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On the asymptotic and oscillatory behavior of solutions of third-order neutral dynamic equations on time scales

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Abstract

The oscillatory and asymptotic behavior results for a class of third-order nonlinear neutral dynamic equations on time scales are presented. The results obtained can be extended to more general third-order neutral dynamic equations of the type considered here. Examples are provided to illustrate the applicability of the results.

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1 Introduction

This work is concerned with oscillation and asymptotic behavior of solutions to a third-order nonlinear neutral dynamic equation

$$\bigl(r(t)\bigl(\bigl(x(t)+p(t)x\bigl(g(t)\bigr)\bigr)^{\Delta\Delta}\bigr)^{\alpha}\bigr)^{\Delta}+f\bigl(t,x\bigl(h(t)\bigr)\bigr)=0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where α is a quotient of odd positive integers. Since we are interested in the oscillation and asymptotic behavior of solutions for large t, we assume that $\sup\mathbb{T}=\infty$ and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$ with $t_0\in\mathbb{T}$. The usual notation and concepts from the time scale calculus as can be found in Bohner and Peterson [1, 2] will be used throughout the paper without further mention.

In the rest of the paper we assume that:

(C1) $r:[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ is a positive rd-continuous function and $\int_{t_0}^{\infty}r^{-1/\alpha}(s)\Delta s=\infty$;

(C2) $p:[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ is an rd-continuous function with $p(t)\geq 1$, and $p(t)\not\equiv 1$ for large t;

(C3) $g,h:[t_0,\infty)_{\mathbb{T}}\to\mathbb{T}$ are rd-continuous functions such that $g(t)<t$, g is strictly increasing, and $\lim_{t\to\infty}g(t)=\lim_{t\to\infty}h(t)=\infty$;

(C4) $f(t,u):[t_0,\infty)_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $uf(t,u)>0$ for all $u\neq 0$, and there exists a positive rd-continuous function $q:[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ such that $f(t,u)/u^{\alpha}\geq q(t)$.

The cases

$$g\bigl(\sigma(t)\bigr)\geq h(t) \tag{1.2}$$

and

$$g(\sigma(t)) \leq h(t) \tag{1.3}$$

are both considered.

Defining the function

$$z(t) = x(t) + p(t)x(g(t)), \tag{1.4}$$

equation (1.1) can be written as

$$\left(r(t)\left(z^{\Delta\Delta}(t)\right)^{\alpha}\right)^{\Delta} + f\left(t, x(h(t))\right) = 0. \tag{1.5}$$

Wherever we write '$t \geq t_n$' we mean '$t \in [t_n, \infty)_{\mathbb{T}}$'.

By a solution of (1.1) we mean a function $x \in C_{\mathrm{rd}}([t_x, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $z \in C^3_{\mathrm{rd}}([t_x, \infty)_{\mathbb{T}}, \mathbb{R})$ and $r(z^{\Delta\Delta})^{\alpha} \in C^1_{\mathrm{rd}}([t_x, \infty)_{\mathbb{T}}, \mathbb{R})$, and which satisfies equation (1.1) on $[t_x, \infty)_{\mathbb{T}}$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[t_x, \infty)_{\mathbb{T}}$, for some $t_x \geq t_0$, and $\sup\{|x(t)| : t \geq T_1\} > 0$ for every $T_1 \geq t_x$. Moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative, and it is nonoscillatory otherwise.

In recent years, there has been much research activity concerning the oscillation of solutions of various functional differential equations and functional dynamic equations on time scales, and we refer the reader to the papers [3–31] and the references therein as examples of recent results on this topic. In reviewing the literature, it becomes apparent that results on the oscillatory and asymptotic behavior of third-order neutral dynamic equations on time scales are relatively scarce and most such results are concerned with the cases where $0 \leq p(t) < p_0 < 1$, $-1 \leq p(t) < 0$, and/or $0 \leq \int_a^b p(t,\mu)\Delta\mu \leq p_0 < 1$; see, for example, [8, 10, 11, 15, 17, 18, 29] and the references cited therein as examples of recent results on this topic.

However, to the best of our knowledge, there does not appear to be any results for the third-order neutral dynamic equations on time scales in the case $p(t) \geq 1$. The main aim of this paper is to establish some new criteria for the oscillation and asymptotic behavior of solutions of (1.1) in the case $p(t) \geq 1$. It should be noted that the results in this paper are new even for $\alpha = 1$, $r(t) = 1$, $f(t, x(h(t))) = q(t)x^{\alpha}(h(t))$, and for the constant delays such as $g(t) = t - b$ and $h(t) = t - c$ with $b > 0$ and $c > 0$. Furthermore, the results in this paper can easily be extended to more general third-order neutral dynamic equations of the type (1.1). It is therefore hoped that the present paper will contribute significantly to the study of oscillatory and asymptotic properties of solutions of third-order neutral dynamic equations on time scales.

2 Main results

The next four lemmas will be used to prove our main results. For convenience, we will use the following notations:

$$\beta^{\Delta}_{+}(t) := \max\{0, \beta^{\Delta}(t)\},$$

$$R_1(t,t_1) := \int_{t_1}^{t} \frac{\Delta s}{r^{1/\alpha}(s)} \quad \text{for } t \geq t_1, \qquad R_2(t,t_2) := \int_{t_2}^{t} R_1(s,t_1)\Delta s \quad \text{for } t \geq t_2 > t_1,$$

$$\psi(t) := \begin{cases} \theta(t), & 0 < \alpha \leq 1, \\ \theta^{\alpha}(t), & \alpha > 1, \end{cases} \qquad \theta(t) := \frac{R_1(t,t_1)}{R_1(\sigma(t),t_1)}.$$

Throughout this paper, we assume that

$$\xi_1(t) := \frac{1}{p(g^{-1}(t))}\left(1 - \frac{1}{p(g^{-1}(g^{-1}(t)))}\right) > 0 \tag{2.1}$$

and

$$\xi_2(t) := \frac{1}{p(g^{-1}(t))}\left(1 - \frac{1}{p(g^{-1}(g^{-1}(t)))} \frac{R_2(g^{-1}(g^{-1}(t)),t_2)}{R_2(g^{-1}(t),t_2)}\right) > 0, \tag{2.2}$$

for all sufficiently large t, where g^{-1} is the inverse function of g.

$$\Psi_1(t) := \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(t)),t_2)}{R_1(\sigma(t),t_1)}\right)^{\alpha}$$

and

$$\Psi_2(t) := \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(\sigma(t),t_2)}{R_1(\sigma(t),t_1)}\right)^{\alpha}.$$

Lemma 2.1 ([32]) *If U and V are nonnegative and $\lambda > 1$, then*

$$\lambda U V^{\lambda-1} - U^{\lambda} \leq (\lambda - 1)V^{\lambda},$$

where equality holds if and only if $U = V$.

Lemma 2.2 *Assume that conditions* (C1)-(C4) *hold, and let $x(t)$ be a positive solution of* (1.1) *with $z(t)$ defined as in* (1.4). *Then, for sufficiently large t, either*

(I) $z(t) > 0$, $z^{\Delta}(t) > 0$, $z^{\Delta\Delta}(t) > 0$, *and* $(r(t)(z^{\Delta\Delta}(t))^{\alpha})^{\Delta} < 0$, *or*

(II) $z(t) > 0$, $z^{\Delta}(t) < 0$, $z^{\Delta\Delta}(t) > 0$, *and* $(r(t)(z^{\Delta\Delta}(t))^{\alpha})^{\Delta} < 0$.

Proof The proof is standard and so the details will be omitted. □

Lemma 2.3 *Assume that conditions* (C1)-(C4) *and* (2.1) *hold, and let $x(t)$ be an eventually positive solution of* (1.1) *with $z(t)$ satisfying Case* (II) *of Lemma* 2.2. *If*

$$\int_{t_0}^{t}\int_{v}^{\infty} \frac{1}{r^{1/\alpha}(u)}\left(\int_{u}^{\infty} q(s)\bigl(\xi_1\bigl(h(s)\bigr)\bigr)^{\alpha}\Delta s\right)^{1/\alpha}\Delta u\Delta v = \infty, \tag{2.3}$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof Let $x(t)$ be an eventually positive solution of (1.1). Then there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) > 0$ for $t \geq t_1$. Now, in view of (C4), equation (1.1) or (1.5) takes the form

$$\bigl(r(t)\bigl(z^{\Delta\Delta}(t)\bigr)^{\alpha}\bigr)^{\Delta} + q(t)x^{\alpha}\bigl(h(t)\bigr) \leq 0 \quad \text{for } t \geq t_1. \tag{2.4}$$

From (1.4) (see also (8.6) in [3]), we have

$$\begin{aligned} x(t) &= \frac{1}{p(g^{-1}(t))}\big(z\big(g^{-1}(t)\big) - x\big(g^{-1}(t)\big)\big) \\ &= \frac{z(g^{-1}(t))}{p(g^{-1}(t))} - \frac{1}{p(g^{-1}(t))p(g^{-1}(g^{-1}(t)))}\big(z\big(g^{-1}\big(g^{-1}(t)\big)\big) - x\big(g^{-1}\big(g^{-1}(t)\big)\big)\big) \\ &\geq \frac{z(g^{-1}(t))}{p(g^{-1}(t))} - \frac{1}{p(g^{-1}(t))p(g^{-1}(g^{-1}(t)))}z\big(g^{-1}\big(g^{-1}(t)\big)\big). \end{aligned} \tag{2.5}$$

Since $z(t)$ is decreasing and $g(t) < t$, we get

$$z\big(g^{-1}(t)\big) \geq z\big(g^{-1}\big(g^{-1}(t)\big)\big),$$

and so from (2.5) we obtain

$$x(t) \geq \xi_1(t)z\big(g^{-1}(t)\big) \quad \text{for } t \geq t_1. \tag{2.6}$$

Since $\lim_{t\to\infty} h(t) = \infty$, we can choose $t_2 \geq t_1$ such that $h(t) \geq t_1$ for all $t \geq t_2$. Thus, from (2.6), we have

$$x\big(h(t)\big) \geq \xi_1\big(h(t)\big)z\big(g^{-1}\big(h(t)\big)\big) \quad \text{for } t \geq t_2. \tag{2.7}$$

In view of (2.7), equation (2.4) can be written as

$$\big(r(t)\big(z^{\Delta\Delta}(t)\big)^{\alpha}\big)^{\Delta} + q(t)\big(\xi_1\big(h(t)\big)\big)^{\alpha} z^{\alpha}\big(g^{-1}\big(h(t)\big)\big) \leq 0 \quad \text{for } t \geq t_2. \tag{2.8}$$

Since $z(t) > 0$ and $z^{\Delta}(t) < 0$, there exists a constant M such that

$$\lim_{t\to\infty} z(t) = M < \infty,$$

where $M \geq 0$. If $M > 0$, then there exists $t_3 \geq t_2$ such that $g^{-1}(h(t)) > t_2$ and

$$z(t) \geq M \quad \text{for } t \geq t_3. \tag{2.9}$$

Integrating (2.8) two times from t to ∞, we have

$$-z^{\Delta}(t) \geq M \int_t^{\infty} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{\infty} q(s)\big(\xi_1\big(h(s)\big)\big)^{\alpha}\,\Delta s\right)^{1/\alpha} \Delta u.$$

An integration of the last inequality from t_3 to t yields

$$z(t_3) \geq M \int_{t_3}^{t} \int_v^{\infty} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{\infty} q(s)\big(\xi_1\big(h(s)\big)\big)^{\alpha}\,\Delta s\right)^{1/\alpha} \Delta u \Delta v,$$

which contradicts (2.3), and so we have $M = 0$. Hence, $\lim_{t\to\infty} z(t) = 0$. Since $0 < x(t) \leq z(t)$ on $[t_1, \infty)_{\mathbb{T}}$, we get $\lim_{t\to\infty} x(t) = 0$. This completes the proof of Lemma 2.3. □

Lemma 2.4 *Assume that conditions* (C1)-(C4) *and* (2.2) *hold, and* $x(t)$ *is an eventually positive solution of* (1.1) *with* $z(t)$ *satisfying Case* (I) *of Lemma* 2.2. *Then* $z(t)$ *satisfies the inequality*

$$\bigl(r(t)\bigl(z^{\Delta\Delta}(t)\bigr)^{\alpha}\bigr)^{\Delta}+q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}z^{\alpha}\bigl(g^{-1}\bigl(h(t)\bigr)\bigr)\le 0 \tag{2.10}$$

for large t.

Proof Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t)>0$, $x(g(t))>0$, and $x(h(t))>0$, and $z(t)$ satisfies Case (I) for $t\ge t_1$ for some $t_1\ge t_0$. Proceeding exactly as in Lemma 2.3, we see that (2.4) and (2.5) hold. Since $r(t)(z^{\Delta\Delta}(t))^{\alpha}$ is decreasing, we have

$$\begin{aligned} z^{\Delta}(t) &= z^{\Delta}(t_1)+\int_{t_1}^{t}\frac{(r(s)(z^{\Delta\Delta}(s))^{\alpha})^{1/\alpha}}{r^{1/\alpha}(s)}\Delta s \\ &\ge \bigl(r(t)\bigl(z^{\Delta\Delta}(t)\bigr)^{\alpha}\bigr)^{1/\alpha}R_1(t,t_1)\quad \text{for } t\ge t_1. \end{aligned} \tag{2.11}$$

Thus

$$\left(\frac{z^{\Delta}(t)}{R_1(t,t_1)}\right)^{\Delta}\le 0. \tag{2.12}$$

Hence there exists $t_2\in[t_1,\infty)_{\mathbb{T}}$ such that

$$\begin{aligned} z(t) &= z(t_2)+\int_{t_2}^{t}\frac{z^{\Delta}(s)}{R_1(s,t_1)}R_1(s,t_1)\Delta s \\ &\ge \frac{R_2(t,t_2)}{R_1(t,t_1)}z^{\Delta}(t), \end{aligned} \tag{2.13}$$

which implies that

$$\left(\frac{z(t)}{R_2(t,t_2)}\right)^{\Delta}\le 0\quad \text{for } t\ge t_2. \tag{2.14}$$

Noting that $g^{-1}(t)\le g^{-1}(g^{-1}(t))$, we have by (2.14) that

$$\frac{R_2(g^{-1}(g^{-1}(t)),t_2)z(g^{-1}(t))}{R_2(g^{-1}(t),t_2)}\ge z\bigl(g^{-1}\bigl(g^{-1}(t)\bigr)\bigr). \tag{2.15}$$

Using (2.15) in (2.5), we obtain

$$x(t)\ge \xi_2(t)z\bigl(g^{-1}(t)\bigr)\quad \text{for } t\ge t_2. \tag{2.16}$$

Since $\lim_{t\to\infty}h(t)=\infty$, we can choose $t_3\ge t_2$ such that $h(t)\ge t_2$ for all $t\ge t_3$. Hence, from (2.16), we have

$$x\bigl(h(t)\bigr)\ge \xi_2\bigl(h(t)\bigr)z\bigl(g^{-1}\bigl(h(t)\bigr)\bigr)\quad \text{for } t\ge t_3. \tag{2.17}$$

Substituting (2.17) into (2.4), we arrive at (2.10) and this completes the proof of Lemma 2.4. □

We now give oscillation results for the case where (1.2) holds.

Theorem 2.1 *Assume that conditions* (C1)-(C4), (1.2), (2.1)-(2.3) *hold and there exists a positive function* $\beta \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_T^t\left[\Psi_1(s)-\frac{\beta_+^\Delta(s)}{(R_1(s,t_1))^\alpha}\right]\Delta s=\infty \tag{2.18}$$

for all sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *and for* $T > t_2 > t_1$. *Then a solution* x *of equation* (1.1) *either oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) > 0$, (2.1)-(2.2) hold, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$. We only consider this case, since the proof when x is eventually negative is similar.

If Case (II) holds, then by Lemma 2.3 we have $\lim_{t\to\infty} x(t) = 0$.

Next, assume that Case (I) holds, and proceeding as in the proof of Lemma 2.4, we again arrive at (2.10), (2.11), (2.12), (2.13), and (2.14). Now define the Riccati-type substitution

$$w(t)=\beta(t)\frac{r(t)(z^{\Delta\Delta}(t))^\alpha}{(z^\Delta(t))^\alpha}\quad\text{for } t\geq t_1. \tag{2.19}$$

Clearly, $w(t) > 0$, and from (2.19) and (2.10), we see that

$$\begin{aligned}w^\Delta(t)\leq{}&\beta_+^\Delta(t)\frac{r(t)(z^{\Delta\Delta}(t))^\alpha}{(z^\Delta(t))^\alpha}-\beta\big(\sigma(t)\big)q(t)\big(\xi_2\big(h(t)\big)\big)^\alpha\frac{z^\alpha(g^{-1}(h(t)))}{(z(\sigma(t)))^\alpha}\frac{(z(\sigma(t)))^\alpha}{(z^\Delta(\sigma(t)))^\alpha}\\&-\beta\big(\sigma(t)\big)\frac{r(t)(z^{\Delta\Delta}(t))^\alpha((z^\Delta(t))^\alpha)^\Delta}{(z^\Delta(t))^\alpha(z^\Delta(\sigma(t)))^\alpha}\quad\text{for } t\geq t_3.\end{aligned} \tag{2.20}$$

From $g^{-1}(h(t)) \leq \sigma(t)$, we obtain from (2.14) that

$$\frac{z(g^{-1}(h(t)))}{z(\sigma(t))}\geq\frac{R_2(g^{-1}(h(t)),t_2)}{R_2(\sigma(t),t_2)}. \tag{2.21}$$

By virtue of (2.13) and the fact that $t \leq \sigma(t)$, we have

$$\frac{z(\sigma(t))}{z^\Delta(\sigma(t))}\geq\frac{R_2(\sigma(t),t_2)}{R_1(\sigma(t),t_1)}. \tag{2.22}$$

Using (2.21) and (2.22) in (2.20), we get

$$\begin{aligned}w^\Delta(t)\leq{}&\beta_+^\Delta(t)\frac{r(t)(z^{\Delta\Delta}(t))^\alpha}{(z^\Delta(t))^\alpha}-\beta\big(\sigma(t)\big)q(t)\big(\xi_2\big(h(t)\big)\big)^\alpha\left(\frac{R_2(g^{-1}(h(t)),t_2)}{R_1(\sigma(t),t_1)}\right)^\alpha\\&-\beta\big(\sigma(t)\big)\frac{r(t)(z^{\Delta\Delta}(t))^\alpha((z^\Delta(t))^\alpha)^\Delta}{(z^\Delta(t))^\alpha(z^\Delta(\sigma(t)))^\alpha}.\end{aligned} \tag{2.23}$$

From [1], Theorem 1.90, we have

$$\big((z^\Delta(t))^\alpha\big)^\Delta\geq\begin{cases}\alpha(z^\Delta)^{\alpha-1}(\sigma(t))z^{\Delta\Delta}(t), & \text{if } 0<\alpha\leq 1,\\ \alpha(z^\Delta)^{\alpha-1}(t)z^{\Delta\Delta}(t), & \text{if } \alpha>1.\end{cases} \tag{2.24}$$

If $0 < \alpha \leq 1$, then we obtain by (2.23) and (2.24) that

$$w^{\Delta}(t) \leq \beta_{+}^{\Delta}(t)\frac{r(t)(z^{\Delta\Delta}(t))^{\alpha}}{(z^{\Delta}(t))^{\alpha}} - \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(t)),t_2)}{R_1(\sigma(t),t_1)}\right)^{\alpha} - \alpha\beta\bigl(\sigma(t)\bigr)\frac{r(t)(z^{\Delta\Delta}(t))^{\alpha+1}}{(z^{\Delta}(t))^{\alpha+1}}\frac{z^{\Delta}(t)}{z^{\Delta}(\sigma(t))}. \tag{2.25}$$

If $\alpha > 1$, then we obtain from (2.23) and (2.24) that

$$w^{\Delta}(t) \leq \beta_{+}^{\Delta}(t)\frac{r(t)(z^{\Delta\Delta}(t))^{\alpha}}{(z^{\Delta}(t))^{\alpha}} - \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(t)),t_2)}{R_1(\sigma(t),t_1)}\right)^{\alpha} - \alpha\beta\bigl(\sigma(t)\bigr)\frac{r(t)(z^{\Delta\Delta}(t))^{\alpha+1}}{(z^{\Delta}(t))^{\alpha+1}}\frac{(z^{\Delta}(t))^{\alpha}}{(z^{\Delta}(\sigma(t)))^{\alpha}}. \tag{2.26}$$

By the fact that $t \leq \sigma(t)$, it follows from (2.12) that

$$\frac{z^{\Delta}(t)}{z^{\Delta}(\sigma(t))} \geq \frac{R_1(t,t_1)}{R_1(\sigma(t),t_1)}. \tag{2.27}$$

Next, in view of (2.27), combining (2.25) and (2.26), we obtain, for $\alpha > 0$,

$$w^{\Delta}(t) \leq \beta_{+}^{\Delta}(t)\frac{r(t)(z^{\Delta\Delta}(t))^{\alpha}}{(z^{\Delta}(t))^{\alpha}} - \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(t)),t_2)}{R_1(\sigma(t),t_1)}\right)^{\alpha} - \alpha\beta\bigl(\sigma(t)\bigr)\psi(t)\frac{r(t)(z^{\Delta\Delta}(t))^{\alpha+1}}{(z^{\Delta}(t))^{\alpha+1}} \quad \text{for } t \geq t_3. \tag{2.28}$$

From (2.11), we have

$$\frac{r(t)(z^{\Delta\Delta}(t))^{\alpha}}{(z^{\Delta}(t))^{\alpha}} \leq \frac{1}{(R_1(t,t_1))^{\alpha}}. \tag{2.29}$$

Thus, from (2.29), $z^{\Delta}(t) > 0$ and $z^{\Delta\Delta}(t) > 0$, inequality (2.28) yields

$$w^{\Delta}(t) \leq \frac{\beta_{+}^{\Delta}(t)}{R_1^{\alpha}(t,t_1)} - \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(t)),t_2)}{R_1(\sigma(t),t_1)}\right)^{\alpha}. \tag{2.30}$$

An integration of (2.30) from t_3 to t yields

$$\int_{t_3}^{t}\left[\beta\bigl(\sigma(s)\bigr)q(s)\bigl(\xi_2\bigl(h(s)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(s)),t_2)}{R_1(\sigma(s),t_1)}\right)^{\alpha} - \frac{\beta_{+}^{\Delta}(s)}{R_1^{\alpha}(s,t_1)}\right]\Delta s \leq w(t_3),$$

which contradicts (2.18), therefore any solution $x(t)$ of equation (1.1) is either oscillatory or tends to zero as $t \to \infty$. This completes the proof of Theorem 2.1. □

Theorem 2.2 *Suppose that conditions* (C1)-(C4), (1.2), *and* (2.1)-(2.3) *are satisfied. If there exists a positive function* $\beta \in C_{\mathrm{rd}}^{1}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{T}^{t}\left[\Psi_1(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)(\beta_{+}^{\Delta}(s))^{\alpha+1}}{[\beta(\sigma(s))\psi(s)]^{\alpha}}\right]\Delta s = \infty \tag{2.31}$$

for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *and for* $T > t_2 > t_1$, *then any solution of* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_{0,}\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) > 0$, (2.1)-(2.2) hold, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$. We only consider this case since the proof when x is eventually negative is similar.

If Case (II) holds, then by Lemma 2.3 we have $\lim_{t\to\infty} x(t) = 0$.

Assume that Case (I) holds. Proceeding exactly as in Theorem 2.1, we again arrive at (2.28). In view of (2.19), inequality (2.28) takes the form

$$\begin{aligned} w^{\Delta}(t) \leq{}& \frac{\beta_+^{\Delta}(t)}{\beta(t)} w(t) - \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(t)), t_2)}{R_1(\sigma(t), t_1)}\right)^{\alpha} \\ &- \alpha\beta\bigl(\sigma(t)\bigr)\psi(t)\frac{1}{\beta^{(\alpha+1)/\alpha}(t)r^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t) \quad \text{for } t \geq t_3. \end{aligned} \tag{2.32}$$

If we apply Lemma 2.1 with

$$U = \frac{[\alpha\beta(\sigma(t))\psi(t)]^{1/\lambda}}{[\beta^{\lambda}(t)r^{1/\alpha}(t)]^{1/\lambda}} w(t), \qquad \lambda = \frac{\alpha+1}{\alpha},$$

and

$$V = \left[\frac{\alpha}{\alpha+1}\frac{[\beta^{\lambda}(t)r^{1/\alpha}(t)]^{1/\lambda}}{[\alpha\beta(\sigma(t))\psi(t)]^{1/\lambda}}\frac{\beta_+^{\Delta}(t)}{\beta(t)}\right]^{\alpha},$$

we see that

$$\frac{\beta_+^{\Delta}(t)}{\beta(t)} w(t) - \alpha\beta\bigl(\sigma(t)\bigr)\psi(t)\frac{1}{\beta^{(\alpha+1)/\alpha}(t)r^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t) \leq \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(t)(\beta_+^{\Delta}(t))^{\alpha+1}}{(\beta(\sigma(t))\psi(t))^{\alpha}},$$

substituting this into (2.32) gives

$$w^{\Delta}(t) \leq \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(t)(\beta_+^{\Delta}(t))^{\alpha+1}}{(\beta(\sigma(t))\psi(t))^{\alpha}} - \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(t)), t_2)}{R_1(\sigma(t), t_1)}\right)^{\alpha}.$$

Integrating the last inequality from t_3 to t yields

$$\int_{t_3}^{t}\left[\Psi_1(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)(\beta_+^{\Delta}(s))^{\alpha+1}}{[\beta(\sigma(s))\psi(s)]^{\alpha}}\right]\Delta s \leq w(t_3),$$

which contradicts (2.31), therefore any solution $x(t)$ of equation (1.1) is either oscillatory or tends to zero as $t \to \infty$. This proves the theorem. □

Theorem 2.3 *Suppose that* $\alpha \geq 1$, *and conditions* (C1)-(C4), (1.2) *and* (2.1)-(2.3) *hold. If there is a positive function* $\beta \in C_{\mathrm{rd}}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_T^t\left[\Psi_1(s) - \frac{r^{1/\alpha}(s)(\beta_+^{\Delta}(s))^2}{4\alpha\beta(\sigma(s))\psi(s)[R_1(s, t_1)]^{\alpha-1}}\right]\Delta s = \infty \tag{2.33}$$

for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *and for* $T > t_2 > t_1$, *then any solution of* (1.1) *either oscillates or converges to zero as* $t \to \infty$.

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_{0,}\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) > 0$, (2.1)-(2.2) hold, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$. We only consider this case, since the proof when x is eventually negative is similar.

If Case (II) holds, then from Lemma 2.3 we have $\lim_{t\to\infty} x(t) = 0$.

Assume that Case (I) holds and proceeding as in the proof of Theorem 2.2, we again arrive at (2.32), which can be rewritten as

$$\begin{aligned} w^{\Delta}(t) \leq{}& \frac{\beta_+^{\Delta}(t)}{\beta(t)} w(t) - \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(t)),t_2)}{R_1(\sigma(t),t_1)}\right)^{\alpha} \\ & - \alpha\beta\bigl(\sigma(t)\bigr)\psi(t)\frac{w^{\frac{1}{\alpha}-1}(t)}{\beta^{(\alpha+1)/\alpha}(t)r^{1/\alpha}(t)} w^2(t) \quad \text{for } t \geq t_3. \end{aligned} \tag{2.34}$$

By (2.11) and (2.19), we see that

$$\begin{aligned} w^{\frac{1}{\alpha}-1}(t) &= \bigl(\beta(t)r(t)\bigr)^{\frac{1}{\alpha}-1}\left(\frac{(z^{\Delta\Delta}(t))^{\alpha}}{(z^{\Delta}(t))^{\alpha}}\right)^{\frac{1}{\alpha}-1} \\ &\geq \bigl(\beta(t)r(t)\bigr)^{\frac{1}{\alpha}-1}\bigl[r^{1/\alpha}(t)R_1(t,t_1)\bigr]^{\alpha-1} \\ &= \beta^{\frac{1}{\alpha}-1}(t)\bigl[R_1(t,t_1)\bigr]^{\alpha-1}. \end{aligned} \tag{2.35}$$

Using (2.35) in (2.34), we conclude that

$$\begin{aligned} w^{\Delta}(t) \leq{}& \frac{\beta_+^{\Delta}(t)}{\beta(t)} w(t) - \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(t)),t_2)}{R_1(\sigma(t),t_1)}\right)^{\alpha} \\ & - \frac{\alpha\beta(\sigma(t))\psi(t)[R_1(t,t_1)]^{\alpha-1}}{\beta^2(t)r^{1/\alpha}(t)} w^2(t) \quad \text{for } t \geq t_3. \end{aligned} \tag{2.36}$$

Completing square with respect to w, it follows from (2.36) that

$$w^{\Delta}(t) \leq -\beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(g^{-1}(h(t)),t_2)}{R_1(\sigma(t),t_1)}\right)^{\alpha} + \frac{r^{1/\alpha}(t)(\beta_+^{\Delta}(t))^2}{4\alpha\beta(\sigma(t))\psi(t)[R_1(t,t_1)]^{\alpha-1}}.$$

Integrating this inequality from t_3 to t gives

$$\int_{t_3}^{t}\left[\Psi_1(s) - \frac{r^{1/\alpha}(s)(\beta_+^{\Delta}(s))^2}{4\alpha\beta(\sigma(s))\psi(s)[R_1(s,t_1)]^{\alpha-1}}\right]\Delta s \leq w(t_3),$$

which contradicts (2.33), therefore any solution $x(t)$ of equation (1.1) is either oscillatory or tends to zero as $t \to \infty$. The proof is complete. □

Next, we give oscillation results in the case where (1.3) holds.

Theorem 2.4 *Assume that conditions* (C1)-(C4), (1.3), *and* (2.1)-(2.3) *are satisfied. If there is a positive function* $\beta \in C_{\mathrm{rd}}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{T}^{t}\left[\Psi_2(s) - \frac{\beta_+^{\Delta}(s)}{(R_1(s,t_1))^{\alpha}}\right]\Delta s = \infty \tag{2.37}$$

for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *and for* $T > t_2 > t_1$, *then a solution* x *of equation* (1.1) *either oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) > 0$, (2.1)-(2.2) hold, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$. We only consider this case, since the proof when x is eventually negative is similar.

If Case (II) holds, then from Lemma 2.3 we have $\lim_{t\to\infty} x(t) = 0$.

Assume that Case (I) holds. Proceeding as in the proof of Theorem 2.1, we again arrive at (2.20) and (2.22). Since

$$\sigma(t) \leq g^{-1}\bigl(h(t)\bigr), \tag{2.38}$$

we get

$$\frac{z(g^{-1}(h(t)))}{z(\sigma(t))} \geq 1. \tag{2.39}$$

Using (2.39) and (2.22) in (2.20), we obtain

$$\begin{aligned} w^{\Delta}(t) \leq{}& \beta_+^{\Delta}(t)\frac{r(t)(z^{\Delta\Delta}(t))^{\alpha}}{(z^{\Delta}(t))^{\alpha}} - \beta\bigl(\sigma(t)\bigr)q(t)\bigl(\xi_2\bigl(h(t)\bigr)\bigr)^{\alpha}\left(\frac{R_2(\sigma(t),t_2)}{R_1(\sigma(t),t_1)}\right)^{\alpha} \\ &- \beta\bigl(\sigma(t)\bigr)\frac{r(t)(z^{\Delta\Delta}(t))^{\alpha}((z^{\Delta}(t))^{\alpha})^{\Delta}}{(z^{\Delta}(t))^{\alpha}(z^{\Delta}(\sigma(t)))^{\alpha}} \quad \text{for } t \geq t_3. \end{aligned} \tag{2.40}$$

The remainder of the proof is similar to that of Theorem 2.1, and so the details are omitted. □

Theorem 2.5 *Assume that conditions* (C1)-(C4), (1.3), (2.1)-(2.3) *hold and there exists a positive function* $\beta \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_T^t \left[\Psi_2(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)(\beta_+^{\Delta}(s))^{\alpha+1}}{[\beta(\sigma(s))\psi(s)]^{\alpha}}\right]\Delta s = \infty \tag{2.41}$$

for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *and for* $T > t_2 > t_1$. *Then any solution of* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof The proof follows easily from (2.39) and Theorem 2.2. □

Theorem 2.6 *Let* $\alpha \geq 1$ *and conditions* (C1)-(C4), (1.3) *and* (2.1)-(2.3) *hold. If there exists a positive function* $\beta \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_T^t \left[\Psi_2(s) - \frac{r^{1/\alpha}(s)(\beta_+^{\Delta}(s))^2}{4\alpha\beta(\sigma(s))\psi(s)[R_1(s,t_1)]^{\alpha-1}}\right]\Delta s = \infty \tag{2.42}$$

for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *and for* $T > t_2 > t_1$, *then a solution of* (1.1) *is either oscillatory or converges to zero as* $t \to \infty$.

Proof The proof follows easily from (2.39) and Theorem 2.3. □

Remark 2.1 In condition (C3), if condition $g(t) < t$ is replaced by $g(t) > t$, then again results analogous to Theorems 2.1-2.6 can easily be modeled.

Example 2.1 Let $\mathbb{T} := \overline{q^{\mathbb{Z}}} = \{q^k : k \in \mathbb{Z}, q > 1\} \cup \{0\}$ and consider the third-order neutral dynamic equation

$$\bigl[\bigl(\bigl(x(t) + 5x(t/2)\bigr)^{\Delta\Delta}\bigr)^3\bigr]^\Delta + t(t-1)^3 x^3(t/2) = 0, \quad t \in \overline{2^{\mathbb{Z}}}, t \geq t_0 := 2. \tag{2.43}$$

Here we have $\alpha = 3$, $g(t) = h(t) = t/2$, $q(t) = t(t-1)^3$, $r(t) = 1$, $f(t,u) = q(t)u^\alpha$, and $p(t) = 8$. It is clear that conditions (C1)-(C4) and (1.2) hold, and

$$\xi_1(t) = 7/64 > 0. \tag{2.44}$$

Since

$$1 - \frac{1}{p(g^{-1}(g^{-1}(t)))} \frac{R_2(g^{-1}(g^{-1}(t)), t_2)}{R_2(g^{-1}(t), t_2)} = \frac{2t-7}{4t-8},$$

we see that

$$\xi_2(t) \geq \frac{1}{64} \quad \text{for } t \geq t_2 = 4. \tag{2.45}$$

By (2.44), condition (2.3) becomes

$$\begin{aligned}
&\int_{t_0}^t \int_v^\infty \frac{1}{r^{1/\alpha}(u)} \left(\int_u^\infty q(s)\bigl(\xi_1\bigl(h(s)\bigr)\bigr)^\alpha \Delta s \right)^{1/\alpha} \Delta u \Delta v \\
&\quad = \int_2^t \int_v^\infty \left(\int_u^\infty s(s-1)^3 (7/64)^3 \Delta s \right)^{1/3} \Delta u \Delta v = \infty
\end{aligned}$$

due to $\int_u^\infty s(s-1)^3 \Delta s = \infty$ for $u \geq 2$, and so condition (2.3) holds.

With $\beta(t) = t$ and the fact that (2.45), we see that

$$\begin{aligned}
&\int_T^t \left[\beta\bigl(\sigma(s)\bigr) q(s) \bigl(\xi_2\bigl(h(s)\bigr)\bigr)^\alpha \left(\frac{R_2(g^{-1}(h(s)), t_2)}{R_1(\sigma(s), t_1)} \right)^\alpha - \frac{\beta_+^\Delta(s)}{(R_1(s,t_1))^\alpha} \right] \Delta s \\
&\quad \geq \int_8^t \left[2s^2(s-1)^3 (1/64)^3 \left(\frac{s^2 - 6s + 8}{6s-6} \right)^3 - \frac{1}{(s-2)^3} \right] \Delta s \\
&\quad = \int_8^t \left[2(1/384)^3 s^2 \bigl(s^2 - 6s + 8\bigr)^3 - \frac{1}{(s-2)^3} \right] \Delta s = \infty,
\end{aligned}$$

due to $\int_8^t \frac{1}{(s-2)^3} \Delta s < \infty$ and $\int_8^t s^2(s^2 - 6s + 8)^3 \Delta s = \infty$, so condition (2.18) holds. Thus, all conditions of Theorem 2.1 are satisfied. Therefore, by Theorem 2.1, any solution of (2.43) is either oscillatory or converges to zero.

Example 2.2 Consider the neutral differential equation

$$\left(\frac{1}{t^{1/3}} \left(\left(x(t) + \frac{16t+17}{t+1} x\left(\frac{t}{2}\right) \right)'' \right)^{1/3} \right)' + \bigl(t^2 + 5\bigr) x^{1/3}(t-1) = 0, \quad t \geq 2. \tag{2.46}$$

Here we have $\mathbb{T} = \mathbb{R}$, $\alpha = 1/3$, $g(t) = t/2$, $h(t) = t-1$, $q(t) = t^2+5$, $r(t) = 1/t^{1/3}$, $f(t,u) = q(t)u^\alpha$, and $p(t) = (16t+17)/(t+1)$. It is clear that conditions (C1)-(C4) and (1.3) hold. In view of the fact that

$$16 \le p(t) < 17,$$

we see that

$$\xi_1(t) \ge \frac{15}{272} > 0. \tag{2.47}$$

Since

$$\frac{1}{p(g^{-1}(g^{-1}(t)))} \frac{R_2(g^{-1}(g^{-1}(t)), t_2)}{R_2(g^{-1}(t), t_2)} \le \frac{64t^3 - 48t + 9}{128t^3 - 384t + 144} \le \frac{177}{272} \quad \text{for } t \ge t_2 = 3,$$

we obtain

$$\xi_2(t) \ge \frac{95}{4{,}624}. \tag{2.48}$$

By (2.47), condition (2.3) becomes

$$\begin{aligned}
&\int_{t_0}^{t} \int_{v}^{\infty} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^\infty q(s)\bigl(\xi_1\bigl(h(s)\bigr)\bigr)^\alpha \Delta s \right)^{1/\alpha} \Delta u \Delta v \\
&\quad \ge (15/272) \int_2^t \int_v^\infty u \left(\int_u^\infty \bigl(s^2+5\bigr)\Delta s \right)^3 \Delta u \Delta v = \infty
\end{aligned}$$

due to $\int_u^\infty (s^2+5)\Delta s = \infty$ for $u \ge 2$, and so condition (2.3) holds.

With $\beta(t) = c > 0$ is a constant and the fact that (2.48), condition (2.41) becomes

$$\begin{aligned}
&\int_T^t \left[\beta(s)q(s)\bigl(\xi_2\bigl(h(s)\bigr)\bigr)^\alpha \left(\frac{R_2(s,3)}{R_1(s,2)} \right)^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(s)(\beta'_+(s))^{\alpha+1}}{(\beta(s)\psi(s))^\alpha} \right] ds \\
&\quad \ge \int_4^t c\bigl(s^2+5\bigr)(95/4{,}624)^{1/3} \left(\frac{s^3 - 12s + 9}{3(s^2-4)} \right)^{1/3} ds \\
&\quad \ge c(95/13{,}872)^{1/3} \int_4^t s^{4/3}\bigl(s^3 - 12s + 9\bigr)^{1/3} ds = \infty,
\end{aligned}$$

so condition (2.41) holds. Hence, by Theorem 2.5, any solution of (2.46) is either oscillatory or converges to zero.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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