Xu et al. *Advances in Difference Equations* (2017) 2017:134
DOI 10.1186/s13662-017-1186-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Global attracting set and exponential decay of second-order neutral stochastic functional differential equations driven by fBm

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Abstract

In this paper, we are concerned with a class of second-order neutral stochastic functional differential equations driven by a fractional Brownian motion with Hurst parameter $1/2 < \hbar < 1$ on the Hilbert space. By combining some stochastic analysis theory and new integral inequality techniques, we identify the global attracting sets of the equations under investigation. Some sufficient conditions ensuring the exponential decay of mild solutions in the pth moment to the stochastic systems are obtained. Last, an example is presented to illustrate our theory in the work.

MSC: 60H15; 60G15; 60H05

Keywords: global attracting set; exponential decay in the pth moment; second-order SDEs; fractional Brownian motion

1 Introduction

This paper is devoted to the study of the global attracting set and exponential decay of a class of second-order neutral stochastic functional differential equations driven by a fractional Brownian motion with Hurst parameter $1/2 < \hbar < 1$. Second-order stochastic systems can capture the dynamic behavior of many natural phenomena. In many real-world scenarios, it seems advantageous to reflect a more complex situation than first-order stochastic differential equations (SDEs). Recently, there has been increasing interest in the study of second-order stochastic differential equations due to their important applications in many areas such as medicine and biology, mathematical physics, electronics and telecommunications. For instance, Ren and Sakthivel [1] investigated the existence, uniqueness and stability of second-order neutral stochastic evolution equations with infinite delay; Liang and Guo [2] probed the behavior for second-order stochastic evolution equations with memory; Arthi et al. [3] discussed the exponential stability for second-order neutral stochastic differential equations with impulses.

One solution for many SDEs is a semimartingale as well as a Markov process. However, in the real world, many objects are not always such processes and they may have long-range aftereffects. Since the work of Mandelbrot and Van Ness [4], the researchers have focused the increasing attention on stochastic models based on the fractional Brownian motion. A fractional Brownian motion (fBm) of Hurst parameter $\hbar \in (0,1)$ is a centered Gaussian process $\beta^{\hbar} = \{\beta^{\hbar}(t), t \geq 0\}$ with the covariance function

$$R_{\hbar}(t,s) = \mathbb{E}\big(\beta^{\hbar}(t)\beta^{\hbar}(s)\big) = \frac{1}{2}\big(t^{2\hbar} + s^{2\hbar} - |t-s|^{2\hbar}\big).$$

When $\hbar = 1/2$, the fBm becomes the standard Brownian motion, and the fBm $\beta^{\hbar}$ is neither a semimartingale nor a Markov process if $\hbar \neq 1/2$. However, the fBm $\beta^{\hbar}$, $\hbar > 1/2$, is a long-memory process and presents an aggregation behavior. The long-memory property makes the fBm a potential candidate to model noise in mathematical finance (see [5]), in biology (see [6, 7]), in communication networks (see, for instance, [8]), the analysis of global temperature anomaly [9] and electricity markets [10] etc. Stochastic differential equations driven by a fractional Brownian motion (fBm) have attracted the interest of many researchers. For more details on this topic, one can refer to the literature [11–17].

On the other hand, analysis found that the research mainly analyzed the stability of second-order equations to prove exponential stability through some strong conditions, which may be difficult to meet in practice. To find some weaker constraints to ensure the stability of a system, by using the inequality technique, Xu [18] investigated the attracting sets of a class of Volterra differential equations, and the relative stability conditions were further weakened. Attracting sets require the solutions enter some sets at a time and not exit, no matter what the solutions begin with. Attracting sets of first-order stochastic dynamical systems have been extensively studied over the last few decades. For instance, Li and Xu [19] investigated the global attracting sets of neutral stochastic partial functional differential equations; Wang and Li [20] obtained the global attracting sets of impulsive stochastic partial differential equations with infinite delays by establishing some impulsive-integral inequalities. Li [21] studied the global attracting sets and exponential decay in the pth moment of impulsive neutral stochastic functional differential equations driven by fBm. However, it is worth mentioning that the problem of determining the attracting sets of second-order stochastic partial differential equations driven by fBm is more complicated owing to its complexity and still remains open for a while. Hence, techniques and methods for the attracting sets of second-order neutral stochastic partial differential equations driven by fBm should be developed and explored.

To this end, in this paper, we will consider a class of second-order neutral stochastic functional differential equations driven by fBm with Hurst parameter $\hbar \in (1/2,1)$. We aim to investigate the global attracting sets, exponential decay in the pth moment of the kind of second-order stochastic differential equation driven by fBm. We will first develop some new integral inequalities. Subsequently, by using the stochastic analysis techniques, the properties of operator semigroup and combining those new integral inequalities, we will attempt to give the global attracting sets of the considered system. Last, some sufficient conditions ensuring exponential decay in the pth moment of the system under investigation are obtained.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. We devote Section 3 to the global attracting sets of the mild solutions to the equations under investigation. In Section 4, we investigate the exponential decay of equations under investigation. In Section 5, we give an example to illustrate the efficiency of the obtained result. In Section 6, we present our conclusion.

2 Preliminary

In this section we collect some notions, conceptions and lemmas which will be used throughout the paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose that there is a one-dimensional fractional Brownian motion $\beta^\hbar = \{\beta^\hbar(t), t \geq 0\}$ with Hurst parameter $\hbar \in (0,1)$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Now we state the stochastic integral of a deterministic H-valued function with respect to the scalar fractional Brownian motion $\beta^\hbar$, $\hbar \in (0,1)$. To this end, let $T > 0$ and denote by $\mathcal{E}_0$ the linear space of all H-valued step functions on $[0,T]$, that is, $f \in \mathcal{E}_0$ if and only if

$$f(t) = \sum_{i=1}^{n-1} x_i \mathbf{1}_{[t_i,t_{i+1})}(t), \quad t \in [0,T],$$

where $\mathbf{1}_{[t_i,t_{i+1})}$ is the indicator function on the set $[t_i, t_{i+1})$, $x_i \in H$, for $i \in \{1,\dots,n-1\}$ and $0 = t_1 < t_2 < \cdots < t_n = T$. For $f \in \mathcal{E}_0$, we define its stochastic integral with respect to $\beta^\hbar$ as

$$\int_0^T f(s)\,d\beta^\hbar(s) = \sum_{i=1}^{n} x_i\bigl(\beta^\hbar(t_{i+1}) - \beta^\hbar(t_i)\bigr). \tag{2.1}$$

Let $L_\hbar : \mathcal{E}_0 \to L^2([0,T];H)$ be the linear map given by

$$(L_\hbar f)(t) = f(t)k_\hbar(T,t) + \int_t^T \bigl(f(s) - f(t)\bigr)\frac{\partial k_\hbar}{\partial s}(s,t)\,ds \quad \forall f \in \mathcal{E}_0, \tag{2.2}$$

where $k_\hbar(\cdot,\cdot)$ is the kernel function given in

$$k_\hbar(t,s) = \begin{cases} \frac{\tilde{c}_\hbar (t-s)^{\hbar-\frac{1}{2}}}{\Gamma(\hbar+\frac{1}{2})} + \frac{\tilde{c}_\hbar(\frac{1}{2}-\hbar)}{\Gamma(\hbar+\frac{1}{2})}\int_s^t (u-s)^{\hbar-\frac{3}{2}}(1-(\frac{s}{u})^{\frac{1}{2}-\hbar})\,du & \text{if } \hbar \in (0,1/2), \\ \frac{\hat{c}_\hbar}{\Gamma(\hbar-\frac{1}{2})} s^{\frac{1}{2}-\hbar}\int_s^t (u-s)^{\hbar-\frac{3}{2}} u^{\hbar-\frac{1}{2}}\,du & \text{if } \hbar \in (1/2,1), \end{cases} \tag{2.3}$$

where $\tilde{c}_\hbar$ and $\hat{c}_\hbar$ are positive constants depending only on $\hbar$.

It is known in this case that

$$\mathbb{E}\left\|\int_0^T f(t)\,d\beta^\hbar(t)\right\|_H^2 = \|L_\hbar f\|^2_{L^2([0,T];H)}. \tag{2.4}$$

Let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ be the Hilbert space obtained as the completion of the pre-Hilbert space $\mathcal{E}_0$ under the inner product

$$\langle f,g\rangle_{\mathcal{E}} := \langle L_\hbar f, L_\hbar g\rangle_{L^2([0,T];H)} \quad \text{for any } f,g \in \mathcal{E}_0.$$

Then stochastic integral (2.1) is extendable to an arbitrary $f \in \mathcal{E}$ by isometry (2.4). If $H = \mathbb{R}^1$, it is known (see [22]) that in terms of $L_\hbar$ the process $\{\beta(t) = \beta^\hbar((L_\hbar)^{-1}(\mathbf{1}_{[0,t]})), t \geq 0\}$ is a standard real Brownian motion, and $\beta^\hbar$ has the following integral representation:

$$\beta^\hbar(t) = \int_0^t k_\hbar(t,s)\,d\beta(s), \quad t \geq 0.$$

Let K be an alternative separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_K$ and the norm $\|\cdot\|_K$, respectively. We denote by $\mathcal{L}(K,H)$ the space of all linear bounded operators from K into H, equipped with the usual operator norm $\|\cdot\|$ topology. If $K=H$, we write $\mathcal{L}(K)$ simply for $\mathcal{L}(K,K)$.

Definition 2.1 Let $Q\in\mathcal{L}(K)$ be a positive self-adjoint operator. A K-valued Gaussian process $(B_Q^{\hbar}(t), t\geq 0)$ on $(\Omega,\mathcal{F},\mathbb{P})$ is called a *fractional Q-Brownian motion with Hurst parameter* $\hbar\in(0,1)$ if it satisfies:

(i) for arbitrary $t\geq 0$, the mapping $B_Q^{\hbar}(t):K\to L^2(\Omega;\mathbb{R}^1)$ is linear;

(ii) for arbitrary $k\in K$, $B_Q^{\hbar}(t)k$ is a real Gaussian process with mean $\mathbb{E}B_Q^{\hbar}(t)k=0$ for any $t\geq 0$;

(iii) $\mathbb{E}(B_Q^{\hbar}(t)k_1\cdot B_Q^{\hbar}(s)k_2)=(1/2)(t^{2\hbar}+s^{2\hbar}-|t-s|^{2\hbar})\langle Qk_1,k_2\rangle_K$ for all $s,t\in\mathbb{R}_+=[0,\infty)$ and $k_1,k_2\in K$.

In particular, if $\operatorname{Tr}Q<\infty$, then $B_Q^{\hbar}(t)$ is called a *(genuine) fractional Brownian motion with Hurst parameter* $\hbar\in(0,1)$*;* and if $Q=I$, this process is called a *cylindrical fractional Brownian motion with Hurst parameter* $\hbar\in(0,1)$.

In general, it is not necessarily true that there exists a K-valued random process $\hat{B}_Q^{\hbar}(\cdot)$ such that $B_Q^{\hbar}(t)(k)=\langle\hat{B}_Q^{\hbar}(t),k\rangle_K$ for each $t\geq 0$, although this is the case when $B_Q^{\hbar}$ is a genuine fBm. Suppose that there is a complete orthonormal basis $(e_n, n\in\mathbb{N})$ of K diagonalizing operator Q, i.e., there exists a sequence of $(\lambda_n>0, n\in\mathbb{N})$ such that $Qe_n=\lambda_n e_n$ for each $n\in\mathbb{N}$, letting $\beta_n^{\hbar}(t)=\frac{1}{\sqrt{\lambda_n}}B_Q^{\hbar}(t)(e_n)$ for $n\in\mathbb{N}$, $t\geq 0$, the scalar processes $(\beta_n^{\hbar}, n\in\mathbb{N})$ are a sequence of fractional Brownian motions which are mutually independent and $B_Q^{\hbar}$ can be uniformly represented as

$$B_Q^{\hbar}(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n^{\hbar}(t)e_n,\quad t\geq 0,$$

with its increment covariance operator Q. This series does not necessarily converge almost surely in K. But if Q is a trace class, that is, $\sum_{n=1}^{\infty}\lambda_n<\infty$, then the series defines a K-valued stochastic process. For a fixed $\hbar\in(0,1)$ and $T>0$, let $\{\mathcal{F}_t^{B_Q^{\hbar}}\}_{0\leq t\leq T}$ be the natural filtration of the fractional Q-Brownian motion $B_Q^{\hbar}(t)$, $0\leq t\leq T$. For the operator Q, let $B_Q(t)$, $t\geq 0$, be a K-valued Q-Wiener process.

We introduce the subspace $K_Q=\operatorname{Ran}Q^{1/2}\subset K$, the range of $Q^{1/2}$, which is a Hilbert space endowed with the inner product

$$\langle k_1,k_2\rangle_{K_Q}=\langle Q^{-1/2}k_1, Q^{-1/2}k_2\rangle_K\quad\text{for any } k_1,k_2\in K_Q.$$

Let $\mathcal{L}_2(K_Q,H)$ denote the space of all Hilbert-Schmidt operators from K_Q into H. Then $\mathcal{L}_2(K_Q,H)$ turns out to be a separable Hilbert space, equipped with the norm

$$\|\Psi\|^2_{\mathcal{L}_2(K_Q,H)}=Tr\big[\Psi Q\Psi^*\big]\quad\text{for any } \Psi\in\mathcal{L}_2(K_Q,H).$$

For $T\geq 0$, we denote by $\mathcal{U}_{\hbar}^2([0,T];\mathcal{L}_2(K_Q,H))$ the space of all Borel measurable mappings $f:[0,T]\to\mathcal{L}_2(K_Q,H)$ such that $f(\cdot)x\in\mathcal{E}$ for each $x\in K$ and $L_{\hbar}f\in L^2([0,T];\mathcal{L}_2(K_Q;H))$.

Let $(e_n, n \in \mathbb{N})$ be a complete orthonormal basis diagonalizing operator Q. For any $f \in \mathcal{U}_{\hbar}^2([0,T];\mathcal{L}_2(K_Q,H))$, the stochastic integral $\int_0^T f(t)\,dB_Q^{\hbar}(t)$ is defined as

$$\int_0^T f(t)\,dB_Q^{\hbar}(t) := \sum_{n=1}^{\infty} \sqrt{\lambda_n} \int_0^T f(t)e_n\,d\beta_n^{\hbar}(t), \tag{2.5}$$

provided the infinite series converges in $L^2(\Omega;H)$. It may be verified that this stochastic integral does not depend on the choice of the complete orthonormal basis $\{e_n\}_{n=1}^{\infty}$.

Let $\mathbb{R}^+ = [0,+\infty)$ and $C(X,Y)$ denote the space of continuous mappings from the topological space X to the topological space Y. Especially, for the given constant $r > 0$, $C := C([-r,0];\mathbb{R})$ denotes the family of all continuous $\mathbb{R}$-valued functions ϕ defined on $[-r,0]$ with the norm $\|\phi\|_C = \sup_{-r\le\theta\le 0} |\phi(\theta)|$. Denote $\mathcal{C} = C([-r,0];H)$ equipped with the norm $\|\phi\|_{\mathcal{C}} = \sup_{-r\le\theta\le 0} \|\phi(\theta)\|_H$. Define $BC_{\mathcal{F}_0}^b$ as the family of all bounded $\mathcal{F}_0$-measurable, $\mathcal{C}$-valued random variables ϕ, satisfying $\|\phi\|_{Lp}^p = \sup_{-r\le\theta\le 0} \mathbb{E}\|\phi(\theta)\|_H^p < \infty$ for $p > 0$.

Consider the following second-order neutral stochastic functional differential equation driven by fBm with Hurst parameter $1/2 < \hbar < 1$:

$$\begin{cases} d[x'(t) + G(t,x_t)] \\ \quad = [Ax(t) + f(t,x_t)]\,dt + g(t,x_t)\,d\omega(t) + \sigma(t)\,dB_Q^{\hbar}(t), & t \ge 0, \\ x_0(t) = \varphi(t) \in BC_{\mathcal{F}_0}^b([-r,0];H), & t \in [-r,0], x'(0) = \phi_1, \end{cases} \tag{2.6}$$

where $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous cosine family on H, $B_Q^{\hbar}$ is a fractional Brownian motion with Hurst parameter $\hbar \in (1/2,1)$ and ω is a standard Wiener process on a real and separable Hilbert space K. The delay term $x_t : [-r,0] \to H$ defined by $x_t(\theta)$ for $t \ge 0$ belongs to the space $\mathcal{C}$. $G, f : [0,+\infty) \times \mathcal{C} \to H$, $g : [0,+\infty) \times \mathcal{C} \to \mathcal{L}_2(K_Q,H)$, $\sigma : [0,+\infty) \to \mathcal{L}_2(K_Q,H)$ are some appropriate mappings specified later.

Now, let us recall some basic concepts and facts on cosine families of operators (see [23]).

Definition 2.2 One parameter family $(T(t))_{t\ge 0}$ is called a strongly continuous cosine family if the following conditions hold:

(i) $T(0) = I$;
(ii) $T(t)x$ is continuous in t on $\mathbb{R}$ for all $x \in H$;
(iii) $T(t+s) + T(t-s) = 2T(t)T(s)$ for all $t, s \in \mathbb{R}$.

We also need consider the corresponding strongly continuous sine family $(S(t))_{t\ge 0}$, which is defined as $S(t)x = \int_0^t T(s)\,ds$, $t \in \mathbb{R}$, $x \in H$. As for the infinitesimal generator $A : D(A) \subset H \to H$ of a cosine family of operators $(T(t))_{t\ge 0}$, define $Ax = \frac{d^2}{dt^2} T(t)x|_{t=0}$. A is also a closed and densely defined operator on H.

Throughout this paper, we impose the following assumptions:

(H1) The cosine family of operators $(T(t))_{t\ge 0}$ and its corresponding sine family $(S(t))_{t\ge 0}$ satisfy the following conditions for all $t \ge 0$:

$$\|T(t)\|_H \le Me^{-\lambda_1 t}, \qquad \|S(t)\|_H \le Me^{-\lambda_2 t}, \quad M \ge 1, \lambda_1, \lambda_2 > 0.$$

(H2) The function $\sigma : [0, +\infty) \to \mathcal{L}_2^0(K, H)$ satisfies the following conditions: for the complete orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in K, we have

(σ_1) $\sum_{n=1}^{\infty} \|\sigma(t)Q^{1/2}e_n\|_{L^2([0,T],H)} < \infty$;
(σ_2) $\sum_{n=1}^{\infty} \|\sigma(t)Q^{1/2}e_n\|_H$ is uniformly convergent for $t \in [0, T]$.

(H3) There exist constants $L_f > 0$, $L_g > 0$, $L_G > 0$, $b_f > 0$, $b_g > 0$ and $b_G > 0$ such that for any $x, y \in \mathcal{C}$ and $t \geq 0$,

$$\begin{aligned}
&\big\|f(t,x_t) - f(t,y_t)\big\|_H \leq L_f\|x-y\|_{\mathcal{C}}, \qquad \big\|f(t,0)\big\| \leq b_f, \\
&\big\|g(t,x_t) - g(t,y_t)\big\|_{\mathcal{L}_2(K_Q,H)} \leq L_g\|x-y\|_{\mathcal{C}}, \qquad \big\|g(t,0)\big\| \leq b_g, \\
&\big\|G(t,x_t) - G(t,y_t)\big\|_H \leq L_G\|x-y\|_{\mathcal{C}}, \qquad \big\|G(t,0)\big\| \leq b_G.
\end{aligned}$$

3 Global attracting set and exponential *p*-stability

In this section, we shall get the global attracting sets of stochastic differential equation (2.6). First, we give the following definition of mild solutions to equation (2.6).

Definition 3.1 An H-valued continuous stochastic process $\{x(t), t \in [-r, T]\}$, $0 < T < \infty$ is called a mild solution of (2.6) if $x_0(t) = \varphi(t) \in BC^b_{\mathcal{F}_0}([-r, 0]; H)$, $t \in [-r, 0]$, $x'(0) = \phi_1$, and the following conditions hold:

(i) $x(t)$ is a measurable, $\mathcal{F}_t$-adapted process with $\mathbb{E}\int_0^T \|x(t)\|_H^2\,dt < \infty$;
(ii) $x(t)$ satisfies the following integral equation:

$$\begin{aligned}
x(t) &= T(t)\varphi(0) + S(t)\big(\phi_1 - G(0,x_0)\big) \\
&\quad + \int_0^t T(t-s)G(s,x_s)\,ds + \int_0^t S(t-s)f(s,x_s)\,ds \\
&\quad + \int_0^t S(t-s)g(s,x_s)\,d\omega(s) + \int_0^t S(t-s)\sigma(s)\,dB_Q^H(s).
\end{aligned} \tag{3.1}$$

Remark 3.1 Under assumptions (H1)-(H3), the existence and uniqueness of mild solution to system (2.6) are easily shown by using Picard's iterative method (see Revathi et al. [24]).

Definition 3.2 (see [18]) A set $S \subset H$ is called the *global attracting set* of (2.6) if for any initial value $\psi \in BC^b_{\mathcal{F}_0}([-r, 0]; H)$, the solution process $x(t, \psi)$ of (2.6) converges to S as $t \to \infty$, i.e.,

$$\operatorname{dist}\big(x(t,\psi), S\big) \to 0 \quad \text{as } t \to \infty,$$

where $\operatorname{dist}(x, S) = \inf_{y\in S} \mathbb{E}\|x - y\|_H$.

In order to get the global attractivity of equation (2.6), we need the following important integral inequalities.

Lemma 3.1 *Let $y(t) \in C(\mathbb{R}, \mathbb{R}^+)$ be a solution of the delay integral inequality*

$$y(t) \leq \begin{cases} b_1e^{-\lambda_1 t} + b_2e^{-\lambda_2 t} & \\ \quad + b_3\int_0^t e^{-\lambda_1(t-s)}\|y_s\|_C\,ds + b_4\int_0^t e^{-\lambda_2(t-s)}\|y_s\|_C\,ds + b_5, & t \geq 0, \\ \varphi(t), & t \in [-r, 0], \end{cases} \tag{3.2}$$

where $\varphi(t) \in C([-r,0];\mathbb{R}^+)$, $b_1,\dots,b_5$ *are nonnegative constants and* $\lambda_1,\lambda_2 > 0$. *If*

$$\rho := \frac{b_3}{\lambda_1} + \frac{b_4}{\lambda_2} < 1, \tag{3.3}$$

then there are constants $\lambda \in (0,\lambda_1 \wedge \lambda_2)$ *and* $N \geq 0$ *such that*

$$y(t) \leq Ne^{-\lambda t} + \frac{\lambda_5}{1-\rho}, \quad \forall t \geq 0, \tag{3.4}$$

where λ *and* N *satisfy that*

$$\|\varphi(0)\|_C < N, \quad and \quad \frac{b_1+b_2}{N} + e^{\lambda r}\frac{b_3}{\lambda_1-\lambda} + e^{\lambda r}\frac{b_4}{\lambda_2-\lambda} < 1. \tag{3.5}$$

Proof To prove (3.4), we first prove that for any $h > 1$,

$$y(t) < hNe^{-\lambda t} + \frac{\lambda_5}{1-\rho}, \quad \forall t \geq 0. \tag{3.6}$$

If (3.6) is not true, by virtue of the continuity of $y(t)$, then there exists $t_1 > 0$ such that

$$y(t_1) = hNe^{-\lambda t_1} + \frac{\lambda_5}{1-\rho}, \tag{3.7}$$

and

$$y(t) \leq hNe^{-\lambda t} + \frac{\lambda_5}{1-\rho}, \quad t \in [-r,t_1]. \tag{3.8}$$

On the other hand, from condition (3.3) and $\varphi(t) \in C([-r,0];\mathbb{R}^+)$, we can verify that there exist positive constants λ and N such that (3.5) holds. Thus, from (3.2), (3.5) and (3.8), we can obtain

$$\begin{aligned}
y(t_1) &\leq b_1e^{-\lambda_1 t_1} + b_2e^{-\lambda_2 t_1} + b_3\int_0^{t_1} e^{-\lambda_1(t_1-s)}\|y_s\|_C\,ds + b_4\int_0^{t_1} e^{-\lambda_2(t_1-s)}\|y_s\|_C\,ds + b_5 \\
&\leq b_1e^{-\lambda_1 t_1} + b_2e^{-\lambda_2 t_1} + b_3\int_0^{t_1} e^{-\lambda_1(t_1-s)}\big[hNe^{\lambda r}e^{-\lambda s} + (1-\rho)^{-1}b_5\big]\,ds \\
&\quad + b_4\int_0^{t_1} e^{-\lambda_2(t_1-s)}\big[hNe^{\lambda r}e^{-\lambda s} + (1-\rho)^{-1}b_5\big]\,ds + b_5 \\
&\leq b_1e^{-\lambda_1 t_1} + b_2e^{-\lambda_2 t_1} + b_3\int_0^{t_1} hNe^{-\lambda_1(t_1-s)}e^{\lambda r}e^{-\lambda s}\,ds \\
&\quad + b_4\int_0^{t_1} hNe^{-\lambda_2(t_1-s)}e^{\lambda r}e^{-\lambda s}\,ds + \left(\frac{b_3}{\lambda_1}+\frac{b_4}{\lambda_2}\right)(1-\rho)^{-1}b_5 + b_5 \\
&\leq hNe^{-\lambda t_1}\frac{b_1+b_2}{N} + hNe^{-\lambda t_1}e^{\lambda r}b_3\int_0^{t_1} e^{-\lambda_1(t_1-s)}e^{-\lambda(t_1-s)}\,ds \\
&\quad + hNe^{-\lambda t_1}e^{\lambda r}b_4\int_0^{t_1} e^{-\lambda_2(t_1-s)}e^{-\lambda(t_1-s)}\,ds + (1-\rho)^{-1}b_5 \\
&\leq hNe^{-\lambda t_1}\left(\frac{b_1+b_2}{N} + e^{\lambda r}\frac{b_3}{\lambda_1-\lambda} + e^{\lambda r}\frac{b_4}{\lambda_2-\lambda}\right) + (1-\rho)^{-1}b_5 \\
&< hNe^{-\lambda t_1} + (1-\rho)^{-1}b_5,
\end{aligned}$$

which contradicts equality (3.7). So, (3.6) holds for all $t \geq 0$. Letting $h \to 1$ in (3.6), we have (3.4). The proof is complete. □

We also need the following lemma.

Lemma 3.2 ([21]) *For any* $\sigma : [0, +\infty) \to \mathcal{L}_2^0(K, H)$ *such that* (H2) *holds and* $\sup_{t\geq 0} \|\sigma(t)\|_{\mathcal{L}_2^0} < \infty$, *and for any* $p \geq 2$, $t > 0$,

$$\mathbb{E}\left\| \int_0^t S(t-s)\sigma(s)\, dB_Q^{\hbar}(s) \right\|_H^p \leq C \sup_{t\geq 0} \|\sigma(t)\|_{\mathcal{L}_2^0}^p,$$

where $C > 0$ *is a constant depending only on* $\hbar$, M, p *and* r.

Theorem 3.1 *Assume that* (H1)-(H3) *and* $\sup_{t\geq 0} \|\sigma(t)\|_{\mathcal{L}_2^0} < \infty$ *hold, then* $S = \{\phi \in H \mid \|\phi\|_H^p \leq (1-\rho)^{-1} J\}$ *is a global attracting set of system* (2.6) *if the following inequality*

$$\begin{aligned} \rho &:= 12^{p-1} M^p \lambda_1^{-p} L_G^p + 12^{p-1} M^p \lambda_2^{-p} L_f^p \\ &\quad + 12^{p-1} M^p L_g^p \lambda_2^{p/2-1} \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \left(\frac{p-2}{2(p-1)} \right)^{\frac{p}{2}-1} < 1 \end{aligned} \tag{3.9}$$

holds for $p > 2$, *and*

$$\begin{aligned} J &:= 12^{p-1} M^p \lambda_1^{-p} b_G^p + 12^{p-1} M^p \lambda_2^{-p} b_f^p \\ &\quad + 12^{p-1} M^p \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \left(\frac{p-2}{2(p-1)} \right)^{\frac{p}{2}-1} \lambda_2^{-\frac{p}{2}} b_g^p + 6^{p-1} C \sup_{t\geq 0} \|\sigma(s)\|_{\mathcal{L}_2^0}^p. \end{aligned}$$

Proof From (3.1), we have

$$\begin{aligned} \mathbb{E}\|x(t)\|^p &= \mathbb{E}\Big\| T(t)\varphi(0) + S(t)\big(\phi_1 - G(0, x_0)\big) \\ &\quad + \int_0^t T(t-s) G(s, x_s)\, ds + \int_0^t S(t-s) f(s, x_s)\, ds \\ &\quad + \int_0^t S(t-s) g(s, x_s)\, d\omega(s) + \int_0^t S(t-s)\sigma(s)\, dB_Q^H(s) \Big\|_H^p \\ &\leq 6^{p-1}\mathbb{E}\|T(t)\varphi(0)\|_H^p + 6^{p-1}\mathbb{E}\big\|S(t)\big(\phi_1 - G(0, x_0)\big)\big\|_H^p \\ &\quad + 6^{p-1}\mathbb{E}\left\| \int_0^t T(t-s) G(s, x_s)\, ds \right\|_H^p + 6^{p-1}\mathbb{E}\left\| \int_0^t S(t-s) f(s, x_s)\, ds \right\|_H^p \\ &\quad + 6^{p-1}\mathbb{E}\|S(t-s) g(s, x_s)\, d\omega(s)\|_H^p + 6^{p-1}\mathbb{E}\left\| \int_0^t S(t-s)\sigma(s)\, dB_Q^H(s) \right\|_H^p \\ &:= 6^{p-1}\big(J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) + J_6(t)\big). \end{aligned} \tag{3.10}$$

First, it is easy to see from (H1) that for any $t \geq 0$,

$$J_1(t) = \mathbb{E}\|T(t)\varphi(0)\|_H^p \leq M^p e^{-\lambda_1 t}\mathbb{E}\|\varphi(0)\|_H^p \tag{3.11}$$

and

$$\begin{aligned}
J_2(t) &= \mathbb{E}\big\|S(t)\big(\phi_1 - G(0,x_0)\big)\big\|_H^p \\
&\leq M^p e^{-p\lambda_2 t}\mathbb{E}\big\|\phi_1 - G(0,x_0)\big\|_H^p \\
&\leq M^p e^{-p\lambda_2 t}3^{p-1}\mathbb{E}\big(\|\phi_1\|_H^p + L_G\big\|\varphi(0)\big\|_C^p + b_G^p\big).
\end{aligned} \tag{3.12}$$

By virtue of (H1), (H3) and Hölder's inequality, we can obtain for any $t \geq 0$ that

$$\begin{aligned}
J_3(t) &= \mathbb{E}\left\|\int_0^t T(t-s)G(s,x_s)\,ds\right\|_H^p \\
&\leq \mathbb{E}\left(\int_0^t Me^{-\lambda_1(t-s)}\big\|G(s,x_s)\big\|_H\,ds\right)^p \\
&\leq \mathbb{E}\left(\int_0^t Me^{-\lambda_1(t-s)}\big(L_G\|x_s\|_C + b_G\big)\,ds\right)^p \\
&\leq 2^{p-1}M^p L_G^p\left(\int_0^t e^{-\lambda_1(t-s)}\right)^{p-1}\cdot\int_0^t e^{-\lambda_1(t-s)}\mathbb{E}\|x_s\|_C^p\,ds + 2^{p-1}M^p\lambda_1^{-p}b_G^p \\
&\leq 2^{p-1}M^p\lambda_1^{1-p}L_G^p\int_0^t e^{-\lambda_1(t-s)}\mathbb{E}\|x_s\|_C^p\,ds + 2^{p-1}M^p\lambda_1^{-p}b_G^p.
\end{aligned} \tag{3.13}$$

For term $J_4(t)$, similarly to $J_3(t)$, by virtue of (H1), (H3) and Hölder's inequality, we also have for any $t \geq 0$ that

$$\begin{aligned}
J_4(t) &= \mathbb{E}\left\|\int_0^t S(t-s)f(s,x_s)\,ds\right\|_H^p \\
&\leq \mathbb{E}\left(\int_0^t Me^{-\lambda_2(t-s)}\big(L_f\|x_s\|_C + b_f\big)\,ds\right)^p \\
&\leq 2^{p-1}M^p\lambda_2^{1-p}L_f^p\int_0^t e^{-\lambda_2(t-s)}\mathbb{E}\|x_s\|_C^p\,ds + 2^{p-1}M^p\lambda_2^{-p}b_f^p.
\end{aligned} \tag{3.14}$$

Now, let us estimate the terms $J_5(t)$. By using Burkhölder-Davis-Gundy's inequality, assumptions (H1) and (H3) and Hölder's inequality, we obtain for $p > 2$

$$\begin{aligned}
J_5(t) &= \mathbb{E}\left\|\int_0^t S(t-s)g(s,x_s)\,d\omega(s)\right\|^p \\
&\leq M^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\int_0^t \big(e^{-2\lambda_2(t-s)}\mathbb{E}\big\|g(s,x_s)\big\|_{\mathcal{L}_2^0}^2\big)\,ds\right)^{\frac{p}{2}} \\
&\leq M^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\int_0^t e^{-\frac{2\lambda_2(p-1)}{p-2}(t-s)}\,ds\right)^{\frac{p}{2}-1}\int_0^t e^{-\lambda_2(t-s)}E\big\|g(s,x_s)\big\|_{\mathcal{L}_2^0}^p\,ds \\
&\leq 2^{p-1}M^p L_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda_2(p-1)}\right)^{\frac{p}{2}-1}\int_0^t e^{-\lambda_2(t-s)}E\|x_s\|_C^p\,ds \\
&\quad + 2^{p-1}M^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2(p-1)}\right)^{\frac{p}{2}-1}\lambda_2^{-\frac{p}{2}}b_g^p.
\end{aligned} \tag{3.15}$$

Last, by Lemma 3.2, we have

$$J_6(t) = \mathbb{E}\left\| \int_0^t S(t-s)\sigma(s)\,dB_Q^{\hbar}(s)\right\|^p \le C\sup_{t\ge 0}\left\|\sigma(s)\right\|_{\mathcal{L}_2^0}^p. \tag{3.16}$$

Substituting (3.11)-(3.16) into (3.10), we obtain

$$\begin{aligned}
\mathbb{E}\left\|x(t)\right\|^p \le{}& 6^{p-1}M^p e^{-\lambda_1 t}\mathbb{E}\left\|\varphi(0)\right\|_H^p \\
&+ 18^{p-1}M^p e^{-\lambda_2 t}\mathbb{E}\left(\|\phi_1\|_H^p + L_G\left\|\varphi(0)\right\|_C^p + b_G^p\right) \\
&+ 12^{p-1}M^p\lambda_1^{1-p}L_G^p\int_0^t e^{-\lambda_1(t-s)}\mathbb{E}\|x_s\|_C^p\,ds + 12^{p-1}M^p\lambda_1^{-p}b_G^p \\
&+ 12^{p-1}M^p\lambda_2^{1-p}L_f^p\int_0^t e^{-\lambda_2(t-s)}\mathbb{E}\|x_s\|_C^p\,ds + 12^{p-1}M^p\lambda_2^{-p}b_f^p \\
&+ 12^{p-1}M^pL_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda_2(p-1)}\right)^{\frac{p}{2}-1}\int_0^t e^{-\lambda_2(t-s)}E\|x_s\|_C^p\,ds \\
&+ 12^{p-1}M^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2(p-1)}\right)^{\frac{p}{2}-1}\lambda_2^{-\frac{p}{2}}b_g^p \\
&+ 6^{p-1}C\sup_{t\ge 0}\left\|\sigma(s)\right\|_{\mathcal{L}_2^0}^p.
\end{aligned} \tag{3.17}$$

Let $b_1 := 6^{p-1}M^p\mathbb{E}\|\varphi(0)\|_H^p$, $b_2 := 18^{p-1}M^p\mathbb{E}(\|\phi_1\|_H^p + L_G\|\varphi(0)\|_C^p + b_G^p)$, $b_3 := 12^{p-1}M^p\lambda_1^{1-p}L_G^p$, and

$$b_4 := 12^{p-1}M^p\lambda_2^{1-p}L_f^p + 12^{p-1}M^pL_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda_2(p-1)}\right)^{\frac{p}{2}-1}.$$

From (3.9), we know $\rho := \frac{b_3}{\lambda_1} + \frac{b_4}{\lambda_2} < 1$. Since $\varphi \in PC_{\mathcal{F}_0}^B([-r,0];H)$, so there exist $N > 0$, $\lambda \in (0, \lambda_1 \wedge \lambda_2)$ such that

$$\left\|\varphi(0)\right\|_C < N, \quad \text{and} \quad \frac{b_1+b_2}{N} + e^{\lambda r}\frac{b_3}{\lambda_1-\lambda} + e^{\lambda r}\frac{b_4}{\lambda_2-\lambda} < 1.$$

It follows from Lemma 3.1 that

$$\mathbb{E}\left\|x(t)\right\|_H^p \le Ne^{-\lambda t} + \frac{b_5}{1-\rho}.$$

So, by Definition 3.2 we know that S is a global attracting set of the mild solution to (2.6). □

Remark 3.2 Notice that (3.15) has no meaning if $p = 2$. But we can re-estimate $J_5(t)$ if $p = 2$. Thus, we have also the following corollary.

Corollary 3.1 *Assume that* (H1)-(H3) *and* $\sup_{t\ge 0}\|\sigma(t)\|_{\mathcal{L}_2^0} < \infty$ *hold. If the following inequality*

$$\rho := 12M^2\lambda_1^{-2}L_G^2 + 12M^2\lambda_2^{-2}L_f^2 + 12M^2\lambda_2^{-1}L_g^2 < 1 \tag{3.18}$$

and

$$J := 12M^2\lambda_1^{-2}b_G^2 + 12M^2\lambda_2^{-2}b_f^2 + 12M^2L_g^2\lambda_2^{-1}b_g^2 + 6C\sup_{t\geq 0}\|\sigma(s)\|^2_{\mathcal{L}_2^0}$$

hold, then $S = \{\phi \in H | \|\phi\|_H^2 \leq (1-\rho)^{-1}J\}$ *is a global attracting set of system* (2.6).

Proof We only need re-estimate $J_5(t)$. By

$$\begin{aligned} J_5(t) &= \mathbb{E}\left\|\int_0^t S(t-s)g(s,x_s)\,d\omega(s)\right\|^2 \\ &\leq M^2\int_0^t e^{-2\lambda_2(t-s)}\mathbb{E}\|g(s,x_s)\|^2_{\mathcal{L}_2^0}\,ds \\ &\leq 2M^2L_g^2\int_0^t e^{-\lambda_2(t-s)}E\|x_s\|_C^2\,ds + 2M^2L_g^2\lambda_2^{-1}b_g^2. \end{aligned} \tag{3.19}$$

Then, by letting $p = 2$ in (3.10)-(3.14) and (3.16) and using (3.19), we can obtain

$$\begin{aligned} \mathbb{E}\|x(t)\|^2 \leq{}& 6M^2e^{-\lambda_1 t}\mathbb{E}\|\varphi(0)\|_H^2 + 18M^2e^{-\lambda_2 t}\mathbb{E}\big(\|\phi_1\|_H^2 + L_G\|\varphi(0)\|_C^2 + b_G^2\big) \\ &+ 12M^2\lambda_1^{-1}L_G^2\int_0^t e^{-\lambda_1(t-s)}\mathbb{E}\|x_s\|_C^2\,ds + 12M^2\lambda_1^{-2}b_G^2 \\ &+ 12M^2\lambda_2^{-1}L_f^2\int_0^t e^{-\lambda_2(t-s)}\mathbb{E}\|x_s\|_C^2\,ds + 12M^2\lambda_2^{-2}b_f^2 \\ &+ 12M^2L_g^2\int_0^t e^{-\lambda_2(t-s)}E\|x_s\|_C^2\,ds + 12M^2L_g^2\lambda_2^{-1}b_g^2 + 6C\sup_{t\geq 0}\|\sigma(s)\|^2_{\mathcal{L}_2^0}. \end{aligned} \tag{3.20}$$

Similar to the proof of Theorem 3.1, by using Lemma 3.1, we can deduce that the desired results are true. The proof is complete. □

4 Exponential decay

In this section, we shall focus on the exponential decay in the pth moment of the mild solution of equation (2.6).

Definition 4.1 The mild solution of (2.6) with initial $\phi \in C([-r,0];H)$ is said to have *exponential decay in the pth moment* if there exists a pair of positive constants $\lambda > 0$ and $M = M(\varphi) \geq 0$ such that

$$\mathbb{E}\|x(t,\varphi)\|_H^p \leq Me^{-\lambda t}, \quad t \geq 0.$$

In particular, system (2.6) is said to have exponential decay in the mean square when $p = 2$.

Theorem 4.1 *Assume that* (H1)-(H3) *with* $b_G = b_f = b_g = 0$ *and*

$$\int_0^{+\infty} e^{\lambda_2 s}\|\sigma(s)\|^2_{\mathcal{L}_2^0}\,ds < \infty \tag{4.1}$$

hold, and the following inequality

$$\rho := 12^{p-1}M^p\lambda_1^{-p}L_G^p + 12^{p-1}M^p\lambda_2^{-p}L_f^p + 12^{p-1}M^pL_g^p\lambda_2^{p/2-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2(p-1)}\right)^{\frac{p}{2}-1} < 1 \tag{4.2}$$

holds for $p > 2$. *Then the mild solution of system* (2.6) *has exponential decay in the pth moment.*

Proof From (3.1), we have

$$\begin{aligned}
\mathbb{E}\|x(t)\|^p &= \mathbb{E}\bigg\| T(t)\varphi(0) + S(t)\big(\phi_1 - G(0,x_0)\big) \\
&\quad + \int_0^t T(t-s)G(s,x_s)\,ds + \int_0^t S(t-s)f(s,x_s)\,ds \\
&\quad + \int_0^t S(t-s)g(s,x_s)\,d\omega(s) + \int_0^t S(t-s)\sigma(s)\,dB_Q^{\hbar}(s)\bigg\|_H^p \\
&\le 6^{p-1}\mathbb{E}\big\|T(t)\varphi(0)\big\|_H^p + 6^{p-1}\mathbb{E}\big\|S(t)\big(\phi_1 - G(0,x_0)\big)\big\|_H^p \\
&\quad + 6^{p-1}\mathbb{E}\bigg\|\int_0^t T(t-s)G(s,x_s)\,ds\bigg\|_H^p + 6^{p-1}\mathbb{E}\bigg\|\int_0^t S(t-s)f(s,x_s)\,ds\bigg\|_H^p \\
&\quad + 6^{p-1}\mathbb{E}\big\|S(t-s)g(s,x_s)\,d\omega(s)\big\|_H^p + 6^{p-1}\mathbb{E}\bigg\|\int_0^t S(t-s)\sigma(s)\,dB_Q^{\hbar}(s)\bigg\|_H^p \\
&:= 6^{p-1}\big(J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) + J_6(t)\big).
\end{aligned} \tag{4.3}$$

Since for every $t > 0$, $\int_0^t S(t-s)\sigma(s)\,dB_Q^{\hbar}(s)$ is a centered Gaussian random variable and by Kahane-Khintchine's inequality, there exists a constant C_p such that

$$J_6(t) = \mathbb{E}\bigg\|\int_0^t S(t-s)\sigma(s)\,dB_Q^{\hbar}(s)\bigg\|^p \le C_p\left(\mathbb{E}\bigg\|\int_0^t S(t-s)\sigma(s)\,dB_Q^{\hbar}(s)\bigg\|^2\right)^{p/2}.$$

Choosing suitable $\varepsilon > 0$ small enough such that $(\lambda_2 - \varepsilon)p \ge 2\lambda_2$ and $\delta := \lambda_2 - \varepsilon > 0$, by assumption (H2) and Lemma 2 of [13], we derive that

$$\begin{aligned}
\mathbb{E}\bigg\|\int_0^t S(t-s)\sigma(s)\,dB_Q^{\hbar}(s)\bigg\|^2 &\le M^2c_\hbar\hbar(2\hbar-1)t^{2\hbar-1}\int_0^t e^{-2\lambda_2(t-s)}\|\sigma(s)\|_{\mathcal{L}_2^0}^2\,ds \\
&\le M^2c_\hbar\hbar(2\hbar-1)t^{2\hbar-1}\int_0^t e^{-\lambda_2(t-s)}\|\sigma(s)\|_{\mathcal{L}_2^0}^2\,ds \\
&\le e^{-\delta t}M^2c_\hbar\hbar(2\hbar-1)t^{2\hbar-1}e^{-\varepsilon t}\int_0^t e^{\lambda_2 s}\|\sigma(s)\|_{\mathcal{L}_2^0}^2\,ds.
\end{aligned} \tag{4.4}$$

Therefore, (4.1) ensures the existence of a positive constant K such that

$$M^2c_\hbar\hbar(2\hbar-1)t^{2\hbar-1}e^{-\varepsilon t}\int_0^t e^{\lambda_2 s}\|\sigma(s)\|_{\mathcal{L}_2^0}^2\,ds \le K \quad \text{for all } t \ge 0.$$

Then

$$J_6(t) \le C_p K^{p/2} e^{-\lambda_2 t}. \tag{4.5}$$

Notice that $b_G = b_f = b_g = 0$, then substituting (3.11)-(3.15) and (4.5) into (4.3), we obtain

$$\begin{aligned}\mathbb{E}\|x(t)\|^p \le{}& 6^{p-1}M^p e^{-\lambda_1 t}\mathbb{E}\|\varphi(0)\|_H^p \\ &+ 12^{p-1}M^p e^{-\lambda_2 t}\mathbb{E}\big(\|\phi_1\|_H^p + L_G\|\varphi(0)\|_C^p\big) + 6^{p-1}C_p K^{p/2}e^{-\lambda_2 t} \\ &+ 12^{p-1}M^p\lambda_1^{1-p}L_G^p\int_0^t e^{-\lambda_1(t-s)}\mathbb{E}\|x_s\|_C^p\,ds \\ &+ 12^{p-1}M^p\lambda_2^{1-p}L_f^p\int_0^t e^{-\lambda_2(t-s)}\mathbb{E}\|x_s\|_C^p\,ds \\ &+ 12^{p-1}M^p L_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda_2(p-1)}\right)^{\frac{p}{2}-1}\int_0^t e^{-\lambda_2(t-s)}E\|x_s\|_C^p\,ds. \end{aligned} \tag{4.6}$$

Let $b_1 := 6^{p-1}M^p\mathbb{E}\|\varphi(0)\|_H^p$, $b_2 := 12^{p-1}M^p\mathbb{E}(\|\phi_1\|_H^p + L_G\|\varphi(0)\|_C^p) + 6^{p-1}C_pK^{p/2}$, $b_3 := 12^{p-1}M^p\lambda_1^{1-p}L_G^p$, and

$$b_4 := 12^{p-1}M^p\lambda_2^{1-p}L_f^p + 12^{p-1}M^pL_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda_2(p-1)}\right)^{\frac{p}{2}-1}.$$

From (4.2), we know $\rho := \frac{b_3}{\lambda_1} + \frac{b_4}{\lambda_2} < 1$. Since $\varphi \in PC_{\mathcal{F}_0}^B([-r,0];H)$, so there exist $N > 0$, $\lambda \in (0, \lambda_1 \wedge \lambda_2)$ such that

$$\|\varphi(0)\|_C < N, \quad \text{and} \quad \frac{b_1 + b_2}{N} + e^{\lambda r}\frac{b_3}{\lambda_1 - \lambda} + e^{\lambda r}\frac{b_4}{\lambda_2 - \lambda} < 1.$$

It follows from Lemma 3.1 that

$$\mathbb{E}\|x(t)\|_H^p \le Ne^{-\lambda t}.$$

The proof is complete. □

Similar to Corollary 3.1, we also have the following Corollary 4.1.

Corollary 4.1 *Assume that* (H1)-(H3) *are satisfied with* $b_G = b_f = b_g = 0$,

$$\int_0^{+\infty} e^{\lambda_2 s}\|\sigma(s)\|_{\mathcal{L}_2^0}^2\,ds < \infty \tag{4.7}$$

and the following inequality

$$\rho := 12M^2\lambda_1^{-2}L_G^2 + 12M^2\lambda_2^{-2}L_f^2 + 12M^2\lambda_2^{-1}L_g^2 < 1 \tag{4.8}$$

holds. Then the mild solution of system (2.6) *has exponential decay in the mean square.*

Proof We only need re-estimate $J_5(t)$. By Burkhölder-Davis-Gundy's inequality, we have

$$\begin{aligned} J_5(t) &= \mathbb{E}\left\| \int_0^t S(t-s)g(s,x_s)\,d\omega(s)\right\|^2 \\ &\le M^2\int_0^t e^{-2\lambda_2(t-s)}\mathbb{E}\left\|g(s,x_s)\right\|^2_{\mathcal{L}_2^0}\,ds \\ &\le 2M^2L_g^2\int_0^t e^{-\lambda_2(t-s)}E\|x_s\|_C^2\,ds. \end{aligned} \tag{4.9}$$

Notice that $b_G = b_f = b_g = 0$, then let $p = 2$ in (3.10)-(3.14) and (4.5), we can obtain

$$\begin{aligned} \mathbb{E}\left\|x(t)\right\|^2 \le{}& 6M^2e^{-\lambda_1 t}\mathbb{E}\left\|\varphi(0)\right\|_H^2 \\ &+ 12M^2e^{-\lambda_2 t}\mathbb{E}\left(\|\phi_1\|_H^2 + L_G\left\|\varphi(0)\right\|_C^2\right) \\ &+ 12M^2\lambda_1^{-1}L_G^2\int_0^t e^{-\lambda_1(t-s)}\mathbb{E}\|x_s\|_C^2\,ds \\ &+ 12M^2\lambda_2^{-1}L_f^2\int_0^t e^{-\lambda_2(t-s)}\mathbb{E}\|x_s\|_C^2\,ds \\ &+ 12M^2L_g^2\int_0^t e^{-\lambda_2(t-s)}E\|x_s\|_C^2\,ds + Ke^{-\lambda_2 t}. \end{aligned} \tag{4.10}$$

Similar to the proof of Theorem 4.1, by using Lemma 3.1, we can deduce that the desired results are true. The proof is complete. □

5 Example

Example 5.1 We consider the following second-order neutral stochastic partial functional differential equation driven by the fractional Brownian motion:

$$\begin{cases} \partial\left[\frac{\partial}{\partial t}u(t,x) + \nu_1 u(t,x(t-1))\right] & \\ \quad = \left[\frac{\partial^2}{\partial x^2}u(t,x) + \nu_2 u(t,x(t-1))\right]dt & \\ \qquad + \nu_3 u(t,x(t-1))\,d\omega(t) + \sigma(t)\,dB_Q^{\hbar}(t), & 0 \le x \le \pi, t \ge 0, \\ u(t,x) = \varphi(t,x), & s \in [-1,0], x \in [0,\pi], \\ & u(t,0) = u(t,\pi) = 0, \\ \frac{\partial}{\partial t}u(0,x) = u_1(x), & x \in [0,\pi], \end{cases} \tag{5.1}$$

where $\nu_i \ge 0$, $i = 1,2,3$, are constants, $\varphi(s,\cdot) \in H$, $\varphi(\cdot,x) \in BC^b_{\mathcal{F}_0}([-1,0];H)$, $H = L^2[0,\pi]$, $\omega(t)$ is a Wiener process and $B_Q^{\hbar}(t)$ is a fractional Brownian motion.

Define the infinitesimal operator $A : H \to H$ by $A = \frac{\partial^2}{\partial X^2}$, whose domain $D(A) = \{x \in H : x, x'$ are absolutely continuous, $x(0) = 0, x' \in H\}$, then

$$Ax = -\sum_{n=1}^{\infty} n^2\langle x, e_n\rangle_H e_n, \quad x \in D(A),$$

where $e_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$, $n = 1, 2, \ldots$, is a complete orthonormal set of eigenvectors of A.

Let

$$G\big(t,x(t-1)\big) = \nu_1 u\big(t,x(t-1)\big),$$
$$f\big(t,x(t-1)\big) = \nu_2 u\big(t,x(t-1)\big),$$
$$g\big(t,x(t-1)\big) = \nu_3 u\big(t,x(t-1)\big).$$

Then we can get (H1) is satisfied with $M = 1$ and $\lambda_1 = \lambda_2 = \pi$, and (H3) is satisfied with $L_G = \nu_1$, $L_f = \nu_2$, $L_g = \nu_3$ and $b_G = b_f = b_g = 0$. Assume that $\sigma(t)$ satisfies assumption (H2) such that $\sup_{t\geq 0} \|\sigma(s)\|^2_{\mathcal{L}_2^0} < \infty$. Thus, by virtue of Corollary 3.1, we know that

$$S_1 = \Big\{x \in H : \|x\|_H \leq \sqrt{(1-\hat{\rho})^{-1}\hat{J}}\Big\}$$

is a global attracting set of system (5.1) with $\hat{J} = 6C \sup_{t\geq 0} \|\sigma(s)\|^2_{\mathcal{L}_2^0}$ provided that $\hat{\rho} := 12\pi^{-2}\nu_1 + 12\pi^{-2}\nu_2 + 12\pi^{-1}\nu_3 < 1$.

In addition, if $\int_0^{+\infty} e^{\lambda_2 s}\|\sigma(s)\|^2_{\mathcal{L}_2^0}\,ds < \infty$ and $\hat{\rho} < 1$, then by Corollary 4.1, we know the mild solution of system (5.1) has exponential decay in the mean square.

6 Conclusion

In this paper, by establishing new integral inequalities, we obtain the global attracting sets and exponential decay of second-order neutral stochastic functional differential equations driven by fBm with Hurst parameter $H \in (1/2,1)$. By estimating the pth moment of fractional Brownian noise and using new integral inequalities, we obtain some sufficient conditions ensuring the exponential decay in the pth moment of the mild solution of the considered equations. In our next paper, we will explore the global attracting sets and the exponential decay in the pth moment of second-order neutral stochastic functional differential equations driven by the fractional Brownian motion with Hurst parameter $H \in (0,1/2)$ and the global attracting sets and the exponential decay in the pth moment of second-order neutral stochastic functional differential equations driven by a multifractional Brownian motion. Besides, we also will attempt to explore the global attracting sets and the exponential decay in the pth moment of second-order stochastic differential inclusions with time-delay.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
For the first time, we attempted to investigate the stability of second-order neutral stochastic functional differential equations driven by the fBm with parameter $\hbar \in (1/2,1)$, where the stochastic disturbance is the fBm rather than the Brownian motion. We obtain the global attracting sets and exponential decay of second-order neutral stochastic functional differential equations driven by the fBm by establishing new integral inequalities. All authors contributed equally to this work. All authors read and approved the final manuscript.

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Acknowledgements
The authors would like to thank the anonymous referees and editors for their careful comments and valuable suggestions on this work. This work is partially supported by the NNSF of China No. 61673006 and the Natural Science Foundation of Hubei Province (No. 2016CFB479).

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Received: 9 February 2017 Accepted: 19 April 2017 Published online: 10 May 2017

References

1. Ren, Y, Sakthivel, R: Existence, uniqueness, and stability of mild solutions for second-order neutral stochastic evolution equations with infinite delay and Poisson jumps. J. Math. Phys. **53**, 073517 (2012)
2. Liang, F, Guo, Z: Asymptotic behavior for second order stochastic evolution equations with memory. J. Math. Anal. Appl. **49**, 1333-1350 (2014)
3. Arthi, G, Park, JH, Jung, HY: Exponential stability for second-order neutral stochastic differential equations with impulse. Int. J. Control **88**(6), 1300-1309 (2015)
4. Mandelbrot, BB, Van Ness, J: Fractional Brownian motion, fractional noises and applications. SIAM Rev. **10**, 422-437 (1968)
5. Comte, F, Renault, E: Long memory continuous time models. J. Econom. **73**, 101-149 (1996)
6. Boudrahem, S, Rougier, PR: Relation between postural control assessment with eyes open and centre of pressure visual feed back effects in healthy individuals. Exp. Brain Res. **195**, 145-152 (2009)
7. De La, F, Perez-Samartin, AL, Matnez, L, Garcia, MA, Vera-Lopez, A: Long-range correlations in rabbit brain neural activity. Ann. Biomed. Eng. **34**, 295-299 (2006)
8. Willinger, W, Leland, W, Taqqu, M, Wilson, D: On self-similar nature of ethernet traffic. IEEE/ACM Trans. Netw. **2**, 1-15 (1994)
9. Rypdal, M, Rypdal, K: Testing hypotheses about sun-climate complexity linking. Phys. Rev. Lett. **104**, 128-151 (2010)
10. Simonsen, I: Measuring anti-correlations in the nordic electricity spot market by wavelets. Physica A **322**, 597-606 (2003)
11. Bahuguna, D, Sakthivel, R, Chadha, A: Asymptotic stability of fractional impulsive neutral stochastic partial integro-differential equations with infinite delay. Stoch. Anal. Appl. **35**(1), 63-68 (2017)
12. Boufoussi, B, Hajji, S: Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space. Stat. Probab. Lett. **82**(8), 1549-1558 (2012)
13. Caraballo, T, Garrido-Atienza, MJ, Taniguchi, T: The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion. Nonlinear Anal. **74**, 3671-3684 (2011)
14. Duncan, TE, Maslowski, B, Pasik-Duncan, B: Fractional Brownian motion and stochastic equations in Hilbert spaces. Stoch. Dyn. **2**, 225-250 (2002)
15. Gu, YF, Ren, Y, Sakthivel, R: Square-mean pseudo almost automorphic mild solutions for stochastic evolution equations driven by G-Brownian motion. Stoch. Anal. Appl. **34**(3), 528-545 (2016)
16. Maslowski, B, Nualart, D: Evolution equations driven by a fractional Brownian motion. J. Funct. Anal. **202**, 277-305 (2003)
17. Ren, Y, Chen, X, Sakthivel, R: Impulsive neutral stochastic functional integro-differential equations with infinite delay driven by fBm. Appl. Math. Comput. **247**(15), 205-212 (2014)
18. Xu, DY: Invariant and attracting sets of Volterra difference equations with delays. Comput. Math. Appl. **45**, 1311-1317 (2003)
19. Li, DS, Xu, DY: Attracting and quasi-invariant sets of stochastic neutral partial functional differential equations. Acta Math. Sci. **33**, 578-588 (2013)
20. Wang, L, Li, D: Impulsive-integral inequalities for attracting and quasi-invariant sets of impulsive stochastic partial differential equations with infinite delays. J. Inequal. Appl. **2013**, 338 (2013)
21. Li, Z: Global attracting and quasi-invariant sets of impulses neutral stochastic functional differential equations driven by fBm. Neurocomputing **117**, 620-627 (2016)
22. Alòs, E, Mazet, O, Nualart, D: Stochastic calculus with respect to Gaussian processes. Ann. Probab. **29**, 766-801 (2001)
23. Henríquez, HR: Existence of solutions of nonautonomous second order functional differential equations with infinite delay. Nonlinear Anal. **74**, 3333-3352 (2011)
24. Revathi, P, Sakthivel, R, Song, DY, Ren, Y, Zhang, P: Existence and stability results for second-order stochastic equations driven by fractional Brownian motion. Transp. Theory Stat. Phys. **42**, 299-317 (2013)