $\lambda = \lambda^*$, respectively. Clearly, the component curve turns to the right at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ and to the left at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ (see Figure 2). That is, $\mathcal{C}$ is a reversed $S$-shaped component, this together with Lemma 3.4 completes the proof of Theorem 1.1. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
RM completed the main study, carried out the results of this article, and drafted the paper. JW checked the proofs and verified the calculation. The two authors read and approved the final manuscript.

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**References**

1. Gupta, CP: Existence and uniqueness theorems for the bending of an elastic beam equation. Appl. Anal. **26**(4), 289-304 (1988)
2. Usmani, RA: A uniqueness theorem for a boundary value problem. Proc. Am. Math. Soc. **77**, 329-335 (1979)

3. Schröder, J: Operator Inequalities. Mathematics in Science and Engineering, vol. 147. Academic Press, New York (1980)
4. Yang, YS: Fourth-order two-point boundary value problems. Proc. Am. Math. Soc. **104**, 175-180 (1988)
5. Agarwal, RP: On fourth-order boundary value problems arising in beam analysis. Differ. Integral Equ. **2**, 91-110 (1989)
6. Del Pino, MA, Manásevich, RF: Existence for a fourth-order boundary value problem under a two-parameter nonresonance condition. Proc. Am. Math. Soc. **112**, 81-86 (1991)
7. Ma, RY, Wang, HY: On the existence of positive solutions of fourth-order ordinary differential equations. Appl. Anal. **59**, 225-231 (1995)
8. Bai, ZB, Wang, HY: On positive solutions of some nonlinear fourth-order beam equations. J. Math. Anal. Appl. **270**, 357-368 (2002)
9. Li, YX: Positive solutions of fourth-order boundary value problems with two parameters. J. Math. Anal. Appl. **281**, 477-484 (2003)
10. Drábek, P, Holubová, G, Matas, A, Nečesal, P: Nonlinear models of suspension bridges: discussion of the results. Appl. Math. **48**, 497-514 (2003)
11. Liu, B: Positive solutions of fourth order two-point boundary value problems. Appl. Math. Comput. **148**, 407-420 (2004)
12. De Coster, C, Habets, P: The lower and upper solutions method for boundary value problems. In: Cañada, A, Drábek, P, Fonda, A (eds.) Handbook of Differential Equations: Ordinary Differential Equations (2004)
13. Yao, QL: Existence, multiplicity and infinite solvability of positive solutions to a nonlinear fourth-order periodic boundary value problem. Nonlinear Anal. TMA **63**, 237-246 (2005)
14. Han, GD, Li, FY: Multiple solutions of some fourth-order boundary value problems. Nonlinear Anal. **66**, 2591-2603 (2007)
15. Cabada, A, Cid, JÁ, Sanchez, L: Positivity and lower and upper solutions for fourth order boundary value problems. Nonlinear Anal. **67**, 1599-1612 (2007)
16. Han, GD, Xu, ZB: Multiple solutions of some nonlinear fourth-order beam equations. Nonlinear Anal. **68**, 3646-3656 (2008)
17. Webb, JRL, Infante, G, Franco, D: Positive solutions of nonlinear fourth-order boundary value problems with local and non-local boundary conditions. Proc. R. Soc. Edinb. A **138**, 427-446 (2008)
18. Rynne, BP: Infinitely many solutions of superlinear fourth order boundary value problems. Topol. Methods Nonlinear Anal. **19**, 303-312 (2002)
19. Ma, RY, Gao, CH, Han, XL: On linear and nonlinear fourth-order eigenvalue problems with indefinite weight. Nonlinear Anal. **74**, 4186-4191 (2011)
20. Dai, GW, Han, XL: Global bifurcation and nodal solutions for fourth-order problems with sign-changing weight. Appl. Math. Comput. **219**, 9399-9407 (2013)
21. Sim, I, Tanaka, S: Three positive solutions for one-dimensional $p$-Laplacian problem with sign-changing weight. Appl. Math. Lett. **49**, 42-50 (2015)
22. Wang, JX, Ma, RY: $S$-Shaped connected component for the fourth-order boundary value problem. Bound. Value Probl. (2016). doi:10.1186/s136610160699z