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An expanded mixed covolume element method for integro-differential equation of Sobolev type on triangular grids

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Abstract

The expanded mixed covolume Element (EMCVE) method is studied for the two-dimensional integro-differential equation of Sobolev type. We use a piecewise constant function space and the lowest order Raviart-Thomas (RT_0) space as the trial function spaces of the scalar unknown u and its gradient $\boldsymbol{\sigma}$ and flux $\boldsymbol{\lambda}$, respectively. The semi-discrete and backward Euler fully-discrete EMCVE schemes are constructed, and the optimal a priori error estimates are derived. Moreover, numerical results are given to verify the theoretical analysis.

MSC: 65M08; 65M60

Keywords: integro-differential equation of Sobolev type; expanded mixed covolume element method; optimal a priori error estimate

1 Introduction

We consider the linear integro-differential equation of Sobolev type

$$c(x)\frac{\partial u}{\partial t} - \operatorname{div}\left(a(x)\nabla u + b(x)\nabla\frac{\partial u}{\partial t} + \int_0^t k(x,t,\tau)\nabla u(x,\tau)\,d\tau\right) = f(x,t), \tag{1}$$

for $(x,t) \in \Omega \times J$, with boundary and initial conditions

$$\begin{cases} u(x,t) = 0, & (x,t) \in \partial\Omega \times \bar{J}, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2}$$

where Ω is a convex and bounded polygonal domain in R^2 with boundary denoted by $\partial\Omega$, $J = (0,T]$ with $0 < T < \infty$, the initial function $u_0(x)$, the source function $f(x,t)$, and coefficients $k(x,t,\tau)$, $a(x)$, $b(x)$ and $c(x)$ are given bounded and smooth functions, and there exist some constants a_0, a_1, b_0, b_1, c_0 and c_1 such that

$$0 < a_0 \le a(x) \le a_1 < \infty, \qquad 0 < b_0 \le b(x) \le b_1 < \infty, \qquad 0 < c_0 \le c(x) \le c_1 < \infty.$$

Partial integro-differential equations are often used to describe various physical processes such as heat conduction behavior in memory material, nuclear reactor dynamics, compression of viscoelastic media and the propagation of sound in viscous media. Various numerical studies have been reported based on the finite element methods [1–3], finite volume element methods [4, 5], mixed finite element methods [6–9], discontinuous mixed covolume methods [10] etc. Numerical solutions for the integro-differential equation of Sobolev type have been given by Cui [11] who constructed a finite element scheme and obtained optimal error estimate by introducing Sobolev-Volterra projection; Che et al. [12] who considered H^1-Galerkin expanded mixed finite element method; and Guezane-Lakoud et al. [13] who developed Rothe's method for one-dimensional problem with integral conditions.

Mixed covolume element (MCVE) method was first introduced by Russell [14] to solve the mixed formulation of linear elliptic problems. Subsequently, Chou et al. [15, 16] considered the MCVE method for the elliptic boundary value problems by using the RT_0 space on the triangular grids and rectangular grids, respectively. This method not only can calculate several different physical quantities (such as pressure and Darcy velocity in [15]) but also maintains the mass local conservation law, and this is very important in fluid numerical computations. The satisfactory numerical simulation results on different test problems were obtained in [15–17]. The MCVE methods have been used to solve quasi-linear second order elliptic equations [18], parabolic equations [19, 20], and so on.

This article proposes an EMCVE scheme to solve the 2D linear integro-differential equation of Sobolev type. We introduce the variables $\boldsymbol{\sigma}(x,t) = -\nabla u(x,t)$ and $\boldsymbol{\lambda}(x,t) = -(a(x)\nabla u(x,t) + b(x)\nabla u_t + \int_0^t k(x,t,\tau)\nabla u(x,\tau)\,\mathrm{d}\tau)$ and write problem (1) as the system of first order PDEs

$$\begin{cases} \text{(a)} \quad \boldsymbol{\sigma}(x,t) = -\nabla u(x,t), \\ \text{(b)} \quad \boldsymbol{\lambda}(x,t) = a(x)\boldsymbol{\sigma}(x,t) + b(x)\frac{\partial\boldsymbol{\sigma}}{\partial t}(x,t) + \int_0^t k(x,t,\tau)\boldsymbol{\sigma}(x,\tau)\,\mathrm{d}\tau, \\ \text{(c)} \quad c(x)\frac{\partial u}{\partial t}(x,t) + \operatorname{div}\boldsymbol{\lambda}(x,t) = f(x,t). \end{cases} \tag{3}$$

The EMCVE scheme is obtained by integrating these equations on local covolume directly and using the Green's formula when proper. And then, the local conservation law with the discrete solution holds. This method skillfully combines finite volume element methods [21, 22] with expanded mixed finite element methods [23, 24], can use the advantage of finite volume element methods to calculate more different physical quantities simultaneously. Rui and Lu [25] applied the EMCVE method to solve the elliptic problem on rectangular grids in the rectangular area. In this article, we propose a semi-discrete and backward Euler fully-discrete EMCVE scheme based on triangular grids and obtain the optimal order error estimates by introducing a Volterra-type generalized EMCVE projection. Moreover, we give numerical results for a model equation to verify the feasibility and effectiveness of the scheme.

The expanded mixed weak formulation of (3) is to solve $(u,\boldsymbol{\sigma},\boldsymbol{\lambda}) \in L^2(\Omega) \times \mathbf{H}(\operatorname{div},\Omega) \times \mathbf{H}(\operatorname{div},\Omega)$ satisfying

$$\begin{cases} (\boldsymbol{\sigma},\mathbf{w}) - (\operatorname{div}\mathbf{w},u) = 0, \quad \forall \mathbf{w} \in \mathbf{H}(\operatorname{div},\Omega), \\ (\boldsymbol{\lambda},\mathbf{z}) = (a\boldsymbol{\sigma},\mathbf{z}) + (b\boldsymbol{\sigma}_t,\mathbf{z}) + (\int_0^t k\boldsymbol{\sigma}\,\mathrm{d}\tau,\mathbf{z}), \quad \forall \mathbf{z} \in \mathbf{H}(\operatorname{div},\Omega), \\ (cu_t,v) + (\operatorname{div}\boldsymbol{\lambda},v) = (f,v), \quad \forall v \in L^2(\Omega), \\ u(x,0) = u_0(x), \qquad \boldsymbol{\sigma}(x,0) = -\nabla u_0(x), \quad \forall x \in \Omega, \end{cases} \tag{4}$$

where $\mathbf{H}(\operatorname{div},\Omega) = \{\mathbf{z} \in (L^2(\Omega))^2 : \operatorname{div}\mathbf{z} \in L^2(\Omega)\}$.

We also use the general notations and definitions of the Sobolev spaces as in [26]. Let $(\cdot,\cdot)$ be the inner product in $L^2(\Omega)$ and $(L^2(\Omega))^2$, that is, $(\psi,\phi)=\int_\Omega \psi\phi\,\mathrm{d}x$ (if $\psi,\phi\in L^2(\Omega)$) and $(\mathbf{z},\mathbf{w})=\int_\Omega \mathbf{z}\cdot\mathbf{w}\,\mathrm{d}x$ (if $\mathbf{z},\mathbf{w}\in(L^2(\Omega))^2$), and either $\|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{(L^2(\Omega))^2}$ is denoted as $\|\cdot\|$. We also use the norm $\|\mathbf{z}\|_{\mathbf{H}(\mathrm{div},\Omega)}=(\|\mathbf{z}\|^2+\|\operatorname{div}\mathbf{z}\|^2)^{\frac{1}{2}}$ of the space $\mathbf{H}(\mathrm{div},\Omega)$. Throughout this paper, the constant $C>0$ does not depend on the spatial and time mesh parameters h and Δt.

2 Expanded mixed covolume element formulation

In order to describe the EMCVE scheme for system (1), we construct the partition $\mathcal{T}_h$ of the domain Ω. As in [15], let $\mathcal{T}_h=\{K_B\}$ be a quasi-uniform triangulation partition, where K_B is the triangle with barycenter point B, and $h=\max\{h_{K_B}\}$, h_{K_B} stands for the diameter of triangle K_B. We define the nodes to be the midpoints on the edges of every triangular element, where $P_1,P_2,\ldots,P_{N_\tau}$ stand for interior nodes, and $P_{N_\tau+1},\ldots,P_N$ stand for boundary nodes.

We use the RT_0 space as the trial function space $\mathbf{H}_h$ for variables $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$, where

$$\mathbf{H}_h=\bigl\{\mathbf{z}_h\in\mathbf{H}(\mathrm{div},\Omega):\mathbf{z}_h|_K=(a+bx_1,c+bx_2),\forall K\in\mathcal{T}_h\bigr\},\tag{5}$$

and use L_h as a trial space for variable u, where

$$L_h=\bigl\{v_h\in L^2(\Omega):v_h|_K\text{ is constant},\forall K\in\mathcal{T}_h\bigr\}.\tag{6}$$

Now the dual partition $\mathcal{T}_h^*$ is constructed by a union of interior quadrilaterals and border triangle. Referring now to Figure 1, and the quadrilateral $A_1B_1A_3B_2$ is the dual element $K_{P_3}^*$ with interior node P_3, which contains two elements K_L (the triangle $\triangle A_1B_1A_3$) and K_R (the triangle $\triangle A_1A_3B_2$); the triangle $\triangle A_4A_5B_3$ is the dual element $K_{P_6}^*$ with boundary node P_6, which contains one element K_I (the triangle $\triangle A_5B_3A_4$).

Integrate (3) on these primal and dual elements to obtain

$$\begin{cases}\text{(a)}\quad \int_{K_P^*\cap K_i}(\boldsymbol{\sigma}+\nabla u)\,\mathrm{d}x=0,\quad i=L,R\text{ or }I,\\ \text{(b)}\quad \int_{K_P^*\cap K_i}\boldsymbol{\lambda}\,\mathrm{d}x=\int_{K_P^*\cap K_i}(a\boldsymbol{\sigma}+b\boldsymbol{\sigma}_t+\int_0^t k\boldsymbol{\sigma}\,\mathrm{d}\tau)\,\mathrm{d}x,\quad i=L,R\text{ or }I,\\ \text{(c)}\quad \int_{K_B}(cu_t+\operatorname{div}\boldsymbol{\lambda})\,\mathrm{d}x=\int_{K_B}f\,\mathrm{d}x.\end{cases}\tag{7}$$

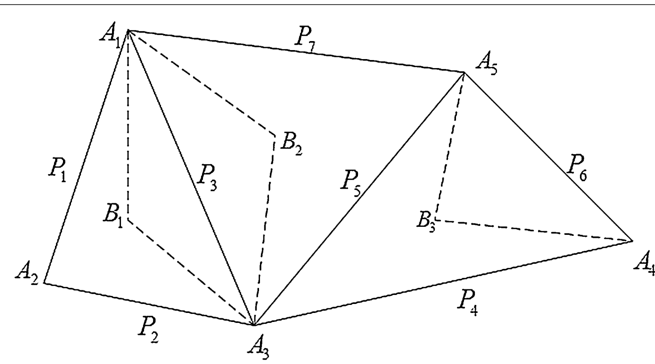


Figure 1 Primal and dual domains.

Similar to [15, 27], we define a transfer operator $\gamma_h : \mathbf{H}_h \to (L^2(\Omega))^2$ by

$$\gamma_h \mathbf{z}_h = \sum_{j=1}^{N_\tau} \big(\mathbf{z}_h|_{K_L}(P_j)\chi_{K_j^* \cap K_L} + \mathbf{z}_h|_{K_R}(P_j)\chi_{K_j^* \cap K_R} \big) + \sum_{j=N_\tau+1}^{N} \mathbf{z}_h|_{K_I}(P_j)\chi_{K_j^*}, \tag{8}$$

where χ_K means the characteristic function of a set K. Then we choose the range of γ_h as the test function space $\mathbf{Y}_h$. By using the transfer operator γ_h, we can rewrite equations (a) and (b) in (7) as

$$(\boldsymbol{\sigma} + \nabla u, \gamma_h \mathbf{w}_h) = 0, \quad \forall \mathbf{w}_h = \mathbf{H}_h, \tag{9}$$

$$(\boldsymbol{\lambda}, \gamma_h \mathbf{z}_h) = \left(a\boldsymbol{\sigma} + b\boldsymbol{\sigma}_t + \int_0^t k\boldsymbol{\sigma}\,\mathrm{d}\tau, \gamma_h \mathbf{z}_h \right), \quad \forall \mathbf{z}_h = \mathbf{H}_h. \tag{10}$$

Applying Green's integral formula, we have

$$\begin{aligned}(\nabla u, \gamma_h \mathbf{w}_h) &= \sum_{j=N_\tau+1}^{N} \mathbf{w}_h|_{K_I}(P_j) \int_{\partial K_{P_j}^* \setminus \partial\Omega} v_h \mathbf{n}\,\mathrm{d}\boldsymbol{\lambda} \\ &\quad + \sum_{j=1}^{N_\tau} \left(\mathbf{w}_h|_{K_L}(P_j) \int_{\partial K_{P_j}^* \cap K_L} v_h \mathbf{n}\,\mathrm{d}\boldsymbol{\lambda} + \mathbf{w}_h|_{K_R}(P_j) \int_{\partial K_{P_j}^* \cap K_R} v_h \mathbf{n}\,\mathrm{d}\boldsymbol{\lambda} \right) \\ &\equiv b(\gamma_h \mathbf{w}_h, u),\end{aligned}$$

for $\forall \mathbf{w}_h \in \mathbf{H}_h$, where $\mathbf{n}$ stands for the unit out-normal direction.

By calculation, it is easy to get the equality $b(\gamma_h \mathbf{w}_h, v_h) = -(\operatorname{div} \mathbf{w}_h, v_h)$, $\forall \mathbf{w}_h \in \mathbf{H}_h$, $\forall v_h \in L_h$. Then we can obtain the semi-discrete EMCVE scheme to find $(u_h, \boldsymbol{\sigma}_h, \boldsymbol{\lambda}_h) \in L_h \times \mathbf{H}_h \times \mathbf{H}_h$ such that

$$\begin{cases} (\boldsymbol{\sigma}_h, \gamma_h \mathbf{w}_h) - (\operatorname{div} \mathbf{w}_h, u_h) = 0, \quad \forall \mathbf{w}_h \in \mathbf{H}_h, \\ (\boldsymbol{\lambda}_h, \gamma_h \mathbf{z}_h) = (a\boldsymbol{\sigma}_h, \gamma_h \mathbf{z}_h) + (b\boldsymbol{\sigma}_{ht}, \gamma_h \mathbf{z}_h) + (\int_0^t k\boldsymbol{\sigma}_h\,\mathrm{d}\tau, \gamma_h \mathbf{z}_h), \quad \forall \mathbf{z}_h \in \mathbf{H}_h, \\ (cu_{ht}, v_h) + (\operatorname{div} \boldsymbol{\lambda}_h, v_h) = (f, v_h), \quad \forall v_h \in L_h, \end{cases} \tag{11}$$

and the initial values $u_h(0)$ and $\boldsymbol{\sigma}_h(0)$ will be defined in Theorems 4.1 and 4.2.

3 Some lemmas

For $\forall \mathbf{z}_h = (z_h^1, z_h^2) \in \mathbf{H}_h$, the discrete norms are defined as follows:

$$|\mathbf{z}_h|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} \left(\|\nabla z_h^1\|_{0,K}^2 + \|\nabla z_h^2\|_{0,K}^2 \right), \qquad \|\mathbf{z}_h\|_{1,h}^2 = \|\mathbf{z}_h\|^2 + |\mathbf{z}_h|_{1,h}^2.$$

Lemma 3.1 ([15]) *The operator γ_h is bounded*

$$\|\gamma_h \mathbf{z}_h\| \le \|\mathbf{z}_h\|, \quad \forall \mathbf{z}_h \in \mathbf{H}_h,$$

and satisfies

$$\|(I-\gamma_h)\mathbf{z}_h\| \le Ch\|\mathbf{z}_h\|_{1,h}, \quad \forall \mathbf{z}_h \in \mathbf{H}_h,$$
$$|(\mathbf{z}_h,(I-\gamma_h)\mathbf{w}_h)| \le Ch\|\mathbf{z}_h\|_{1,h}\|\mathbf{w}_h\|, \quad \forall \mathbf{z}_h,\mathbf{w}_h \in \mathbf{H}_h,$$
$$|(\mathbf{z},(I-\gamma_h)\mathbf{w}_h)| \le Ch\|\mathbf{z}\|_1\|\mathbf{w}_h\|, \quad \forall \mathbf{z} \in (H^1(\Omega))^2, \forall \mathbf{w}_h \in \mathbf{H}_h.$$

Lemma 3.2 ([20]) *The following symmetry relation*

$$(\gamma_h\mathbf{z}_h,\mathbf{w}_h) = (\mathbf{z}_h,\gamma_h\mathbf{w}_h), \quad \forall \mathbf{z}_h,\mathbf{w}_h \in \mathbf{H}_h,$$

holds, and there is a constant $\mu_0 > 0$ *independent of h such that*

$$(\gamma_h\mathbf{z}_h,\mathbf{z}_h) \ge \mu_0\|\mathbf{z}_h\|^2, \quad \forall \mathbf{z}_h \in \mathbf{H}_h.$$

For $\forall x \in K_B$, we define $\bar{a}(x) = a(B)$, $\bar{b}(x) = b(B)$, $\bar{k}(x,t,\tau) = k(B,t,\tau)$.

Lemma 3.3 ([20]) *The following symmetry relation*

$$(\bar{a}\gamma_h\mathbf{z}_h,\mathbf{w}_h) = (\bar{a}\mathbf{z}_h,\gamma_h\mathbf{w}_h), \quad \forall \mathbf{z}_h,\mathbf{w}_h \in \mathbf{H}_h,$$
$$(\bar{b}\gamma_h\mathbf{z}_h,\mathbf{w}_h) = (\bar{b}\mathbf{z}_h,\gamma_h\mathbf{w}_h), \quad \forall \mathbf{z}_h,\mathbf{w}_h \in \mathbf{H}_h,$$

holds, and there are constants $\mu_1 > 0$, $\mu_2 > 0$ *independent of h such that*

$$|(a\mathbf{z}_h,\gamma_h\mathbf{w}_h) - (\bar{a}\mathbf{z}_h,\gamma_h\mathbf{w}_h)| \le Ch\|\mathbf{z}_h\|\|\mathbf{w}_h\|, \quad \forall \mathbf{z}_h,\mathbf{w}_h \in \mathbf{H}_h,$$
$$|(b\mathbf{z}_h,\gamma_h\mathbf{w}_h) - (\bar{b}\mathbf{z}_h,\gamma_h\mathbf{w}_h)| \le Ch\|\mathbf{z}_h\|\|\mathbf{w}_h\|, \quad \forall \mathbf{z}_h,\mathbf{w}_h \in \mathbf{H}_h,$$
$$(\bar{a}\mathbf{z}_h,\gamma_h\mathbf{z}_h) \ge \mu_1\|\mathbf{z}_h\|^2, \qquad (a\mathbf{z}_h,\gamma_h\mathbf{z}_h) \ge \mu_1\|\mathbf{z}_h\|^2, \quad \forall \mathbf{z}_h \in \mathbf{H}_h,$$
$$(\bar{b}\mathbf{z}_h,\gamma_h\mathbf{z}_h) \ge \mu_2\|\mathbf{z}_h\|^2, \qquad (b\mathbf{z}_h,\gamma_h\mathbf{z}_h) \ge \mu_2\|\mathbf{z}_h\|^2, \quad \forall \mathbf{z}_h \in \mathbf{H}_h.$$

Lemma 3.4 ([20]) *The following estimates hold*:

$$|(a\mathbf{z}_h,(I-\gamma_h)\mathbf{w}_h)| \le Ch\|\mathbf{z}_h\|_{1,h}\|\mathbf{w}_h\|, \quad \forall \mathbf{z}_h,\mathbf{w}_h \in \mathbf{H}_h,$$
$$|(a\mathbf{z},(I-\gamma_h)\mathbf{w}_h)| \le Ch\|\mathbf{z}\|_1\|\mathbf{w}_h\|, \quad \forall \mathbf{z} \in (H^1(\Omega))^2, \forall \mathbf{w}_h \in \mathbf{H}_h,$$
$$|(b\mathbf{z}_h,(I-\gamma_h)\mathbf{w}_h)| \le Ch\|\mathbf{z}_h\|_{1,h}\|\mathbf{w}_h\|, \quad \forall \mathbf{z}_h,\mathbf{w}_h \in \mathbf{H}_h,$$
$$|(b\mathbf{z},(I-\gamma_h)\mathbf{w}_h)| \le Ch\|\mathbf{z}\|_1\|\mathbf{w}_h\|, \quad \forall \mathbf{z} \in (H^1(\Omega))^2, \forall \mathbf{w}_h \in \mathbf{H}_h.$$

The Raviart-Thomas projection $\Pi_h : \mathbf{H}(\mathrm{div},\Omega) \to \mathbf{H}_h$ is defined in [29] such that

$$(\mathrm{div}(\mathbf{z} - \Pi_h\mathbf{z}), v_h) = 0, \quad \forall \mathbf{z} \in \mathbf{H}(\mathrm{div},\Omega), \forall v_h \in L_h,$$

and the L^2 projection $R_h : L^2(\Omega) \to L_h$ is defined by

$$(\chi - R_h\chi, v_h) = 0, \quad \forall \chi \in L^2(\Omega), \forall v_h \in L_h.$$

Then the properties of Π_h and R_h are known from [28–30]

$$\|\mathbf{w} - \Pi_h\mathbf{w}\| \le Ch\|\mathbf{w}\|_1, \quad \forall \mathbf{w} \in \big(H^1(\Omega)\big)^2, \tag{12}$$

$$\big\|\operatorname{div}(\mathbf{w} - \Pi_h\mathbf{w})\big\| \le Ch\|\operatorname{div}\mathbf{w}\|_1, \quad \forall \mathbf{w} \in \mathbf{H}^1(\operatorname{div},\Omega), \tag{13}$$

$$\|\chi - R_h\chi\| \le Ch\|\chi\|_1, \quad \forall \chi \in H^1(\Omega), \tag{14}$$

where $\mathbf{H}^1(\operatorname{div},\Omega) = \{\mathbf{w} \in (L^2(\Omega))^2 : \operatorname{div}\mathbf{w} \in H^1(\Omega)\}$.

Lemma 3.5 ([20]) *The following estimate holds*:

$$\|\mathbf{z} - \gamma_h\Pi_h\mathbf{z}\| \le Ch\|\mathbf{z}\|_1, \quad \forall \mathbf{z} \in \big(H^1(\Omega)\big)^2.$$

Lemma 3.6 *The following symmetry relation*

$$\left(\int_0^t \bar{k}\mathbf{z}_h\,\mathrm{d}\tau, \gamma_h\mathbf{w}_h\right) = \left(\int_0^t \bar{k}\gamma_h\mathbf{z}_h\,\mathrm{d}\tau, \mathbf{w}_h\right), \quad \forall \mathbf{w}_h, \mathbf{z}_h \in \mathbf{H}_h, \tag{15}$$

holds, and we have

$$\left|\left(\int_0^t k\mathbf{z}_h\,\mathrm{d}\tau, \gamma_h\mathbf{w}_h\right) - \left(\int_0^t k\gamma_h\mathbf{z}_h\,\mathrm{d}\tau, \mathbf{w}_h\right)\right| \le Ch\int_0^t \|\mathbf{z}_h\|\,\mathrm{d}\tau \cdot \|\mathbf{w}_h\|. \tag{16}$$

Proof Let $K = \triangle A_1A_2A_3$, $\triangle_j = \triangle A_{j+1}BA_{j+2}$ $(j = 1,2,3)$, and $A_4 = A_1$ (see Figure 1). Denote $\mathbf{w}_h = (w_h^1, w_h^2)$ and $\mathbf{z}_h = (z_h^1, z_h^2)$, then

$$\begin{aligned}
&\left(\int_0^t \bar{k}\mathbf{z}_h\,\mathrm{d}\tau, \gamma_h\mathbf{w}_h\right)_{\triangle_j} - \left(\int_0^t \bar{k}\gamma_h\mathbf{z}_h\,\mathrm{d}\tau, \mathbf{w}_h\right)_{\triangle_j} \\
&\quad = \int_{\triangle_j}\left(w_h^1(P_j)\cdot\int_0^t \bar{k}z_h^1\,\mathrm{d}\tau - w_h^1\cdot\int_0^t \bar{k}z_h^1(P_j)\,\mathrm{d}\tau\right)\mathrm{d}x \\
&\qquad + \int_{\triangle_j}\left(w_h^2(P_j)\cdot\int_0^t \bar{k}z_h^2\,\mathrm{d}\tau - w_h^2\cdot\int_0^t \bar{k}z_h^2(P_j)\,\mathrm{d}\tau\right)\mathrm{d}x = M_{j1} + M_{j2}.
\end{aligned}$$

By applying the numerical quadrature formula, we get

$$\begin{aligned}
\sum_{j=1}^3 M_{j1} &= \sum_{j=1}^3\left\{w_h^1(P_j)\frac{1}{3}\left[\int_0^t \bar{k}z_h^1(B)\,\mathrm{d}\tau + 2\int_0^t \bar{k}z_h^1(P_j)\,\mathrm{d}\tau\right]\right. \\
&\qquad \left. - \frac{1}{3}\big[w_h^1(B) + 2w_h^1(P_j)\big]\int_0^t \bar{k}z_h^1(P_j)\,\mathrm{d}\tau\right\}\frac{|K|}{3} \\
&= \sum_{j=1}^3\left\{w_h^1(P_j)\frac{1}{3}\int_0^t \bar{k}z_h^1(B)\,\mathrm{d}\tau - \frac{1}{3}w_h^1(B)\int_0^t \bar{k}z_h^1(P_j)\,\mathrm{d}\tau\right\}\frac{|K|}{3} \\
&= \left[w_h^1(B)\int_0^t \bar{k}z_h^1(B)\,\mathrm{d}\tau - w_h^1(B)\int_0^t \bar{k}z_h^1(B)\,\mathrm{d}\tau\right]\frac{|K|}{3} = 0.
\end{aligned}$$

Similarly, we get $\sum_{j=1}^3 M_{j2} = 0$. Summing over all j and K, then we complete the proof of (15).

To prove (16), using (15), we have

$$\left(\int_0^t k\mathbf{z}_h\,\mathrm{d}\tau, \gamma_h\mathbf{w}_h\right) - \left(\int_0^t k\gamma_h\mathbf{z}_h\,\mathrm{d}\tau, \mathbf{w}_h\right)$$
$$= \sum_K \sum_{j=1}^3 \left[\left(\int_0^t (k-\bar{k})\mathbf{z}_h\,\mathrm{d}\tau, \gamma_h\mathbf{w}_h\right)_{\triangle_j} - \left(\int_0^t (k-\bar{k})\gamma_h\mathbf{z}_h\,\mathrm{d}\tau, \mathbf{w}_h\right)_{\triangle_j}\right].$$

Noting that $k(x,t,\tau)$ is Lipschitz continuous with variable x, we get the desired conclusion. □

Lemma 3.7 *For* $\forall\mathbf{z}_h, \mathbf{w}_h \in \mathbf{H}_h, \forall\mathbf{z} \in (H^1(\Omega))^2$, *we have*

$$\left|\left(\int_0^t k\mathbf{z}_h\,\mathrm{d}\tau, (I-\gamma_h)\mathbf{w}_h\right)\right| \le Ch\int_0^t \|\mathbf{z}_h\|_{1,h}\,\mathrm{d}\tau \cdot \|\mathbf{w}_h\|, \tag{17}$$

$$\left|\left(\int_0^t k\mathbf{z}\,\mathrm{d}\tau, (I-\gamma_h)\mathbf{w}_h\right)\right| \le Ch\int_0^t \|\mathbf{z}\|_1\,\mathrm{d}\tau \cdot \|\mathbf{w}_h\|. \tag{18}$$

Proof To prove (17), we obtain

$$\left(\int_0^t k\mathbf{z}_h\,\mathrm{d}\tau, (I-\gamma_h)\mathbf{w}_h\right) = \left(\int_0^t k(I-\gamma_h)\mathbf{z}_h\,\mathrm{d}\tau, \mathbf{w}_h\right) + \left[\left(\int_0^t k\gamma_h\mathbf{z}_h\,\mathrm{d}\tau, \mathbf{w}_h\right) - \left(\int_0^t k\mathbf{z}_h\,\mathrm{d}\tau, \gamma_h\mathbf{w}_h\right)\right].$$

By using Lemmas 3.1 and 3.6, we complete the proof of (17).

Next we prove (18). Using (12), we have

$$\begin{aligned}\left(\int_0^t k\mathbf{z}\,\mathrm{d}\tau, (I-\gamma_h)\mathbf{w}_h\right) &= \left(\int_0^t k(\mathbf{z}-\Pi_h\mathbf{z})\,\mathrm{d}\tau, (I-\gamma_h)\mathbf{w}_h\right) + \left(\int_0^t k\Pi_h\mathbf{z}\,\mathrm{d}\tau, \mathbf{w}_h\right)\\ &\le C\int_0^t \|\mathbf{z}-\Pi_h\mathbf{z}\|\,\mathrm{d}\tau \cdot \left\|(I-\gamma_h)\mathbf{w}_h\right\| + Ch\int_0^t \|\Pi_h\mathbf{z}\|\,\mathrm{d}\tau \cdot \|\mathbf{w}\|\\ &\le Ch\int_0^t \|\mathbf{z}\|_1\,\mathrm{d}\tau \cdot \|\mathbf{w}\|.\end{aligned}$$

This ends the proof of Lemma 3.7. □

Now, we introduce the Volterra-type generalized EMCVE projection. Define $(\tilde{u}_h, \tilde{\boldsymbol{\sigma}}_h, \tilde{\boldsymbol{\lambda}}_h) : [0,T] \to L_h \times \mathbf{H}_h \times \mathbf{H}_h$ such that

$$\left(\operatorname{div}(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}_h), v_h\right) = 0, \quad \forall v_h \in L_h, \tag{19a}$$

$$(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \gamma_h\mathbf{w}_h) - (\operatorname{div}\mathbf{w}_h, u - \tilde{u}_h) = -\left(\boldsymbol{\sigma}, (I-\gamma_h)\mathbf{w}_h\right), \quad \forall\mathbf{w}_h \in \mathbf{H}_h, \tag{19b}$$

$$\begin{aligned}(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}_h, \gamma_h\mathbf{z}_h) &= \left(a(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h), \gamma_h\mathbf{z}_h\right) - \left(\boldsymbol{\lambda}, (I-\gamma_h)\mathbf{z}_h\right)\\ &\quad + \left(a\boldsymbol{\sigma}, (I-\gamma_h)\mathbf{z}_h\right) + \left(\int_0^t k(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)\,\mathrm{d}\tau, \gamma_h\mathbf{z}_h\right)\\ &\quad + \left(\int_0^t k\boldsymbol{\sigma}\,\mathrm{d}\tau, (I-\gamma_h)\mathbf{z}_h\right), \quad \forall\mathbf{z}_h \in \mathbf{H}_h.\end{aligned} \tag{19c}$$

Theorem 3.1 *Suppose* $(\tilde{u}_h, \tilde{\boldsymbol{\sigma}}_h, \tilde{\boldsymbol{\lambda}}_h)$ *satisfies* (19a)-(19c), *then there is a constant* $C > 0$ *independent of h and t such that*

$$\|\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}_h\| \le Ch\|\boldsymbol{\lambda}\|_1, \tag{20}$$

$$\|\operatorname{div}(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}_h)\| \le Ch\|\operatorname{div}\boldsymbol{\lambda}\|_1, \tag{21}$$

$$\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\| \le Ch\left(\|\boldsymbol{\sigma}\|_1 + \|\boldsymbol{\lambda}\|_1 + \int_0^t \|\boldsymbol{\sigma}\|_1 \,\mathrm{d}\tau\right), \tag{22}$$

$$\|u - \tilde{u}_h\| \le Ch\left(\|\boldsymbol{\sigma}\|_1 + \|\boldsymbol{\lambda}\|_1 + \|u\|_1 + \int_0^t \|\boldsymbol{\sigma}\|_1 \,\mathrm{d}\tau\right). \tag{23}$$

Proof Noting that $\tilde{\boldsymbol{\lambda}}_h = \Pi_h\boldsymbol{\lambda}$, we have estimates (20) and (21).

Splitting $\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma} + \Pi_h\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h$ in (19c) yields

$$\begin{aligned}\left(a(\Pi_h\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h), \gamma_h\mathbf{z}_h\right) &= (\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}_h, \gamma_h\mathbf{z}_h) + \left(\boldsymbol{\lambda}, (I - \gamma_h)\mathbf{z}_h\right) \\ &\quad - \left(\int_0^t k(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)\,\mathrm{d}\tau, \gamma_h\mathbf{z}_h\right) - \left(\int_0^t k\boldsymbol{\sigma}\,\mathrm{d}\tau, (I - \gamma_h)\mathbf{z}_h\right) \\ &\quad - \left(a\boldsymbol{\sigma}, (I - \gamma_h)\mathbf{z}_h\right) - \left(a(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}), \gamma_h\mathbf{z}_h\right), \quad \forall \mathbf{z}_h \in \mathbf{H}_h.\end{aligned} \tag{24}$$

Choose $\mathbf{z}_h = \Pi_h\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h$ in (24) and use the Cauchy-Schwarz inequality to get

$$\begin{aligned}\mu_1\|\Pi_h\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\|^2 &\le C\left(\|\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}_h\|^2 + \|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|^2\right) + Ch^2\left(\|\boldsymbol{\lambda}\|_1^2 + \|\boldsymbol{\sigma}\|_1^2\right) \\ &\quad + C\int_0^t \left(\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|^2 + h^2\|\boldsymbol{\sigma}\|_1^2 + \|\Pi_h\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\|^2\right)\mathrm{d}\tau \\ &\quad + \frac{\mu_1}{2}\|\Pi_h\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\|^2.\end{aligned} \tag{25}$$

Using (12) and (20), applying Gronwall's inequality, we obtain estimate (22).

Noting that $\operatorname{div}(\mathbf{H}_h) = L_h$, we have $(\operatorname{div}\mathbf{w}_h, u - R_hu) = 0, \forall \mathbf{w}_h \in \mathbf{H}_h$, and rewrite (19b) as

$$(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \gamma_h\mathbf{w}_h) - (\operatorname{div}\mathbf{w}_h, R_hu - \tilde{u}_h) = -\left(\boldsymbol{\sigma}, (I - \gamma_h)\mathbf{w}_h\right), \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \tag{26}$$

Next we introduce an auxiliary elliptic problem. Given $\varphi \in L^2(\Omega)$, let ψ satisfy the following elliptic problem:

$$\begin{cases} -\Delta\psi = \varphi, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega. \end{cases} \tag{27}$$

And we have the following elliptic regularity result:

$$\|\psi\|_2 \le C\|\varphi\|. \tag{28}$$

Using the projection Π_h and R_h, and (26)-(28), we have

$$\begin{aligned}(R_hu - \tilde{u}_h, g) &= (R_hu - \tilde{u}_h, -\Delta\psi) = -\left(\operatorname{div}\left(\Pi_h(\nabla\psi)\right), R_hu - \tilde{u}_h\right) \\ &= -\left(\boldsymbol{\sigma}, (I - \gamma_h)\left(\Pi_h(\nabla\psi)\right)\right) - \left(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \gamma_h\Pi_h(\nabla\psi)\right)\end{aligned}$$

$$= -\big(\boldsymbol{\sigma}, (I-\gamma_h)\big(\Pi_h(\nabla\psi)\big)\big) + \big(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, (I-\gamma_h\Pi_h)(\nabla\psi)\big)$$
$$- (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \nabla\psi). \tag{29}$$

Noting that

$$\big(\boldsymbol{\sigma}, (I-\gamma_h)\big(\Pi_h(\nabla\psi)\big)\big) = \big(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}, (I-\gamma_h)\big(\Pi_h(\nabla\psi)\big)\big) + \big(\Pi_h\boldsymbol{\sigma}, \Pi_h(\nabla\psi) - \nabla\psi\big) + \big(\Pi_h\boldsymbol{\sigma}, \nabla\psi - \gamma_h\Pi_h(\nabla\psi)\big),$$

using (12), (28) and Lemma 3.5, we have

$$\big|\big(\boldsymbol{\sigma}, (I-\gamma_h)\big(\Pi_h(\nabla\psi)\big)\big)\big| \le Ch\|\boldsymbol{\sigma}\|_1\|\varphi\|, \tag{30}$$

and

$$\big|(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \nabla\psi)\big| \le Ch\bigg(\|\boldsymbol{\sigma}\|_1 + \|\boldsymbol{\lambda}\|_1 + \int_0^t \|\boldsymbol{\sigma}\|_1\,\mathrm{d}\tau\bigg)\|\varphi\|, \tag{31}$$

$$\big|\big(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, (I-\gamma_h\Pi_h)(\nabla\psi)\big)\big| \le Ch^2\bigg(\|\boldsymbol{\sigma}\|_1 + \|\boldsymbol{\lambda}\|_1 + \int_0^t \|\boldsymbol{\sigma}\|_1\,\mathrm{d}\tau\bigg)\|\varphi\|. \tag{32}$$

Using (30)-(32) in (29) yields

$$\|R_hu - \tilde{u}_h\| \le Ch\bigg(\|\boldsymbol{\sigma}\|_1 + \|\boldsymbol{\lambda}\|_1 + \int_0^t \|\boldsymbol{\sigma}\|_1\,\mathrm{d}\tau\bigg). \tag{33}$$

Apply the triangle inequality with (14) and (33) to obtain (23). □

Differentiating (19a)-(19c) with respect to time variable t, we can also obtain the following projection estimates.

Theorem 3.2 *Suppose* $(\tilde{u}_h, \tilde{\boldsymbol{\sigma}}_h, \tilde{\boldsymbol{\lambda}}_h)$ *satisfies* (19a)-(19c), *then there is a constant* $C > 0$ *independent of h and t such that*

$$\bigg\|\frac{\partial\boldsymbol{\lambda}}{\partial t} - \frac{\partial\tilde{\boldsymbol{\lambda}}_h}{\partial t}\bigg\| \le Ch\bigg\|\frac{\partial\boldsymbol{\lambda}}{\partial t}\bigg\|_1, \tag{34}$$

$$\bigg\|\frac{\partial\boldsymbol{\sigma}}{\partial t} - \frac{\partial\tilde{\boldsymbol{\sigma}}_h}{\partial t}\bigg\| \le Ch\Bigg(\sum_{i=0}^{1}\bigg(\bigg\|\frac{\partial^i\boldsymbol{\sigma}}{\partial t^i}\bigg\|_1 + \bigg\|\frac{\partial^i\boldsymbol{\lambda}}{\partial t^i}\bigg\|_1\bigg) + \int_0^t \big(\|\boldsymbol{\sigma}\|_1 + \|\boldsymbol{\lambda}\|_1\big)\,\mathrm{d}\tau\Bigg), \tag{35}$$

$$\bigg\|\frac{\partial u}{\partial t} - \frac{\partial\tilde{u}_h}{\partial t}\bigg\| \le Ch\Bigg(\bigg\|\frac{\partial u}{\partial t}\bigg\|_1 + \sum_{i=0}^{1}\bigg(\bigg\|\frac{\partial^i\boldsymbol{\sigma}}{\partial t^i}\bigg\|_1 + \bigg\|\frac{\partial^i\boldsymbol{\lambda}}{\partial t^i}\bigg\|_1\bigg) + \int_0^t \big(\|\boldsymbol{\sigma}\|_1 + \|\boldsymbol{\lambda}\|_1\big)\,\mathrm{d}\tau\Bigg). \tag{36}$$

4 The error estimates of semi-discrete expanded mixed covolume element formulation

In this section, we first discuss the existence and uniqueness of solution for the semi-discrete EMCVE scheme (11).

Theorem 4.1 *Set* $u_h(0) = \tilde{u}_h(0)$, $\boldsymbol{\sigma}_h(0) = \tilde{\boldsymbol{\sigma}}_h(0)$, *then there is a unique solution for system* (11).

Proof Let $\{\chi_j\}_{j=1}^{N_1}$ and $\{\varphi_j\}_{j=1}^{N}$ be the basis of L_h and $\mathbf{H}_h$, respectively. Then $\boldsymbol{\sigma}_h, \boldsymbol{\lambda}_h, \tilde{\boldsymbol{\sigma}}_h(0) \in \mathbf{H}_h$, $\tilde{u}_h(0), u_h \in L_h$ can be expressed as

$$\boldsymbol{\sigma}_h(x,t) = \sum_{j=1}^{N} \boldsymbol{\sigma}_j(t)\varphi_j(x), \qquad \boldsymbol{\lambda}_h(x,t) = \sum_{j=1}^{N} \boldsymbol{\lambda}_j(t)\varphi_j(x),$$

$$\tilde{\boldsymbol{\sigma}}_h(x,0) = \sum_{j=1}^{N} \tilde{\boldsymbol{\sigma}}_j(0)\varphi_j(x), \qquad \tilde{u}_h(x,0) = \sum_{j=1}^{N_1} \tilde{u}_j(0)\chi_j, \qquad u_h(x,t) = \sum_{j=1}^{N_1} u_j(t)\chi_j.$$

Substitute the above expressions into system (11) and set $\mathbf{w}_h, \mathbf{z}_h = \phi_i$ $(i = 1,2,\dots,N)$, $v_h = \chi_i$ $(i = 1,2,\dots,N_1)$, then we write system (11) as the following matrix form:

$$\begin{cases} \text{(a)} \quad AZ(t) - BU(t) = 0, \\ \text{(b)} \quad AL(t) = A_1Z(t) + A_2\frac{\mathrm{d}}{\mathrm{d}t}Z(t) + \int_0^t A_3(\tau)Z(\tau)\,\mathrm{d}\tau, \\ \text{(c)} \quad C\frac{\mathrm{d}}{\mathrm{d}t}U(t) + B^TL(t) = F(t), \\ \text{(d)} \quad U(0) = \tilde{U}_0, \qquad Z(0) = \tilde{Z}_0, \end{cases} \tag{37}$$

where

$$\begin{aligned} &Z(t) = \bigl(\boldsymbol{\sigma}_1(t), \boldsymbol{\sigma}_2(t), \dots, \boldsymbol{\sigma}_N(t)\bigr)^T, \qquad L(t) = \bigl(\boldsymbol{\lambda}_1(t), \boldsymbol{\lambda}_2(t), \dots, \boldsymbol{\lambda}_N(t)\bigr)^T, \\ &U(t) = \bigl(u_1(t), u_2(t), \dots, u_{N_1}(t)\bigr)^T, \qquad A = \bigl((\phi_j, \gamma_h\phi_i)\bigr)_{i=1,\dots,N;j=1,\dots,N}, \\ &B = \bigl((\chi_j, \operatorname{div}\phi_i)\bigr)_{i=1,\dots,N;j=1,\dots,N_1}, \qquad A_1 = \bigl((a\phi_j, \gamma_h\phi_i)\bigr)_{i=1,\dots,N;j=1,\dots,N}, \\ &A_2 = \bigl((b\phi_j, \gamma_h\phi_i)\bigr)_{i=1,\dots,N;j=1,\dots,N}, \qquad A_3 = \bigl((k\phi_j, \gamma_h\phi_i)\bigr)^T_{i=1,\dots,N;j=1,\dots,N}, \\ &C = \bigl((\chi_j, \chi_i)\bigr)_{i=1,\dots,N_1;j=1,\dots,N_1}, \qquad F(t) = \bigl((f, \chi_i)\bigr)^T_{i=1,\dots,N_1}, \\ &\tilde{U}_0 = \bigl(\tilde{u}_j(0)\bigr)^T_{j=1,2,\dots,N_1}, \qquad \tilde{Z}_0 = \bigl(\tilde{\boldsymbol{\sigma}}_j(0)\bigr)^T_{j=1,2,\dots,N_1}. \end{aligned}$$

It is easy to see that A and C are symmetric positive definite matrixes, and A_1 and A_2 are invertible matrixes. We rewrite equation (c) in (37) as

$$\begin{cases} (A_2 + G^{-1})\frac{\mathrm{d}}{\mathrm{d}t}Z(t) + A_1Z(t) + \int_0^t A_3Z(\tau)\,\mathrm{d}\tau = G^{-1}A^{-1}BC^{-1}F(t), \\ Z(0) = \tilde{Z}_0, \end{cases} \tag{38}$$

where $G = A^{-1}BC^{-1}B^TA^{-1}$.

Using quadratic form theory, we can know that $(A_2 + G^{-1})$ is an invertible matrix, and problem (38) has a unique solution by the theory of differential equations. Thus, systems (37) and (11) have a unique solution. □

Now we write the errors as

$$\begin{aligned} &\boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h + \tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h = \tilde{\xi} + \xi, \\ &\boldsymbol{\lambda} - \boldsymbol{\lambda}_h = \boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}_h + \tilde{\boldsymbol{\lambda}}_h - \boldsymbol{\lambda}_h = \tilde{\zeta} + \zeta, \\ &u - u_h = u - \tilde{u}_h + \tilde{u}_h - u_h = \tilde{\phi} + \phi, \end{aligned}$$

where $(\tilde{u}_h, \tilde{\boldsymbol{\sigma}}_h, \tilde{\boldsymbol{\lambda}}_h)$ is the Volterra-type generalized EMCVE projection of $(u, \boldsymbol{\sigma}, \boldsymbol{\lambda})$. Using (11) and (4), we have the error equations

$$(\xi, \gamma_h \mathbf{w}_h) - (\operatorname{div} \mathbf{w}_h, \phi) = 0, \quad \forall \mathbf{w}_h \in \mathbf{H}_h, \tag{39a}$$

$$(\zeta, \gamma_h \mathbf{z}_h) = (a\xi, \gamma_h \mathbf{z}_h) + (b\xi_t, \gamma_h \mathbf{z}_h) + (b\tilde{\xi}_t, \gamma_h \mathbf{z}_h) + \bigl(b\boldsymbol{\sigma}_t, (I - \gamma_h)\mathbf{z}_h\bigr) + \biggl(\int_0^t k\xi \,\mathrm{d}\tau, \gamma_h \mathbf{z}_h\biggr), \quad \forall \mathbf{z}_h \in \mathbf{H}_h, \tag{39b}$$

$$(c\phi_t, v_h) + (\operatorname{div} \zeta, v_h) = -(c\tilde{\phi}_t, v_h), \quad \forall v_h \in L_h. \tag{39c}$$

Theorem 4.2 *Let* $(u, \boldsymbol{\sigma}, \boldsymbol{\lambda})$, $(u_h, \boldsymbol{\sigma}_h, \boldsymbol{\lambda}_h)$ *be the solutions of* (4) *and* (11), *respectively, and set that* $u_h(0) = \tilde{u}_h(0)$, $\boldsymbol{\sigma}_h(0) = \tilde{\boldsymbol{\sigma}}_h(0)$. *Then there is a constant* $C > 0$ *independent of* h *and* t *such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \le Ch\bigl(\|*\|\bigr), \tag{40}$$

$$\|u - u_h\| \le Ch\bigl(\|u\|_1 + \|*\|\bigr), \tag{41}$$

$$\bigl\|(u - u_h)_t\bigr\| + \bigl\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)_t\bigr\| \le Ch\bigl(\|\boldsymbol{\sigma}_t\|_1 + \|\boldsymbol{\lambda}_t\|_1 + \|u_t\|_1 + \|*\|\bigr), \tag{42}$$

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\| \le Ch\bigl(\|\boldsymbol{\sigma}_t\|_1 + \|\boldsymbol{\lambda}_t\|_1 + \|u_t\|_1 + \|*\|\bigr), \tag{43}$$

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathbf{H}(\operatorname{div},\Omega)} \le Ch\bigl(\|\operatorname{div} \boldsymbol{\lambda}\|_1 + \|\boldsymbol{\sigma}_t\|_1 + \|\boldsymbol{\lambda}_t\|_1 + \|u_t\|_1 + \|*\|\bigr), \tag{44}$$

where

$$\|*\| = \|\boldsymbol{\sigma}\|_1 + \|\boldsymbol{\lambda}\|_1 + \biggl(\int_0^t \biggl\|\frac{\partial u}{\partial t}\biggr\|_1^2 \,\mathrm{d}t\biggr)^{\frac{1}{2}} + \sum_{i=0}^{1}\biggl(\biggl(\int_0^t \biggl\|\frac{\partial^i \boldsymbol{\sigma}}{\partial t^i}\biggr\|_1^2 \,\mathrm{d}t\biggr)^{\frac{1}{2}} + \biggl(\int_0^t \biggl\|\frac{\partial^i \boldsymbol{\lambda}}{\partial t^i}\biggr\|_1^2 \,\mathrm{d}t\biggr)^{\frac{1}{2}}\biggr).$$

Proof Differentiating (39a) with respect to variable t, we have

$$(\xi_t, \gamma_h \mathbf{w}_h) - (\operatorname{div} \mathbf{w}_h, \phi_t) = 0, \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \tag{45}$$

Setting $v_h = \phi_t$ in (39c), $\mathbf{w}_h = \zeta$ in (45), and $\mathbf{z}_h = \xi_t$ in (39b), we have

$$(c\phi_t, \phi_t) + (a\xi, \gamma_h \xi_t) + (b\xi_t, \gamma_h \xi_t) = -(c\tilde{\phi}_t, \phi_t) - (b\tilde{\xi}_t, \gamma_h \xi_t) - \bigl(b\boldsymbol{\sigma}_t, (I - \gamma_h)\xi_t\bigr) + \biggl(\int_0^t k\xi \,\mathrm{d}\tau, \gamma_h \xi_t\biggr). \tag{46}$$

Noting that

$$(a\xi, \gamma_h \xi_t) = (\bar{a}\xi, \gamma_h \xi_t) + \bigl[(a\xi, \gamma_h \xi_t) - (\bar{a}\xi, \gamma_h \xi_t)\bigr],$$

and $(\bar{a}\xi, \gamma_h \xi_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{a}\xi, \gamma_h \xi)$, using Lemmas 3.3-3.5 and Lemma 3.7, we get

$$c_0\|\phi_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{a}\xi, \gamma_h \xi) + \mu_2\|\xi_t\|^2 \le \frac{\mu_2}{2}\|\xi_t\|^2 + \frac{c_0}{2}\|\phi_t\|^2 + C\bigl(\|\tilde{\phi}_t\|^2 + \|\tilde{\xi}_t\|^2 + h^2\|\boldsymbol{\sigma}_t\|_1^2 + \|\xi\|^2\bigr) + C\int_0^t \|\xi\|^2 \,\mathrm{d}\tau.$$

Integrating the above inequality from 0 to t, we get

$$(\bar{a}\xi,\gamma_h\xi)+c_0\int_0^t\|\phi_t\|^2\,\mathrm{d}t+\mu_2\int_0^t\|\xi_t\|^2\,\mathrm{d}t \\ \le\big(\bar{a}\xi(0),\gamma_h\xi(0)\big)+C\int_0^t\big(\|\tilde{\phi}_t\|^2+\|\tilde{\xi}_t\|^2+h^2\|\boldsymbol{\sigma}_t\|_1^2+\|\xi\|^2\big)\,\mathrm{d}t. \tag{47}$$

Noting that $\xi(0)=0$, $(\bar{a}\xi,\gamma_h\xi)\ge\mu_1\|\xi\|^2$, applying Gronwall's inequality, we have

$$\mu_1\|\xi\|^2+c_0\int_0^t\|\phi_t\|^2\,\mathrm{d}t+\mu_2\int_0^t\|\xi_t\|^2\,\mathrm{d}t \\ \le C\int_0^t\big(\|\tilde{\phi}_t\|^2+\|\tilde{\xi}_t\|^2+h^2\|\boldsymbol{\sigma}_t\|_1^2\big)\,\mathrm{d}t. \tag{48}$$

Now, we set $v_h=\phi$ in (39c), $\mathbf{w}_h=\zeta$ in (39a), and $\mathbf{z}_h=\xi$ in (39b) to obtain

$$(c\phi_t,\phi)+(a\xi,\gamma_h\xi)=-(c\tilde{\phi}_t,\phi)-(b\xi_t,\gamma_h\xi) \\ -(b\tilde{\xi}_t,\gamma_h\xi)-\big(b\boldsymbol{\sigma}_t,(I-\gamma_h)\xi\big)-\left(\int_0^t k\xi\,\mathrm{d}\tau,\gamma_h\xi\right). \tag{49}$$

Using Lemmas 3.3, 3.4 and 3.7, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|c^{\frac{1}{2}}\phi\big\|^2+\mu_1\|\xi\|^2\le\frac{\mu_1}{2}\|\xi\|^2+C\left(\|\phi\|^2+\int_0^t\|\xi\|^2\,\mathrm{d}\tau\right) \\ +C\big(\|\tilde{\phi}_t\|^2+\|\tilde{\xi}_t\|^2+h^2\|\boldsymbol{\sigma}_t\|_1^2+\|\xi_t\|^2\big). \tag{50}$$

Integrating (50) from 0 to t yields

$$\big\|c^{\frac{1}{2}}\phi\big\|^2-\big\|c^{\frac{1}{2}}\phi(0)\big\|^2+\mu_1\int_0^t\|\xi\|^2\,\mathrm{d}t \\ \le C\int_0^t\|\phi\|^2\,\mathrm{d}t+C\int_0^t\big(\|\tilde{\phi}_t\|^2+\|\tilde{\xi}_t\|^2+h^2\|\boldsymbol{\sigma}_t\|_1^2+\|\xi_t\|^2+\|\xi\|^2\big)\,\mathrm{d}t. \tag{51}$$

Noting that $\phi(0)=0$, and substituting (48) into (51), we get that

$$c_0\|\phi\|^2+\mu_1\int_0^t\|\xi\|^2\,\mathrm{d}t\le C\int_0^t\|\phi\|^2\,\mathrm{d}t+C\int_0^t\big(\|\tilde{\phi}_t\|^2+\|\tilde{\xi}_t\|^2+h^2\|\boldsymbol{\sigma}_t\|_1^2\big)\,\mathrm{d}t.$$

Using Gronwall's inequality yields

$$c_0\|\phi\|^2+\mu_1\int_0^t\|\xi\|^2\,\mathrm{d}t\le C\int_0^t\big(\|\tilde{\phi}_t\|^2+\|\tilde{\xi}_t\|^2+h^2\|\boldsymbol{\sigma}_t\|_1^2\big)\,\mathrm{d}t. \tag{52}$$

Next, using Lemmas 3.3 and 3.4 in (46), we get

$$c_0\|\phi_t\|^2+\mu_2\|\xi_t\|^2\le C\|\xi\|^2+\frac{\mu_2}{2}\|\xi_t\|^2+\frac{c_0}{2}\|\phi_t\|^2 \\ +C\big(\|\tilde{\phi}_t\|^2+\|\tilde{\xi}_t\|^2+h^2\|\boldsymbol{\sigma}_t\|_1^2\big)+C\int_0^t\|\xi\|^2\,\mathrm{d}\tau. \tag{53}$$

Substituting (48) and (52) into (53) yields

$$c_0\|\phi_t\|^2 + \mu_2\|\xi_t\|^2 \le C\big(\|\tilde{\phi}_t\|^2 + \|\tilde{\xi}_t\|^2 + h^2\|\boldsymbol{\sigma}_t\|_1^2\big) + C\int_0^t \big(\|\tilde{\phi}_t\|^2 + \|\tilde{\xi}_t\|^2 + h^2\|\boldsymbol{\sigma}_t\|_1^2\big)\,\mathrm{d}t. \tag{54}$$

To estimate $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|$ and $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathbf{H}(\mathrm{div},\Omega)}$, we choose $\mathbf{z}_h = \zeta$ in (39b) to see that

$$(\zeta, \gamma_h\zeta) = (a\xi, \gamma_h\zeta) + (b\xi_t, \gamma_h\zeta) + \big(b\boldsymbol{\sigma}_t, (I - \gamma_h)\zeta\big) + (b\tilde{\xi}_t, \gamma_h\zeta) + \left(\int_0^t k\xi\,\mathrm{d}\tau, \gamma_h\zeta\right).$$

Using Lemmas 3.2 and 3.4, we get

$$\mu_0\|\zeta\|^2 \le C\big(\|\xi_t\|^2 + \|\xi\|^2 + \|\tilde{\xi}_t\|^2 + h^2\|\boldsymbol{\sigma}_t\|_1^2\big) + C\int_0^t \|\xi\|^2\,\mathrm{d}\tau + \frac{\mu_0}{2}\|\zeta\|^2. \tag{55}$$

Substituting (48), (52) and (54) into (55), we have that

$$\|\zeta\|^2 \le C\big(\|\tilde{\phi}_t\|^2 + \|\tilde{\xi}_t\|^2 + h^2\|\boldsymbol{\sigma}_t\|_1^2\big) + C\int_0^t \big(\|\tilde{\phi}_t\|^2 + \|\tilde{\xi}_t\|^2 + h^2\|\boldsymbol{\sigma}_t\|_1^2\big)\,\mathrm{d}t. \tag{56}$$

Choosing $v_h = \operatorname{div}\zeta$ in (39c) yields

$$(\operatorname{div}\zeta, \operatorname{div}\zeta) = -(c\phi_t, \operatorname{div}\zeta) - (c\tilde{\phi}_t, \operatorname{div}\zeta).$$

And we have

$$\|\operatorname{div}\zeta\|^2 \le C\big(\|\tilde{\phi}_t\|^2 + \|\phi_t\|^2\big).$$

Using (48) and (54), we have

$$\|\operatorname{div}\zeta\|^2 \le C\big(\|\tilde{\xi}_t\|^2 + \|\tilde{\phi}_t\|^2 + h^2\|\boldsymbol{\sigma}_t\|_1^2\big) + C\int_0^t \big(\|\tilde{\xi}_t\|^2 + \|\tilde{\phi}_t\|^2 + h^2\|\boldsymbol{\sigma}_t\|_1^2\big)\,\mathrm{d}t. \tag{57}$$

Thus, combine (48), (52), (54) and (57), apply the triangle inequality to complete the proof. □

5 The fully-discrete expanded mixed covolume element formulation

Let Δt be the time step length, and $t_n = n\Delta t$ $(n = 0, 1, 2, \ldots, M)$ for some positive integer M. Define $\varphi^n = \varphi(t_n)$ and $\partial_t\varphi^n = \frac{\varphi^n - \varphi^{n-1}}{\Delta t}$ for a function φ. To approximate the integral term, we select the left rectangle quadrature formula

$$\int_0^{t_n} \varphi(s)\,\mathrm{d}s \approx \Delta t\sum_{j=0}^{n-1}\varphi(t_j),$$

and the quadrature error $\varepsilon^n(\varphi) = \int_0^{t_n}\varphi(s)\,\mathrm{d}s - \Delta t\sum_{j=0}^{n-1}\varphi(t_j)$ satisfies

$$\big|\varepsilon^n(\varphi)\big| \le C\Delta t\int_0^{t_n}\big|\varphi_t(s)\big|\,\mathrm{d}s.$$

Now, we define the backward Euler fully-discrete scheme: find $(u_h^n, \boldsymbol{\sigma}_h^n, \boldsymbol{\lambda}_h^n) \in L_h \times \mathbf{H}_h \times \mathbf{H}_h$, $n = 0, 1, \ldots, N$, such that

$$(u_h^0, v_h) = (u_0, v_h), \quad \forall v_h \in L_h, \tag{58a}$$

$$(\boldsymbol{\sigma}_h^0, \gamma_h \mathbf{w}_h) - (\operatorname{div} \mathbf{w}_h, u_h^0) = 0, \quad \forall \mathbf{w}_h \in \mathbf{H}_h, \tag{58b}$$

$$(\boldsymbol{\sigma}_h^n, \gamma_h \mathbf{w}_h) - (\operatorname{div} \mathbf{w}_h, u_h^n) = 0, \quad \forall \mathbf{w}_h \in \mathbf{H}_h, n \geq 1, \tag{58c}$$

$$\begin{aligned}(\boldsymbol{\lambda}_h^n, \gamma_h \mathbf{z}_h) &= (a\boldsymbol{\sigma}_h^n, \gamma_h \mathbf{z}_h) + (b\partial_t \boldsymbol{\sigma}_h^n, \gamma_h \mathbf{z}_h) \\ &\quad + \Delta t \sum_{j=0}^{n-1} (k^{n,j} \boldsymbol{\sigma}_h^j, \gamma_h \mathbf{z}_h), \quad \forall \mathbf{z}_h \in \mathbf{H}_h, n \geq 1,\end{aligned} \tag{58d}$$

$$(c\partial_t u_h^n, v_h) + (\operatorname{div} \boldsymbol{\lambda}_h^n, v_h) = (f^n, v_h), \quad \forall v_h \in L_h, n \geq 1, \tag{58e}$$

where $k^{n,j} = k(x, t_n, t_j)$.

The above calculation of $\{u_h^n, \boldsymbol{\sigma}_h^n, \boldsymbol{\lambda}_h^n\}$ $(n = 1, 2, \ldots, M)$ only involves the inverse operation of stiffness matrix with the spaces $\mathbf{H}_h$ and L_h. u_h^0 and $\boldsymbol{\sigma}_h^0$ are calculated by solving (58a) and (58b). The calculation proceeds by solving (58c), (58d) and (58e) equations for $\{\boldsymbol{\sigma}_h^n, \boldsymbol{\lambda}_h^n, u_h^n\}$ with using already calculated $\{\boldsymbol{\sigma}_h^{n-1}, u_h^{n-1}\}$. It is easy to get that there is a unique solution for the fully-discrete scheme (58a)-(58e).

We now rewrite the errors as

$$\begin{aligned}\boldsymbol{\sigma}(t_n) - \boldsymbol{\sigma}_h^n &= \boldsymbol{\sigma}(t_n) - \tilde{\boldsymbol{\sigma}}_h(t_n) + \tilde{\boldsymbol{\sigma}}_h(t_n) - \boldsymbol{\sigma}_h^n = \tilde{\xi}^n + \xi^n, \\ \boldsymbol{\lambda}(t_n) - \boldsymbol{\lambda}_h^n &= \boldsymbol{\lambda}(t_n) - \tilde{\boldsymbol{\lambda}}_h(t_n) + \tilde{\boldsymbol{\lambda}}_h(t_n) - \boldsymbol{\lambda}_h^n = \tilde{\zeta}^n + \zeta^n, \\ u(t_n) - u_h^n &= u(t_n) - \tilde{u}_h(t_n) + \tilde{u}_h(t_n) - u_h^n = \tilde{\phi}^n + \phi^n,\end{aligned}$$

where $(\tilde{u}_h, \tilde{\boldsymbol{\sigma}}_h, \tilde{\boldsymbol{\lambda}}_h)$ is the Volterra-type generalized EMCVE projection of $(u, \boldsymbol{\sigma}, \boldsymbol{\lambda})$.

Using (19a)-(19c), we obtain the following error equations:

$$(\tilde{\phi}^0 + \phi^0, v_h) = 0, \quad \forall v_h \in L_h, \tag{59a}$$

$$\begin{aligned}&(\boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_h^0, \gamma_h \mathbf{w}_h) - (\operatorname{div} \mathbf{w}_h, u(0) - u_h^0) \\ &\quad = -(\boldsymbol{\sigma}(0), (I - \gamma_h)\mathbf{w}_h), \quad \forall \mathbf{w}_h \in \mathbf{H}_h,\end{aligned} \tag{59b}$$

$$(\xi^n, \gamma_h \mathbf{w}_h) - (\operatorname{div} \mathbf{w}_h, \phi^n) = 0, \quad \forall \mathbf{w}_h \in \mathbf{H}_h, n \geq 1, \tag{59c}$$

$$\begin{aligned}(\zeta^n, \gamma_h \mathbf{z}_h) &= (a\xi^n, \gamma_h \mathbf{z}_h) + (b\partial_t \xi^n, \gamma_h \mathbf{z}_h) + (b\partial_t \tilde{\xi}^n, \gamma_h \mathbf{z}_h) \\ &\quad + (b\alpha^n, \gamma_h \mathbf{z}_h) + (b\boldsymbol{\sigma}_t^n, (I - \gamma_h)\mathbf{z}_h) + (\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h), \gamma_h \mathbf{z}_h) \\ &\quad + \Delta t \sum_{j=0}^{n-1} (k^{n,j} \xi^j, \gamma_h \mathbf{z}_h), \quad \forall \mathbf{z}_h \in \mathbf{H}_h, n \geq 1,\end{aligned} \tag{59d}$$

$$(c\partial_t \phi^n, v_h) + (\operatorname{div} \zeta^n, v_h) = -(c\partial_t \tilde{\phi}^n, v_h) - (c\beta^n, v_h), \quad \forall v_h \in L_h, n \geq 1, \tag{59e}$$

where

$$\begin{aligned}\alpha^n &= \boldsymbol{\sigma}_t^n - \partial_t \boldsymbol{\sigma}^n, \qquad \beta^n = u_t^n - \partial_t u^n, \\ \varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h) &= \int_0^{t_n} k(t_n, \tau)\tilde{\boldsymbol{\sigma}}_h(\tau)\, \mathrm{d}\tau - \Delta t \sum_{j=0}^{n-1} k^{n,j} \tilde{\boldsymbol{\sigma}}_h^j.\end{aligned}$$

Theorem 5.1 *Let $(u_h^n, \boldsymbol{\sigma}_h^n, \boldsymbol{\lambda}_h^n)$ be the solution of scheme* (58a)-(58e), *and suppose that the solution $(u, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ of system* (4) *has properties that $\boldsymbol{\sigma}, \boldsymbol{\lambda} \in L^\infty((H^1(\Omega))^2)$, $\boldsymbol{\sigma}_t, \boldsymbol{\lambda}_t \in L^2((H^1(\Omega))^2)$, $u \in L^\infty(H^1(\Omega))$, $u_t \in L^2(H^1(\Omega))$, $\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_{tt} \in L^2((L^2(\Omega))^2)$, $u_{tt} \in L^2(L^2(\Omega))$, then there is a constant $C > 0$ independent of h and Δt such that*

$$\max_{0\le n\le M}\left(\|u(t_n) - u_h^n\| + \|\boldsymbol{\sigma}(t_n) - \boldsymbol{\sigma}_h^n\|\right) \le C(h + \Delta t),$$

$$\max_{1\le n\le M}\left(\|u_t(t_n) - \partial_t u_h^n\| + \|\boldsymbol{\sigma}_t(t_n) - \partial_t \boldsymbol{\sigma}_h^n\|\right) \le C(h + \Delta t),$$

$$\max_{0\le n\le M}\left(\|\boldsymbol{\lambda}(t_n) - \boldsymbol{\lambda}_h^n\| + \|\boldsymbol{\lambda}(t_n) - \boldsymbol{\lambda}_h^n\|_{\mathbf{H}(\mathrm{div},\Omega)}\right) \le C(h + \Delta t).$$

Proof Using (19a)-(19c), we rewrite (59b) as

$$\left(\xi^0, \gamma_h \mathbf{w}_h\right) - \left(\operatorname{div} \mathbf{w}_h, \phi^0\right) = 0, \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \tag{60}$$

Then using (59c) and (60), we have

$$\left(\partial_t \xi^n, \gamma_h \mathbf{w}_h\right) - \left(\operatorname{div} \mathbf{w}_h, \partial_t \phi^n\right) = 0, \quad \forall \mathbf{w}_h \in \mathbf{H}_h, n \ge 1. \tag{61}$$

Choosing $v_h = \partial_t \phi^n$ in (59e), $\mathbf{w}_h = \zeta^n$ in (61), and $\mathbf{z}_h = \partial_t \xi^n$ in (59d), we have

$$\begin{aligned}
&\left(c\partial_t \phi^n, \partial_t \phi^n\right) + \left(a\xi^n, \gamma_h \partial_t \xi^n\right) + \left(b\partial_t \xi^n, \gamma_h \partial_t \xi^n\right) \\
&\quad = -\left(c\partial_t \tilde{\phi}^n, \partial_t \phi^n\right) - \left(c\beta^n, \partial_t \phi^n\right) - \left(b\partial_t \tilde{\xi}^n, \gamma_h \partial_t \xi^n\right) - \left(b\boldsymbol{\sigma}_t^n, (I - \gamma_h)\partial_t \xi^n\right) \\
&\qquad - \left(b\alpha^n, \gamma_h \partial_t \xi^n\right) - \Delta t \sum_{j=0}^{n-1} \left(k^{n,j}\xi^j, \gamma_h \partial_t \xi^n\right) - \left(\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h), \gamma_h \partial_t \xi^n\right).
\end{aligned} \tag{62}$$

Noting the fact that $(a\xi^n, \gamma_h \partial_t \xi^n) = (\bar{a}\xi^n, \gamma_h \partial_t \xi^n) + [(a\xi^n, \gamma_h \partial_t \xi^n) - (\bar{a}\xi^n, \gamma_h \partial_t \xi^n)]$, and $(\bar{a}\xi^n, \gamma_h \partial_t \xi^n) \ge \frac{1}{2\Delta t}[(\bar{a}\xi^n, \gamma_h \xi^n) - (\bar{a}\xi^{n-1}, \gamma_h \xi^{n-1})]$, we have

$$\begin{aligned}
&c_0 \|\partial_t \phi^n\|^2 + \frac{1}{2\Delta t}\left[\left(\bar{a}\xi^n, \gamma_h \xi^n\right) - \left(\bar{a}\xi^{n-1}, \gamma_h \xi^{n-1}\right)\right] + \mu_2 \|\partial_t \xi^n\|^2 \\
&\quad \le C\left(\|\beta^n\|^2 + \|\partial_t \tilde{\phi}^n\|^2 + \|\partial_t \tilde{\xi}^n\|^2 + \|\alpha^n\|^2 + h^2 \|\boldsymbol{\sigma}_t^n\|_1^2\right) \\
&\qquad + C\Delta t \sum_{j=0}^{n-1} \|\xi^j\|^2 + C\left(\|\xi^n\|^2 + \|\varepsilon^n(k\tilde{\boldsymbol{\sigma}})\|^2\right) \\
&\qquad + \frac{c_0}{2}\|\partial_t \phi^n\|^2 + \frac{\mu_2}{2}\|\partial_t \xi^n\|^2.
\end{aligned} \tag{63}$$

Summing from $n = 1$ to m and multiplying (63) by $2\Delta t$, we have

$$\begin{aligned}
\left(\bar{a}\xi^m, \gamma_h \xi^m\right) &\le C\|\xi^0\|^2 + C\Delta t \sum_{n=0}^{m} \|\xi^n\|^2 + C\Delta t \sum_{n=1}^{m} \left(\|\alpha^n\|^2 + \|\beta^n\|^2\right) \\
&\quad + C\Delta t \sum_{n=1}^{m} \left(h^2 \|\boldsymbol{\sigma}_t^n\|_1^2 + \|\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h)\|^2 + \|\partial_t \tilde{\phi}^n\|^2 + \|\partial_t \tilde{\xi}^n\|^2\right).
\end{aligned} \tag{64}$$

Note that $(\bar{a}\xi^m, \gamma_h\xi^m) \geq \mu_1\|\xi^m\|^2$, choose Δt in (64) to satisfy $C\Delta t < \frac{\mu_1}{2}$, and use Gronwall's inequality to get

$$\begin{aligned}\|\xi^m\|^2 &\leq C\|\xi^0\|^2 + C\Delta t\sum_{n=1}^{m}\left(\|\alpha^n\|^2 + \|\beta^n\|^2\right)\\&\quad + C\Delta t\sum_{n=1}^{m}\left(h^2\|\boldsymbol{\sigma}_t^n\|_1^2 + \|\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h)\|^2 + \|\partial_t\tilde{\phi}^n\|^2 + \|\partial_t\tilde{\xi}^n\|^2\right).\end{aligned} \tag{65}$$

Now, by Lemma 3.3, it follows from (62) that

$$\begin{aligned}&c_0\|\partial_t\phi^n\|^2 + \mu_2\|\partial_t\xi^n\|^2\\&\quad\leq C\left(\|\xi^n\|^2\right) + C\Delta t\sum_{j=0}^{n-1}\|\xi^j\|^2\\&\qquad + \frac{c_0}{2}\|\partial_t\phi^n\|^2 + \frac{\mu_2}{2}\|\partial_t\xi^n\|^2 + C\|\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h)\|^2\\&\qquad + C\left(h^2\|\boldsymbol{\sigma}_t^n\|_1^2 + \|\partial_t\tilde{\phi}^n\|^2 + \|\beta^n\|^2 + \|\partial_t\tilde{\xi}^n\|^2 + \|\alpha^n\|^2\right).\end{aligned} \tag{66}$$

Substituting (65) into (66), we have that

$$\begin{aligned}&c_0\|\partial_t\phi^n\|^2 + \mu_2\|\partial_t\xi^n\|^2\\&\quad\leq C\|\xi^0\|^2 + C\|\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h)\|^2\\&\qquad + C\left(h^2\|\boldsymbol{\sigma}_t^n\|_1^2 + \|\partial_t\tilde{\phi}^n\|^2 + \|\beta^n\|^2 + \|\partial_t\tilde{\xi}^n\|^2 + \|\alpha^n\|^2\right)\\&\qquad + C\Delta t\sum_{j=1}^{n}\left(\|\partial_t\tilde{\phi}^j\|^2 + \|\beta^j\|^2 + \|\partial_t\tilde{\xi}^j\|^2 + \|\alpha^j\|^2 + h^2\|\boldsymbol{\sigma}_t^j\|_1^2 + \|\varepsilon^j(k\tilde{\boldsymbol{\sigma}}_h)\|^2\right).\end{aligned} \tag{67}$$

To estimate $\|\boldsymbol{\lambda}(t_n) - \mathbf{Z}^n\| + \|\boldsymbol{\lambda}(t_n) - \mathbf{Z}^n\|_{\mathbf{H}(\mathrm{div},\Omega)}$, we set $\mathbf{z}_h = \zeta^n$ in (59d) and get that

$$\begin{aligned}\mu_0\|\zeta^n\|^2 &\leq C\left(\|\partial_t\tilde{\xi}^n\|^2 + \|\alpha^n\|^2 + h^2\|\boldsymbol{\sigma}_t\|_1^2\right) + C\left(\|\xi^n\|^2 + \|\partial_t\xi^n\|^2\right)\\&\quad + C\Delta t\sum_{j=1}^{n-1}\|\xi^j\|^2 + C\|\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h)\|^2 + \frac{\mu_0}{2}\|\zeta^n\|^2.\end{aligned} \tag{68}$$

Substituting (65) and (67) into (68), we have

$$\begin{aligned}\|\zeta^n\|^2 &\leq C\left(\|\xi^0\|^2 + \|\alpha^n\|^2 + \|\beta^n\|^2\right) + C\Delta t\sum_{j=1}^{n}\left(\|\alpha^j\|^2 + \|\beta^j\|^2\right)\\&\quad + C\left(\|\partial_t\tilde{\phi}^n\|^2 + \|\partial_t\tilde{\xi}^n\|^2 + h^2\|\boldsymbol{\sigma}_t^n\|_1^2 + \|\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h)\|^2\right)\\&\quad + C\Delta t\sum_{j=1}^{n}\left(h^2\|\boldsymbol{\sigma}_t^j\|_1^2 + \|\varepsilon^j(k\tilde{\boldsymbol{\sigma}}_h)\|^2 + \|\partial_t\tilde{\phi}^j\|^2 + \|\partial_t\tilde{\xi}^j\|^2\right).\end{aligned} \tag{69}$$

Choose $v_h = \operatorname{div}\zeta^n$ in (59e) to obtain

$$\|\operatorname{div}\zeta^n\|^2 \leq C\left(\|\partial_t\phi^n\|^2 + \|\partial_t\tilde{\phi}^n\|^2 + \|\beta^n\|^2\right). \tag{70}$$

Substituting (67) into (70), we get

$$\begin{aligned}\|\operatorname{div}\zeta^n\|^2 &\le C(\|\xi^0\|^2+\|\alpha^n\|^2+\|\beta^n\|^2)+C\Delta t\sum_{j=1}^{n}(\|\alpha^j\|^2+\|\beta^j\|^2)\\&\quad+C(\|\partial_t\tilde{\phi}^n\|^2+\|\partial_t\tilde{\xi}^n\|^2+h^2\|\boldsymbol{\sigma}_t^n\|_1^2+\|\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h)\|^2)\\&\quad+C\Delta t\sum_{j=1}^{n}(h^2\|\boldsymbol{\sigma}_t^j\|_1^2+\|\varepsilon^j(k\tilde{\boldsymbol{\sigma}}_h)\|^2+\|\partial_t\tilde{\phi}^j\|^2+\|\partial_t\tilde{\xi}^j\|^2).\end{aligned}\tag{71}$$

Finally, we estimate $\|u(t_m)-U^m\|$. Setting $v_h=\phi^n$ in (59e), $\mathbf{w}_h=\zeta^n$ in (59c), and $\mathbf{z}_h=\xi^n$ in (59d), we get

$$\begin{aligned}&(c\partial_t\phi^n,\phi^n)+(a\xi^n,\gamma_h\xi^n)+(b\partial_t\xi^n,\gamma_h\xi^n)\\&\quad=-(c\partial_t\tilde{\phi}^n,\phi^n)-(c\beta^n,\phi^n)-(b\partial_t\tilde{\xi}^n,\gamma_h\xi^n)-(b\boldsymbol{\sigma}_t^n,(I-\gamma_h)\xi^n)\\&\qquad-(b\alpha^n,\gamma_h\xi^n)-\Delta t\sum_{j=0}^{n-1}(k^{n,j}\xi^j,\gamma_h\xi^n)-(\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h),\gamma_h\xi^n).\end{aligned}\tag{72}$$

Noting that $(c\partial_t\phi^n,\phi^n)\ge\frac{1}{2\Delta t}(\|c^{\frac12}\phi^n\|^2-\|c^{\frac12}\phi^{n-1}\|^2)$, and using Lemmas 3.3 and 3.4, we obtain

$$\begin{aligned}&\frac{1}{2\Delta t}(\|c^{\frac12}\phi^n\|^2-\|c^{\frac12}\phi^{n-1}\|^2)+\mu_1\|\xi^n\|^2\\&\quad\le C(\|\phi^n\|^2+\|\partial_t\xi^n\|^2)+C(\|\partial_t\tilde{\phi}^n\|^2+\|\beta^n\|^2+\|\partial_t\tilde{\xi}^n\|^2+\|\alpha^n\|^2+h^2\|\boldsymbol{\sigma}_t^n\|_1^2)\\&\qquad+C\Delta t\sum_{j=0}^{n-1}\|\xi^j\|^2+C\|\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h)\|^2+\frac{\mu_1}{2}\|\xi^n\|^2.\end{aligned}\tag{73}$$

Summing from $n=1$ to m, multiplying (73) by $2\Delta t$, and using (67), we get

$$\begin{aligned}c_0\|\phi^m\|^2 &\le C(\|\phi^0\|^2+\|\xi^0\|^2)+C\Delta t\sum_{n=1}^{m}(\|\alpha^n\|^2+\|\beta^n\|^2)+C\Delta t\sum_{n=1}^{m}\|\phi^n\|^2\\&\quad+C\Delta t\sum_{n=1}^{m}(h^2\|\boldsymbol{\sigma}_t^n\|_1^2+\|\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h)\|^2+\|\partial_t\tilde{\phi}^n\|^2+\|\partial_t\tilde{\xi}^n\|^2).\end{aligned}\tag{74}$$

Choose Δt in (74) to satisfy $C\Delta t<\frac{c_0}{2}$, and use Gronwall's inequality to get

$$\begin{aligned}\|\phi^m\|^2 &\le C(\|\phi^0\|^2+\|\xi^0\|^2)+C\Delta t\sum_{n=1}^{m}(\|\alpha^n\|^2+\|\beta^n\|^2)\\&\quad+C\Delta t\sum_{n=1}^{m}(h^2\|\boldsymbol{\sigma}_t^n\|_1^2+\|\varepsilon^n(k\tilde{\boldsymbol{\sigma}}_h)\|^2+\|\partial_t\tilde{\phi}^n\|^2+\|\partial_t\tilde{\xi}^n\|^2).\end{aligned}\tag{75}$$

Now, we note that

$$\|\alpha^n\|^2\le C\Delta t\int_{t_{n-1}}^{t_n}\|\boldsymbol{\sigma}_{tt}\|^2\,\mathrm{d}t,\qquad\|\partial_t\tilde{\phi}^n\|^2\le C\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|\tilde{\phi}_t\|^2\,\mathrm{d}t,\tag{76}$$

$$\left\|\beta^n\right\|^2 \le C\Delta t\int_{t_{n-1}}^{t_n}\|u_{tt}\|^2\,\mathrm{d}t, \qquad \left\|\partial_t\tilde{\xi}^n\right\|^2 \le C\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|\tilde{\xi}_t\|^2\,\mathrm{d}t. \tag{77}$$

Using (19a)-(19c), we have

$$\|\tilde{\boldsymbol\sigma}_h\|^2 \le C\left(\|\boldsymbol\sigma\|^2 + h^2\|\boldsymbol\lambda\|_1^2 + \int_0^t\|\boldsymbol\sigma\|^2\,\mathrm{d}t\right),$$

$$\|\tilde{\boldsymbol\sigma}_{ht}\|^2 \le C\left(\sum_{i=0}^{1}\left(\left\|\frac{\partial^i\boldsymbol\sigma}{\partial t^i}\right\|^2 + h^2\left\|\frac{\partial^i\boldsymbol\lambda}{\partial t^i}\right\|_1^2\right) + \int_0^t\left(\|\boldsymbol\sigma\|^2 + h^2\|\boldsymbol\lambda\|_1^2\right)\mathrm{d}t\right),$$

and we have

$$\left\|\varepsilon^n(k\tilde{\boldsymbol\sigma}_h)\right\|^2 \le C\Delta t^2\sum_{i=0}^{1}\int_0^{t_n}\left(\left\|\frac{\partial^i\boldsymbol\sigma}{\partial t^i}\right\|^2 + h^2\left\|\frac{\partial^i\boldsymbol\lambda}{\partial t^i}\right\|_1^2\right)\mathrm{d}t. \tag{78}$$

Further, using (59a) and (59b), we get

$$\left\|\xi^0\right\|^2 \le Ch^2\left(\left\|\boldsymbol\sigma(0)\right\|_1^2 + \left\|\boldsymbol\lambda(0)\right\|_1^2\right), \tag{79}$$

$$\left\|\phi^0\right\| \le \left\|\tilde{\phi}^0\right\| \le Ch\left(\left\|u(0)\right\|_1 + \left\|\boldsymbol\sigma(0)\right\|_1 + \left\|\boldsymbol\lambda(0)\right\|_1\right). \tag{80}$$

Finally, apply the triangle inequality to obtain the error estimates. □

6 Numerical example

For confirming the above theoretical analysis, we give a numerical example and consider the spatial and temporal domain $\Omega = (0,1)\times(0,1)$, $J = (0,1]$, the coefficients $a(x) = 1 + 2x_1^2 + x_2^2$, $b(x) = 1 + x_1^2 + 2x_2^2$, $c(x) = 1$, $k(x,t,\tau) = (1 + x_1^2 + x_2^2 + t^2)\tau$, and the initial function

$$u(x,0) = x_1(x_1-1)x_2(x_2-1).$$

The exact solution is

$$u(x,t) = e^{-t}x_1(x_1-1)x_2(x_2-1),$$

and the source function $f(x,t)$, auxiliary variables $\boldsymbol\sigma(x,t) = -\nabla u(x,t)$ and $\boldsymbol\lambda(x,t) = -(a(x)\nabla u(x,t) + b(x)\nabla u_t(x,t) + \int_0^t k(x,t,\tau)\nabla u(x,\tau)\,\mathrm{d}\tau)$ are determined by the above functions.

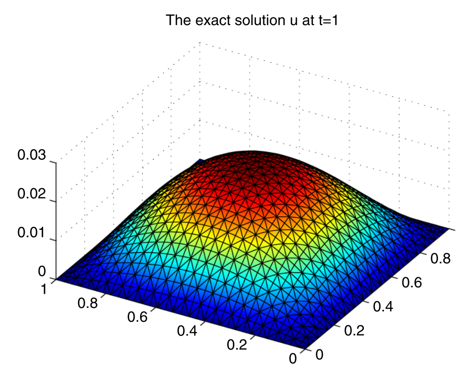
We use the fifth order Gauss quadrature rule to calculate the errors $\|u - u_h\|_{L^\infty(L^2(\Omega))}$, $\|\boldsymbol\sigma - \boldsymbol\sigma_h\|_{L^\infty((L^2(\Omega))^2)}$, $\|\boldsymbol\lambda - \boldsymbol\lambda\|_{L^\infty(H(\mathrm{div},\Omega))}$ and $\|\boldsymbol\lambda - \boldsymbol\lambda_h\|_{L^\infty((L^2(\Omega))^2)}$. The simulation results for the backward Euler fully-discrete scheme are given in Table 1 by using RT_0 space with different mesh sizes $h = \sqrt{2}\Delta t = \frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{16}, \frac{\sqrt{2}}{32}, \frac{\sqrt{2}}{64}$. Based on the error results and convergence rates, we can verify the theoretical analysis.

The graphs of exact solutions for u, $\boldsymbol\sigma$ and $\boldsymbol\lambda$ at $t = 1$ are drawn on Figures 2, 3 and 4, respectively. The graphs of the corresponding discrete solutions for u_h^n, $\boldsymbol\sigma_h^n$ and $\boldsymbol\lambda_h^n$ with the mesh $h = \frac{\sqrt{2}}{32}$ and $\Delta t = \frac{1}{32}$ are drawn on Figures 5, 6 and 7, respectively. The numerical results and figures show that the EMCVE scheme is feasible and efficient.

Table 1 Error estimates and convergence rates

$h, \Delta t$	$\Vert u - u_h \Vert_{L^\infty(L^2(\Omega))}$	Rate	$\Vert \sigma - \sigma_h \Vert_{L^\infty((L^2(\Omega))^2)}$	Rate
$(\frac{\sqrt{2}}{8}, \frac{1}{8})$	3.8464e–003		1.6043e–002	
$(\frac{\sqrt{2}}{16}, \frac{1}{16})$	2.0591e–003	0.90	8.6877e–003	0.88
$(\frac{\sqrt{2}}{32}, \frac{1}{32})$	1.0637e–003	0.95	4.5181e–003	0.94
$(\frac{\sqrt{2}}{64}, \frac{1}{64})$	5.4043e–004	0.98	2.3037e–003	0.97
$h, \Delta t$	$\Vert \lambda - \lambda_h \Vert_{L^\infty((L^2(\Omega))^2)}$	Rate	$\Vert \lambda - \lambda \Vert_{L^\infty(H(\text{div},\Omega))}$	Rate
$(\frac{\sqrt{2}}{8}, \frac{1}{8})$	1.5432e–002		5.1929e–002	
$(\frac{\sqrt{2}}{16}, \frac{1}{16})$	8.3902e–003	0.88	2.7873e–002	0.90
$(\frac{\sqrt{2}}{32}, \frac{1}{32})$	4.3491e–003	0.95	1.4411e–002	0.95
$(\frac{\sqrt{2}}{64}, \frac{1}{64})$	2.2108e–003	0.98	7.3231e–003	0.98

Figure 2 The exact solution of *u*.



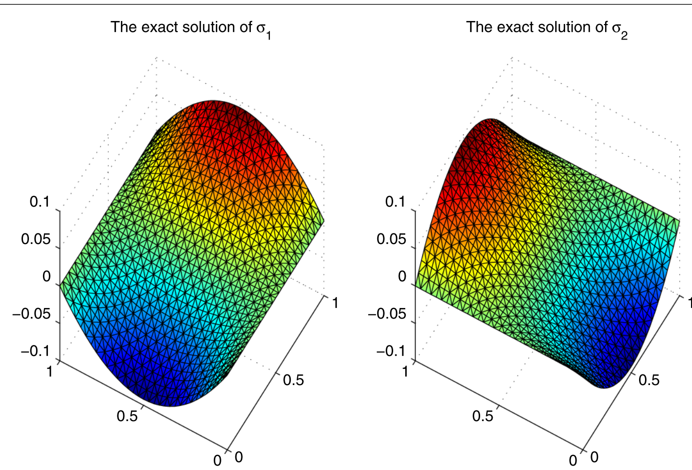


Figure 3 The exact solution of $\sigma = (\sigma_1, \sigma_2)$.

7 Conclusions

We present the EMCVE method for the 2D linear integro-differential equation of Sobolev type. We introduce the transfer operator γ_h and construct the semi-discrete, backward Euler fully-discrete EMCVE schemes. We obtain the optimal order error estimates for the scalar unknown u (in $L^2(\Omega)$-norm), gradient $\boldsymbol{\sigma}$ (in $(L^2(\Omega))^2$-norm) and flux $\boldsymbol{\lambda}$ (in $(L^2(\Omega))^2$-norm and $\mathbf{H}(\text{div}, \Omega)$-norm) by introducing the Volterra-type generalized EMCVE projection. Moreover, we give the numerical experiment to verify the theoretical analysis.

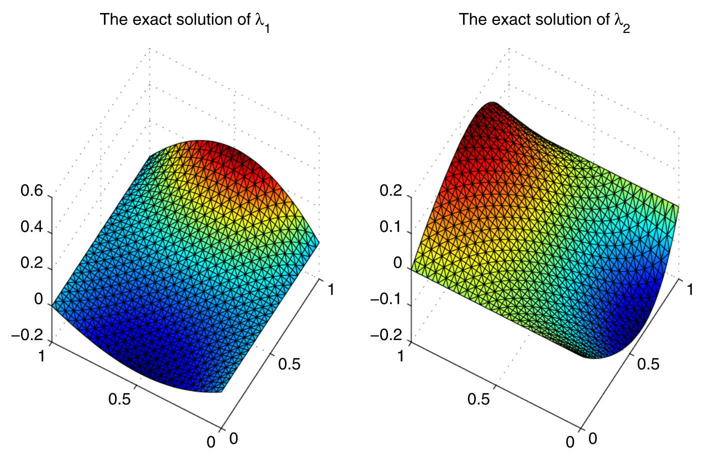
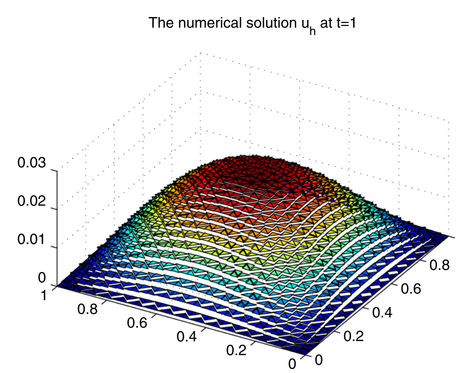


Figure 4 The exact solution of $\lambda = (\lambda_1, \lambda_2)$.

Figure 5 The numerical solution of u_h.



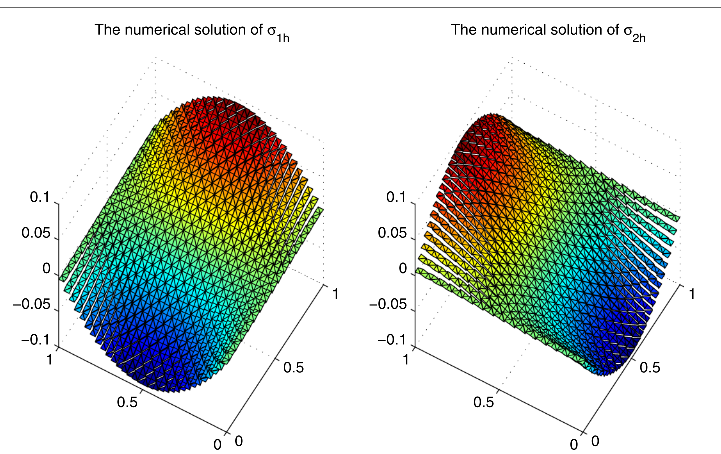


Figure 6 The numerical solution of $\sigma_h = (\sigma_{1h}, \sigma_{2h})$.

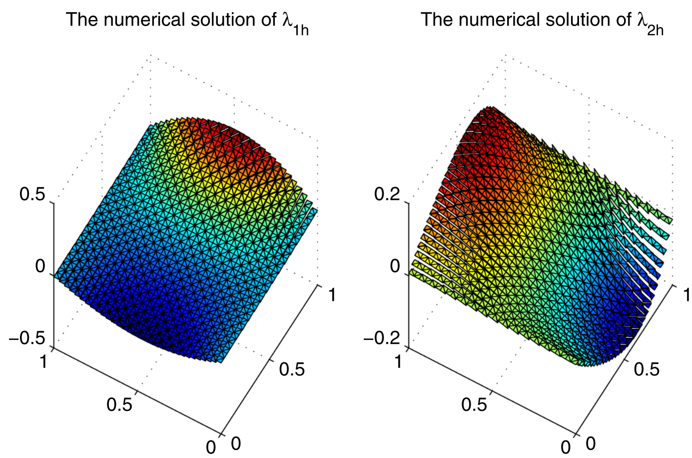


Figure 7 The numerical solution of $\lambda_h = (\lambda_{1h}, \lambda_{2h})$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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