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On nonlocal boundary value problems of nonlinear nth-order q-difference equations

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Abstract

In this paper, we study the existence and uniqueness of the solution of nonlocal boundary value problems of nonlinear nth-order q-difference equations. The uniqueness follows from the well-known Banach contraction principle. We prove that those q-solutions, under some conditions, converge to the classical solution when q approaches 1^-. A new numerical algorithm is introduced via definition of q-calculus for solving the nonlocal boundary value problem of nonlinear nth-order q-difference equations. The numerical experiments show that the algorithm is quite accurate and efficient. Moreover, numerical results are carried out to confirm the accuracy of our theoretical results of the algorithm.

1 Introduction and preliminary

Ordinary differential equations (ODEs) give a description of phenomena that change continuously. They play an important role in physics, engineering and mathematics. Generally, a solution of a system of ODEs is determined by specifying some conditions on some points in a domain. One of the interesting problems which arises in many subjects of physical science is a boundary value problem (BVP). Specific conditions of BVPs are imposed at different values of the independent variable, for instance, consider the Robin problem

$$u''(t) + f\big(t, u(t), u'(t)\big) = 0 \tag{1}$$

with the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u'(1) = 0. \tag{2}$$

Equations (1) and (2) are commonly called 'the local problem'. A further generalization of BVPs is those with nonlocal conditions. If we replace the condition $u'(1) = 0$ in (2) by $u(1) = u(\eta)$, then (1) with the conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = u(\eta), \tag{3}$$

where $\eta \in (0, 1)$, is called the nonlocal problem. Note that (1), (2) is a particular case of (1), (3) when we have the limit $\eta \to 1^-$.

Nonlocal BVPs seem to be more interesting than the local ones not only they are more natural but also because of their numerous applications. Moreover, in the numerical experiment, the calculation of the value of a local boundary condition, such as $u'(1)$ in (2), is more difficult than that of the nonlocal condition $(u(\eta) - u(1))/(\eta - 1)$ in (3). For more information as regards nonlocal BVPs see [1–4] and [5].

Recently, the study of q-difference equations has played an important role in various fields of physics and mathematics, especially in quantum mechanics, due to its numerous applications. Starting from the re-introduction of the q-difference operator by Jackson [6] in 1908, the subject of q-difference equations has been deeply studied by several authors; some examples of results can be found in [7–11] and [12].

The existence and uniqueness of solutions of q-difference BVPs have been studied by several authors; see [7, 8, 13]. For nonlocal q-difference problems, Ahmad and Nieto [14] recently proved that a solution to the problems given by

$$\begin{cases} D_q^3 u(t) = f(t, u(t)), t \in [0,1]_q, \\ u(0) = 0, D_q u(0) = 0, u(1) = \alpha u(\eta), \end{cases} \tag{4}$$

where $f \in C([0,1]_q \times \mathbb{R}, \mathbb{R})$, $\eta \in \{q^n : n \in \mathbb{N}\}$ and $\alpha \neq \frac{1}{\eta^2}$, does exist and it is unique.

In this paper, for a given positive integer n, we study the existence and uniqueness of the solution of the following nonlocal boundary value problem of nonlinear singular nth-order q-difference equations:

$$\begin{cases} D_q^n u(t) = f(t, u, D_q u, \dots, D_q^m), 0 \leq t \leq 1, \\ D_q^k u(0) = 0, u(1) = \alpha u(\eta), k = 0, 1, 2, \dots, n-2, \end{cases} \tag{5}$$

where $\eta \in [0,1]_q$ and m is a non-negative integer with $m \leq n-1$, $\alpha \neq \frac{1}{\eta^{n-1}}$, and f is a continuous function in $C([0,1]_q \times \mathbb{R}^{m+1}, \mathbb{R})$, $[0,1]_q = \{q^n : n \in \mathbb{N}\} \cup \{0,1\}$.

In the last section of this paper, we proceed to its numerical solution and give the comparison with the ordinary difference equations.

2 Preliminary

In order to introduce our results, we recall some needed concepts about q-calculus and q-difference equations. From now let $0 < q < 1$. The q-difference operator, re-introduced by Jackson, is defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0;$$

while the q-derivative at zero is defined by

$$D_q f(0) = \lim_{t \to 0} D_q f(t).$$

Note that when the left limit as q approaches 1 of the q-derivative is a classical derivative $\frac{df}{dt}$. The higher order q-derivatives are defined inductively as

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t).$$

Further, for a multivariable real continuous function $f(x_1,x_2,\dots,x_n)$, the partial q-derivative with respect to x_i is defined as

$$\frac{\partial_q f(x_1,x_2,\dots,x_n)}{\partial_q x_i}=\frac{f(x_1,x_2,\dots,x_n)-f(x_1,x_2,\dots,x_{i-1},qx_i,x_{i+1},\dots,x_n)}{(1-q)x_i}.$$

For $x_i=0$, the partial q-derivative with respect to x_i at zero is defined by

$$\left.\frac{\partial_q f(x_1,x_2,\dots,x_n)}{\partial_q x_i}\right|_{x_i=0}=\lim_{x_i\to 0}\frac{\partial_q f(x_1,x_2,\dots,x_n)}{\partial_q x_i}.$$

Also, the higher order partial q-derivative with respect to x_i is defined as

$$\frac{\partial_q^n f(x_1,x_2,\dots,x_n)}{\partial_q x_i^n}=\frac{\partial_q}{\partial_q x_i}\frac{\partial_q^{n-1} f(x_1,x_2,\dots,x_n)}{\partial_q x_i^{n-1}}.$$

In 1910, Jackson [15] has generalized the so-called q-integral, firstly introduced by Thomae [16], in the following way:

$$\int_0^t f(s)\,d_qs=\sum_{n=0}^{\infty}t(1-q)q^n f\bigl(tq^n\bigr),$$

provided that the series converges, and generally

$$\int_a^b f(s)\,d_qs=\int_0^b f(s)\,d_qs-\int_0^a f(s)\,d_qs.$$

Denote by

$$I_q^0 f(t)=f(t),\qquad I_q^n f(t)=I_qI_q^{n-1}f(t)=\int_0^t\int_0^{t_{n-1}}\cdots\int_0^{t_1}f(s)\,d_qs\,d_qt_1\cdots d_qt_{n-1},\quad n\in\mathbb{N}.$$

Remark that the fundamental theorem of q-calculus is $D_qI_qf(t)=f(t)$, and if f is continuous at $x=0$, then $I_qD_qf(x)=f(x)-f(0)$.

Let k be a real or complex number, the kth basic number is defined by $[k]_q=\frac{1-q^k}{1-q}$. For a non-negative integer n, $[0]_q!=1$, $[n-1]_q!=[n-1]_q[n-2]_q\cdots[1]_q$. The q-analog of the binomial coefficient can be defined as

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[n-k]_q![k]_q!}.$$

We will also use the notation

$$(a;q)_0=1,\qquad (a;q)_n=\prod_{k=0}^{n-1}\bigl(1-aq^k\bigr),\quad n\in\mathbb{N}.$$

The q-analog of the power function $(a-b)_q^{(n)}$ with $n\in\mathbb{N}_0$ is

$$(a-b)_q^{(0)}=1,\qquad (a-b)_q^{(n)}=\prod_{k=0}^{n-1}\bigl(a-bq^k\bigr).$$

Al-Salam [17] proved the q-analog of Cauchy's formula as follows:

$$I_q^n f(t) = \frac{t^{n-1}}{[n-1]_q!} \int_0^t \left(\frac{qs}{t};q\right)_{n-1} f(s)\, d_q s.$$

The q-Leibniz product rule and q-integration by parts formula are

$$D_q(gh)(t) = D_q g(t)h(t) + g(qt)D_q h(t)$$

and

$$\int_0^t f(s)D_q g(s)\, d_q s = \big[f(s)g(s)\big]_0^t - \int_0^t D_q f(s)g(qs)\, d_q s,$$

respectively.

3 Existence and uniqueness of solutions

Let C_q^m be the space of all real-valued continuous functions defined on $[0,1]_q$ such that the first m q-derivatives $D_q f(t), D_q^2 f(t), \ldots, D_q^m f(t)$ exist. We equip this space with the norm

$$\|u\|_q = \max_{0\le k\le m} \left\{ \left\| D_q^k u \right\|_{q,\infty} \right\}, \quad u \in C_q, m \le n-1,$$

where

$$\|u\|_{q,\infty} = \max_{t\in[0,1]_q} \left\{ \left|u(t)\right| \right\}.$$

Lemma 3.1 *Define the Green function $G(t,s;q)$ by*

$$\begin{aligned} G(t,s;q) &= \frac{1}{[n-1]_q!}(t-qs)_q^{(n-1)} + \frac{t^{n-1}[\alpha(\eta-qs)_q^{(n-1)} - (1-qs)_q^{(n-1)}]}{1-\alpha\eta^{n-1}}, \quad 0 \le s < \min\{\eta,t\}; \\ &= \frac{1}{[n-1]_q!} \frac{t^{n-1}[\alpha(\eta-qs)_q^{(n-1)} - (1-qs)_q^{(n-1)}]}{1-\alpha\eta^{n-1}}, \quad 0 \le t < s < \eta < 1; \\ &= \frac{1}{[n-1]_q!}(t-qs)_q^{(n-1)} - \frac{t^{n-1}(1-qs)_q^{(n-1)}}{1-\alpha\eta^{n-1}}, \quad 0 \le \eta < s < t \le 1; \\ &= \frac{1}{[n-1]_q!}\left(-\frac{t^{n-1}(1-qs)_q^{(n-1)}}{(1-\alpha\eta^{n-1})}\right), \quad 0 \le \max\{\eta,t\} < s \le 1. \end{aligned} \tag{6}$$

Then $u(t)$ is a solution of (5) *if and only if*

$$\begin{aligned} u(t) &= \frac{1}{[n-1]_q!}\int_0^t (t-qs)_q^{(n-1)} f\bigl(s,u,D_q u,\ldots,D_q^m u\bigr)\,d_q s \\ &\quad + \frac{t^{n-1}}{[n-1]_q!(1-\alpha\eta^{n-1})}\biggl[\alpha\int_0^\eta (\eta-qs)_q^{(n-1)} f\bigl(s,u,D_q u,\ldots,D_q^m u\bigr)\,d_q s \\ &\quad - \int_0^1 (1-qs)_q^{(n-1)} f\bigl(s,u,D_q u,\ldots,D_q^m u\bigr)\,d_q s\biggr] \\ &:= \int_0^1 G(t,s;q) f\bigl(s,u,D_q u,\ldots,D_q^m u\bigr)\,d_q s. \end{aligned} \tag{7}$$

Proof Integrating the equation $D_q^n u = f(t,u,D_q u,\ldots,D_q^m u)$ n-times and using the q-analog of the Cauchy formula, we obtain

$$u(t) = \frac{1}{[n-1]_q!}\int_0^t (t-qs)_q^{(n-1)} f\bigl(s,u,D_q u,\ldots,D_q^m u\bigr)\,d_q s + \sum_{j=0}^{n-1} a_j t^j, \tag{8}$$

where a_j, $j = 0,1,2,\ldots,n-1$, are arbitrary constants. In fact, we have

$$\begin{aligned} D_q^m u(t) &= I_q^{n-m} f\bigl(t,u,D_q u,\ldots,D_q^m u\bigr) \\ &= \frac{1}{[n-m-1]_q!}\int_0^t (t-qs)_q^{(n-m-1)} f\bigl(s,u,D_q u,\ldots,D_q^m u\bigr) d_q s \\ &\quad + \sum_{j=0}^{n-m-1} a_{n-j-1} t^{n-m-j-1}. \end{aligned} \tag{9}$$

Using the boundary conditions $D_q^k u(0) = 0$, for $k = 0,1,2,\ldots,n-2$, in (9) and $u(1) = \alpha u(\eta)$, we find that $a_0 = a_1 = \cdots = a_{n-2} = 0$ and

$$\begin{aligned} a_{n-1} &= \frac{1}{[n-1]_q!(1-\alpha\eta^{n-1})}\biggl[\alpha\int_0^\eta (\eta-qs)_q^{(n-1)} f\bigl(s,u,D_q u,\ldots,D_q^m u\bigr)\,d_q s \\ &\quad - \int_0^1 (1-qs)_q^{(n-1)} f\bigl(s,u,D_q u,\ldots,D_q^m u\bigr)\,d_q s\biggr]. \end{aligned}$$

Substituting the values of $a_0, a_1, \ldots, a_{n-1}$ in (8), we obtain (7). This completes the proof. □

Remark 3.2 If q approaches 1 on the left, then equation (7) takes the form

$$u(t) = \int_0^1 G(t,s) f\bigl(s,u,u',\ldots,u^{(m)}\bigr)\,ds, \tag{10}$$

with the associated form of Green's function for the classical case defined by

$$\begin{aligned} &G(t,s) \\ &\quad = \frac{1}{(n-1)!}(t-s)^{n-1} + \frac{t^{n-1}[\alpha(\eta-s)^{n-1} - (1-s)^{n-1}]}{1-\alpha\eta^{n-1}}, \quad 0 \le s < \min\{\eta,t\}; \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(n-1)!}\frac{t^{n-1}[\alpha(\eta-s)^{n-1}-(1-s)^{n-1}]}{1-\alpha\eta^{n-1}}, \quad 0 \le t < s < \eta < 1;\\
&= \frac{1}{(n-1)!}\{(t-s)^{n-1} - \frac{t^{n-1}(1-s)^{n-1}}{1-\alpha\eta^{n-1}}, \quad 0 \le \eta < s < t \le 1;\\
&= \frac{1}{(n-1)!}\left(-\frac{t^{n-1}(1-s)^{n-1}}{1-\alpha\eta^{n-1}}\right), \quad 0 \le \max\{\eta,t\} < s \le 1.
\end{aligned}
\tag{11}
$$

This solution is equivalent to the solution of a classical nonlinear nth-order boundary value problem,

$$
\begin{aligned}
&u^{(n)}(t) = f\big(t,u,u',\dots,u^{(m)}\big), \qquad u^{(k)}(0) = 0,\\
&u(1) = \alpha u(\eta), \quad k = 0,1,2,\dots,n-2, m \le n-1,
\end{aligned}
\tag{12}
$$

where f is a continuous function in $C([0,1] \times \mathbb{R}^{m+1},\mathbb{R})$.

To accomplish the main results, we define an integral operator $T_q : C_q^m \to C_q^m$ by

$$
\begin{aligned}
T_q u(t) &= \int_0^1 G(t,s;q) f\big(s,u,D_q u,\dots,D_q^m u\big)\, d_q s,\\
&= \frac{1}{[n-1]_q!}\int_0^t (t-qs)_q^{(n-1)} f\big(s,u,D_q u,\dots,D_q^m u\big)\, d_q s\\
&\quad + \frac{t^{n-1}}{[n-1]_q!(1-\alpha\eta^{n-1})}\bigg[\alpha\int_0^\eta (\eta-qs)_q^{(n-1)} f\big(s,u,D_q u,\dots,D_q^m u\big)\, d_q s\\
&\quad - \int_0^1 (1-qs)_q^{(n-1)} f\big(s,u,D_q u,\dots,D_q^m u\big)\, d_q s\bigg].
\end{aligned}
$$

Obviously, T_q is well defined and it is easy to see that $u \in C_q^m$ is a solution of BVP (5) if and only if u is a fixed point of T_q.

The following lemma will be required in our investigation.

Lemma 3.3 *For $k = 0,1,\dots,m$,*

$$
D_q^k T_q u(t) = \int_0^1 \frac{\partial_q^k G(t,s;q)}{\partial_q t} f\big(s,u,D_q u,\dots,D_q^m u\big)\, d_q s,
$$

where the function $\partial_q^k G(t,s;q)/\partial_q t$ is defined by

$$
\begin{aligned}
&\frac{\partial_q^k G(t,s;q)}{\partial_q t}\\
&= \frac{1}{[n-k-1]_q!}(t-qs)_q^{(n-k-1)}\\
&\quad + \frac{t^{n-k-1}[\alpha(\eta-qs)_q^{(n-1)} - (1-qs)_q^{(n-1)}]}{1-\alpha\eta^{n-1}}, \quad 0 \le s < \min\{\eta,t\};\\
&= \frac{1}{[n-k-1]_q!}\frac{t^{n-k-1}[\alpha(\eta-qs)_q^{(n-1)} - (1-qs)_q^{(n-1)}]}{1-\alpha\eta^{n-1}}, \quad 0 \le t < s < \eta < 1;
\end{aligned}
$$

$$= \frac{1}{[n-k-1]_q!}(t-qs)_q^{(n-k-1)} - \frac{t^{n-k-1}(1-qs)_q^{(n-1)}}{1-\alpha\eta^{n-1}}, \quad 0 \le \eta < s < t \le 1;$$

$$= \frac{1}{[n-k-1]_q!}\left(-\frac{t^{n-k-1}(1-qs)_q^{(n-1)}}{(1-\alpha\eta^{n-1})}\right), \quad 0 \le \max\{\eta, t\} < s \le 1. \tag{13}$$

Proof The result is directly obtained by equation (9). □

Lemma 3.4 *For any positive integer m, we have*

$$\int_0^x (x-qs)_q^{(m)}\, d_q s = x^{m+1}\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j+1]_q}.$$

Proof Since

$$\begin{aligned}(1-qs)_q^{(m)} &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q q^{\frac{j(j+1)}{2}}(-s)^j \\ &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q (-1)^j q^{\frac{j(j+1)}{2}} s^j,\end{aligned}$$

it follows that

$$\begin{aligned}\int_0^x (x-qs)_q^{(m)}\, d_q s &= \int_0^x x^m \left(1-q\frac{s}{x}\right)_q^{(m)} d_q s \\ &= \int_0^x x^m \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q (-1)^j q^{\frac{j(j+1)}{2}} \left(\frac{s}{x}\right)^j d_q s \\ &= x^{m+1}\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j+1]_q}.\end{aligned}$$

□

In order to state our main result, we firstly define

$$\mathcal{G}_{q,k+1} = \max_{t\in[0,1]_q}\left|\int_0^1 \frac{\partial_q^k G(t,s;q)}{\partial_q t}\, d_q s\right|$$

and

$$\mathcal{G}_q = \max\{\mathcal{G}_{q,1}, \mathcal{G}_{q,2}, \ldots, \mathcal{G}_{q,m+1}\}.$$

By applying Lemma 3.3 and Lemma 3.4, we can estimate the constants $\mathcal{G}_{q,k+1}$, $k = 0, 1, \ldots, m$, by

$$\mathcal{G}_{q,k+1} = \frac{1}{[n-k-1]_q!}\max\left\{|C_{k+1} - \gamma C_1|, \frac{q^{n-k-1}[n-k-1]_q^{n-k-1}|\gamma|^{n-k}C_1^{n-k}}{C_{k+1}^{n-k-1}[n-k]_q^{n-k}}\right\}.$$

In the case $m = n-1$, we have

$$\mathcal{G}_{q,n} = \max\{1, |1-\gamma C_1|\},$$

where

$$C_k = \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j+1]_q},$$
$$\gamma = \eta + \frac{1-\eta}{1-\alpha\eta^{n-1}}.$$

Based on Banach's fixed point theorem [18], we obtain the following result.

Theorem 3.5 *Let $f : [0,1]_q \times \mathbb{R}^{m+1} \to \mathbb{R}$ be a continuous function, and there exist positive functions $L_j(t) \in C([0,1]_q)$, $j = 1,2,\ldots,m$, such that*

$$\bigl|f(t,u_0,u_1,\ldots,u_m) - f(t,v_0,v_1,\ldots,v_m)\bigr| \le \sum_{j=0}^{m} L_j(t)|u_j - v_j|, \quad t \in [0,1]_q. \tag{14}$$

Then the boundary value problem (5) *has a unique solution, provided $\Lambda_q \mathcal{G}_q < 1$, where $\Lambda_q = \sum_{j=1}^{m} \|L_j\|_{q,\infty}$.*

Proof According to the Banach contraction theorem, if T_q is a contractive mapping then it has a unique fixed point which coincides with the unique solution of problem (5). To obtain the contractive property of T_q, let $u, v \in C_q^m$ and for each $t \in I_q$, we have

$$\begin{aligned}
|T_q u - T_q v| &= \left| \int_0^1 G(t,s;q)\bigl(f\bigl(s,u,D_q u,\ldots,D_q^m u\bigr) - f\bigl(s,v,D_q v,\ldots,D_q^m v\bigr)\bigr)\, d_q s \right| \\
&\le \left| \int_0^1 G(t,s;q) \left(\sum_{j=0}^{m} L_j(t)\bigl|D_q^j(u-v)\bigr| \right) d_q s \right| \\
&\le \left| \bigl(\Lambda_q \|u-v\|_q\bigr) \int_0^1 G(t,s;q)\, d_q s \right| \\
&\le \Lambda_q \mathcal{G}_{q,1} \|u-v\|_q \le \Lambda_q \mathcal{G}_q \|u-v\|_q.
\end{aligned} \tag{15}$$

It was shown in Lemma 3.3 that for each $k = 0,1,\ldots,m$

$$\begin{aligned}
\bigl|D_q^k(T_q u - T_q v)\bigr| &= \left| \int_0^1 \frac{\partial_q^k G(t,s;q)}{\partial_q t} \bigl(f\bigl(s,u,D_q u,\ldots,D_q^m u\bigr) - f\bigl(s,v,D_q v,\ldots,D_q^m v\bigr)\bigr)\, d_q s \right| \\
&\le \left| \int_0^1 \frac{\partial_q^k G(t,s;q)}{\partial_q t} \left(\sum_{j=0}^{m} L_j(t)\bigl|D_q^j(u-v)\bigr| \right) d_q s \right| \\
&\le \left| \bigl(\Lambda_q \|u-v\|_q\bigr) \int_0^1 \frac{\partial_q^k G(t,s;q)}{\partial_q t}\, d_q s \right| \\
&\le \Lambda_q \mathcal{G}_{q,k+1} \|u-v\|_q \le \Lambda_q \mathcal{G}_q \|u-v\|_q.
\end{aligned} \tag{16}$$

Therefore, we obtain $\|Tu - Tv\|_q \le \Lambda_q \mathcal{G}_q \|u-v\|_q$. The required result comes from the assumption $\Lambda_q \mathcal{G}_q < 1$. □

By letting q approach 1, we obtain the following existence and uniqueness result for the classical nonlinear nth-order boundary value problem.

Theorem 3.6 *Let $f : [0,1] \times \mathbb{R}^{m+1} \to \mathbb{R}$ be a continuous function, and there exist positive functions $L_j(t) \in C([0,1])$, $j = 0,1,\dots,m$, such that*

$$\bigl|f(t,u_0,u_1,\dots,u_m) - f(t,v_0,v_1,\dots,v_m)\bigr| \le \sum_{j=0}^{m} L_j(t)|u_j - v_j|, \quad t \in [0,1]. \tag{17}$$

Then the boundary value problem (12) *has a unique solution, provided $\Lambda\mathcal{G} < 1$, where $\Lambda = \sum_{j=0}^{m} \|L_j\|_\infty$ and $\mathcal{G} = \lim_{q\to 1^-} \mathcal{G}_q$.*

4 Numerical experiments

In this section, we prove the error estimate between the q-difference solution and the classical solution. Let u_q be the solution of (5) and u be the solution of (12). We note here that, for any function $f : [0,1] \times \mathbb{R}^{m+1} \to \mathbb{R}$ in problem (12), for convenience its restriction $f|_{[0,1]_q \times \mathbb{R}^{m+1}}$ will be written as f when we consider problem (5), similarly to the function u, so that $\|u - u_q\|_{q,\infty}$ is well defined.

Theorem 4.1 *Let $f : [0,1] \times \mathbb{R}^{m+1} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition* (17). *If $\max\{\Lambda_q\mathcal{G}_q, \Lambda\mathcal{G}\} < 1$, then*

$$\|u - u_q\|_{q,\infty} \le \|Tu_0 - T_q u_0\|_{q,\infty} + \frac{\Lambda\mathcal{G}}{1-\Lambda\mathcal{G}}\|Tu_0 - u_0\|_\infty + \frac{\Lambda_q\mathcal{G}_q}{1-\Lambda_q\mathcal{G}_q}\|T_q u_0 - u_0\|_{q,\infty}. \tag{18}$$

Moreover, if $u_0 = u$, we have

$$\|u - u_q\|_{q,\infty} \le \frac{1}{1-\Lambda_q\mathcal{G}_q}\|Tu_0 - T_q u_0\|_{q,\infty}. \tag{19}$$

Proof To prove the error estimate, the following bounds are valid by Theorem 3.5 and Theorem 3.6:

$$\begin{aligned}
\bigl\|T_q^n u_0 - u_q\bigr\|_\infty &\le \frac{(\Lambda_q\mathcal{G}_q)^n}{1-\Lambda_q\mathcal{G}_q}\|T_q u_0 - u_q\|_{q,\infty},\\
\bigl\|T^n u_0 - u\bigr\|_\infty &\le \frac{(\Lambda\mathcal{G})^n}{1-\Lambda\mathcal{G}}\|Tu_0 - u\|_\infty,
\end{aligned}$$

respectively. It is easy to see that

$$\bigl\|T^n u_0 - u\bigr\|_{q,\infty} \le \bigl\|T^n u_0 - u\bigr\|_\infty.$$

Thus, we have

$$\begin{aligned}
\|u - u_q\|_{q,\infty} &\le \bigl\|T^n u_0 - u\bigr\|_{q,\infty} + \bigl\|T^n u_0 - T_q^n u_0\bigr\|_{q,\infty} + \bigl\|T_q^n u_0 - u_q\bigr\|_{q,\infty}\\
&\le \frac{(\Lambda\mathcal{G})^n}{1-\Lambda\mathcal{G}}\|Tu_0 - u\|_\infty + \bigl\|T^n u_0 - T_q^n u_0\bigr\|_{q,\infty}\\
&\quad + \frac{(\Lambda_q\mathcal{G})^n}{1-\Lambda_q\mathcal{G}}\|T_q u_0 - u_q\|_{q,\infty}.
\end{aligned} \tag{20}$$

Considering the middle term $\|T^n u_0 - T_q^n u_0\|_{q,\infty}$, we obtain

$$\begin{aligned}
&\left\| T^n u_0 - T_q^n u_0 \right\|_{q,\infty} \\
&\quad = \left\| \sum_{j=2}^{n} \bigl(T^j u_0 - T^{j-1} u_0\bigr) + (Tu_0 - T_q u_0) - \sum_{j=2}^{n} \bigl(T_q^j u_0 - T_q^{j-1} u_0\bigr) \right\|_{q,\infty} \\
&\quad \leq \sum_{j=2}^{n} \left\| T^j u_0 - T^{j-1} u_0 \right\|_{\infty} + \|Tu_0 - T_q u_0\|_{q,\infty} + \sum_{j=2}^{n} \left\| T_q^j u_0 - T_q^{j-1} u_0 \right\|_{q,\infty} \\
&\quad \leq \|Tu_0 - u_0\|_{\infty} \sum_{j=2}^{n} (\Lambda \mathcal{G})^{j-1} + \|Tu_0 - T_q u_0\|_{q,\infty} \\
&\qquad + \|T_q u_0 - u_0\|_{q,\infty} \sum_{j=2}^{n} (\Lambda_q \mathcal{G}_q)^{j-1} \\
&\quad = \frac{\Lambda \mathcal{G}[1 - (\Lambda \mathcal{G})^{n-2}]}{1 - \Lambda \mathcal{G}} \|Tu_0 - u_0\|_{\infty} + \|Tu_0 - T_q u_0\|_{q,\infty} \\
&\qquad + \frac{\Lambda_q \mathcal{G}_q [1 - (\Lambda_q \mathcal{G}_q)^{n-2}]}{1 - \Lambda_q \mathcal{G}_q} \|T_q u_0 - u_0\|_{q,\infty}. \qquad (21)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|u - u_q\|_{q,\infty} \\
&\quad \leq \|Tu_0 - T_q u_0\|_{q,\infty} + \frac{(\Lambda \mathcal{G})^n}{1 - \Lambda \mathcal{G}} \|Tu_0 - u\|_{\infty} + \frac{(\Lambda_q \mathcal{G})^n}{1 - \Lambda_q \mathcal{G}} \|T_q u_0 - u_q\|_{q,\infty} \\
&\qquad + \frac{\Lambda \mathcal{G}[1 - (\Lambda \mathcal{G})^{n-2}]}{1 - \Lambda \mathcal{G}} \|Tu_0 - u_0\|_{\infty} + \frac{\Lambda_q \mathcal{G}_q [1 - (\Lambda_q \mathcal{G}_q)^{n-2}]}{1 - \Lambda_q \mathcal{G}_q} \|T_q u_0 - u_0\|_{q,\infty}.
\end{aligned}$$

The inequality (18) follows by letting $n \to \infty$, we obtain (18). Moreover, if $u_0 = u$, we have $Tu_0 = u_0$ and this implies

$$\begin{aligned}
\|u_0 - u_q\|_{q,\infty} &\leq \|Tu_0 - T_q u_0\|_{q,\infty} + \frac{\Lambda_q \mathcal{G}_q}{1 - \Lambda_q \mathcal{G}_q} \|T_q u_0 - u_0\|_{q,\infty} \\
&= \frac{1}{1 - \Lambda_q \mathcal{G}_q} \|T_q u_0 - Tu_0\|_{q,\infty}.
\end{aligned}$$

This completes the proof. □

Next, we will apply the q-Picard iterative method to solve some q-nonlinear boundary value problems and compare the approximate solutions with their exact solutions. In order to identify the method, we first establish the correctional function for (5) as

$$\begin{aligned}
u_{n+1}(t) &= \frac{1}{[n-1]_q!} \int_0^t (t - qs)_q^{(n-1)} f\bigl(s, u_n(s), \ldots, D_q^m u_n(s)\bigr) \, d_q s \\
&\quad + \frac{t^{n-1}}{[n-1]_q!(1 - \alpha \eta^{n-1})}
\end{aligned}$$

$$\times \left(\alpha \int_0^\eta (\eta - qs)_q^{(n-1)} f\big(s, u_n(s), \ldots, D_q^m u_n(s)\big) \, d_q s \right.$$
$$\left. - \int_0^1 (1 - qs)_q^{(n-1)} f\big(s, u_n(x), \ldots, D_q^m u_n(s)\big) \, d_q s \right), \tag{22}$$

where u_n represents the nth-order approximate.

Starting from the initial iteration $u_0(t)$, the successive approximate solutions can be obtained by calculating the q-integral appeared in (22). Occasionally, it might be difficult to calculate the q-integration directly due to the nonlinearity of $f(t, \cdot)$.

Now we shall present the explicit algorithm for solving an approximate solution of the q-integral via operator T_q defined above. Let $\overline{q^{\mathbb{N}}}$ be the time scale which given as $\{q^n | n \in \mathbb{N}\} \cup \{0\}$, where 0 is the cluster point of $\overline{q^{\mathbb{N}}}$. For the numerical computations, the interval $[0, q]$ is partitioned into N subintervals. Recall that the analog maximum errors are defined as

$$\big\| u_n - u^* \big\|_{L_\infty} = \max_{t \in I_q} \big| u_n(t) - u^*(t) \big|,$$

where u_n is the approximate solution with n iterations and u^* is the analytic solution.

Recall that the q-derivative of $u_n(t)$ at $t = q^j$ is defined as

$$D_q u_n\big(q^j\big) = \frac{u_n(q^j) - u_n(q^{j+1})}{(1-q)q^j}, \quad j = 1, 2, \ldots, N-1,$$

and hence the boundary condition gives

$$D_q u_n\big(q^N\big) = \frac{u_n(q^N) - un(q^{N+1})}{(1-q)q^N} \approx \frac{u_n(q^N)}{(1-q)q^N},$$

where we use $u_n(q^{N+1}) \approx 0$. By the inductive step, we obtain

$$D_q^k u_n\big(q^j\big) = \frac{D_q^{k-1} u_n(q^j) - D_q^{k-1} u_n(q^{j+1})}{(1-q)q^j}, \quad j = 1, 2, \ldots, N-1,$$

and

$$D_q^k u_n\big(q^N\big) = \frac{D_q^{k-1} u_n(q^N)}{(1-q)q^N},$$

for $k = 2, 3, \ldots, n-2$. In the case $k = n-1$, the $n-1$ times derivative of $u_n(t)$ at $t = 0$ can be defined by

$$D_q^{n-1} u_n(0) = \lim_{j \to \infty} \frac{D_q^{n-2} u_n(q^j) - D_q^{n-2} u_n(0)}{q^j}, \quad q < 1. \tag{23}$$

Then we can estimate $D_q^{n-1} u_n(0) = \frac{D_q^{n-2} u_n(q^{N-1})}{q^{N-1}}$.

We can calculate the $u_1(q^k) = Tu_0(q^k)$, where $k = 0,1,2,\dots,N-1$, and we have the quantity

$$\begin{aligned}
&\int_0^{\eta}(\eta - qs)_q^{(n-1)} f\bigl(s,u_0,D_q u_0,\dots,D_q^m u_0\bigr)\,d_q s\\
&= \int_0^{q^{N_0}} \bigl(q^{N_0} - qs\bigr)_q^{(n-1)} f\bigl(s,u_0,D_q u_0,\dots,D_q^m u_0\bigr)\,d_q s\\
&= \sum_{j=0}^{\infty}(1-q)q^{j+N_0}\bigl(q^{N_0} - q\cdot q^{j+N_0}\bigr)_q^{(n-1)} f\bigl(q^{j+N_0},u_0\bigl(q^{j+N_0}\bigr),\dots,D_q^{n-1}u_0\bigl(q^{j+N_0}\bigr)\bigr)\\
&= \sum_{j=0}^{\infty}\sum_{i=0}^{m}\begin{bmatrix} n-1\\ i\end{bmatrix}_q (-1)^i(1-q)q^{\frac{i(i+1)}{2}}q^{j(j+1)+nN_0} f\bigl(q^{j+N_0},u_0\bigl(q^{j+N_0}\bigr),\dots,D_q^{n-1}u_0\bigl(q^{j+N_0}\bigr)\bigr)\\
&\approx \sum_{j=0}^{N-N_0-1}\sum_{i=0}^{m}\begin{bmatrix} n-1\\ i\end{bmatrix}_q (-1)^i(1-q)q^{\frac{i(i+1)}{2}}q^{j(j+1)+nN_0}\\
&\quad\times f\bigl(q^{j+N_0},u_0\bigl(q^{j+N_0}\bigr),\dots,D_q^{n-1}u_0\bigl(q^{j+N_0}\bigr)\bigr),
\end{aligned}$$

where $\eta = q^{N_0}$ and the term $D_q u_0(q^j),\dots,D_q^{n-1}u_0(q^j)$ can be estimated by the above method. Similarly, the following estimations are obtained:

$$\begin{aligned}
&\int_0^{1}(1 - qs)_q^{(n-1)} f\bigl(s,u_0,D_q u_0,\dots,D_q^m u_0\bigr)\,d_q s\\
&\quad\approx \sum_{j=0}^{N-1}\sum_{i=0}^{m}\begin{bmatrix} n-1\\ i\end{bmatrix}_q (-1)^i(1-q)q^{\frac{i(i+1)}{2}}q^{j(j+1)} f\bigl(q^{j},u_0\bigl(q^{j}\bigr),\dots,D_q^{n-1}u_0\bigl(q^{j}\bigr)\bigr),\\
&\int_0^{q^k}(t - qs)_q^{(n-1)} f\bigl(s,u_0,D_q u_0,\dots,D_q^m u_0\bigr)\,d_q s\\
&\quad\approx \sum_{j=0}^{N-k-1}\sum_{i=0}^{m}\begin{bmatrix} n-1\\ i\end{bmatrix}_q (-1)^i(1-q)q^{\frac{i(i+1)}{2}}q^{j(j+1)+nk} f\bigl(q^{j+k},u_0\bigl(q^{j+k}\bigr),\dots,D_q^{n-1}u_0\bigl(q^{j+k}\bigr)\bigr).
\end{aligned}$$

By the definition of T_q and the time scale $[0,q]$, we can obtain $u_1(q^k) = T_q u_0(q^k)$ by assuming the value of $q^N \approx 0$. That is,

$$\begin{aligned}
&T_q u_0\bigl(q^k\bigr)\\
&\quad\approx \frac{1}{[n-1]_q!}\sum_{j=0}^{N-k-1}\sum_{i=0}^{m}\begin{bmatrix} n-1\\ i\end{bmatrix}_q (-1)^i(1-q)q^{\frac{i(i+1)}{2}}q^{j(j+1)+nk}\\
&\qquad\times f\bigl(q^{j+k},u_0\bigl(q^{j+k}\bigr),\dots,D_q^{n-1}u_0\bigl(q^{j+k}\bigr)\bigr)\\
&\qquad+\frac{q^{k(n-1)}}{[n-1]_q!(1-\alpha\eta^{n-1})}\left(\alpha\sum_{j=0}^{N-N_0-1}\sum_{i=0}^{m}\begin{bmatrix} n-1\\ i\end{bmatrix}_q\right.\\
&\qquad\times(-1)^i(1-q)q^{\frac{i(i+1)}{2}}q^{j(j+1)+nN_0} f\bigl(q^{j+N_0},\dots,D_q^{n-1}u_0\bigl(q^{j+N_0}\bigr)\bigr)\\
&\qquad\left.-\sum_{j=0}^{N-1}\sum_{i=0}^{m}\begin{bmatrix} n-1\\ i\end{bmatrix}_q (-1)^i(1-q)q^{\frac{i(i+1)}{2}}q^{j(j+1)} f\bigl(q^{j},u_0\bigl(q^{j}\bigr),\dots,D_q^{n-1}u_0\bigl(q^{j}\bigr)\bigr)\right).
\end{aligned}$$

By using the values of $u_{k-1}(t)$, for all $t \in [0,1]_q$, we can repeatedly calculate the $u_k(t)$, for $k = 1, 2, \ldots, n$. Next, we present two numerical experiments to illustrate the efficiency of the proposed algorithm. The computer programs are written in MATLAB by choosing $N = 1{,}000$.

Example 4.2 Let us first consider the q-difference BVP

$$\begin{cases} D_q^2 u = \frac{t}{[2]_q[4]_q} D_q u - \frac{t}{[4]_q} u + 1 - \frac{qt}{[2]_q[4]_q}, \\ u(0) = 0, u(1) = u(q), \end{cases}$$

with $f(t, u, D_q u) = \frac{t}{[2]_q[4]_q} D_q u - \frac{t}{[4]_q} u + 1 - \frac{qt}{[2]_q[4]_q}$, $n = 2$, $\alpha = 1$ and $\eta = q$.

Clearly

$$\left| f(t, v_1, v_2) - f(t, w_1, w_2) \right| \le \frac{t}{[2]_q[4]_q} |v_1 - w_1| + \frac{t}{[4]_q} |v_2 - w_2|,$$

for $0 < q < 1$, we choose $\Lambda_q = 1/[2]_q[4]_q + 1/[4]_q < 1$, $C_1 = 1/[2]_q$ and $\gamma = (1 - q^2)/(1 - q) = [2]_q$. Then we have

$$\mathcal{G}_q = \max\left\{1, |1 - \gamma C_1|\right\} = 1.$$

By using Theorem 3.5, we only can obtain the existence and uniqueness of solution of problem (4.2), however, it is not difficult to show this, and the exact solution is $u(t) = \frac{t^2}{[2]_q} - t$.

With $u_0 = 0$ and $N = 1000$, we apply the numerical schemes for solving this problem. Table 1 shows the errors for $q = 0.6, 0.7, 0.8, 0.9$ at the various values of iteration. It seems that the approximation u_n converges faster when q is large. Figure 1 presents the q-approximate

Table 1 Numerical comparison of the errors in Example 4.2

n	***q***			
	0.6	**0.7**	**0.8**	**0.9**
1	5.14630E–02	5.03004E–02	4.73555E–02	4.34192E–02
2	6.99387E–03	6.04295E–03	4.93923E–03	3.89799E–03
3	9.43402E–04	7.16415E–04	5.04360E–04	3.38723E–04
4	1.27086E–04	8.47161E–05	5.13017E–05	2.93003E–05
5	1.71157E–05	1.00129E–05	5.21482E–06	2.53324E–06
6	2.30502E–06	1.18335E–06	5.30033E–07	2.19010E–07
7	3.10421E–07	1.39850E–07	5.38716E–08	1.89344E–08
8	4.18050E–08	1.65275E–08	5.47540E–09	1.63698E–09
9	5.62994E–09	1.95323E–09	5.56510E–10	1.41532E–10
10	7.58194E–10	2.30834E–10	5.65640E–11	1.22410E–11
11	1.02108E–10	2.72809E–11	5.74957E–12	1.06670E–12
12	1.37516E–11	3.22392E–12	5.85199E–13	9.96425E–14
13	1.85246E–12	3.81251E–13	6.02296E–14	1.83742E–14
14	2.49800E–13	4.51861E–14	6.55032E–15	8.43769E–15
15	3.42504E–14	6.10623E–15	1.72085E–15	8.32667E–15
16	4.88498E–15	1.05471E–15	1.55431E–15	8.32667E–15
17	8.88178E–16	7.21645E–16	1.55431E–15	8.32667E–15
18	4.44089E–16	6.10623E–16	1.55431E–15	8.32667E–15
19	3.33067E–16	6.10623E–16	1.55431E–15	8.32667E–15
20	3.33067E–16	6.10623E–16	1.55431E–15	8.32667E–15

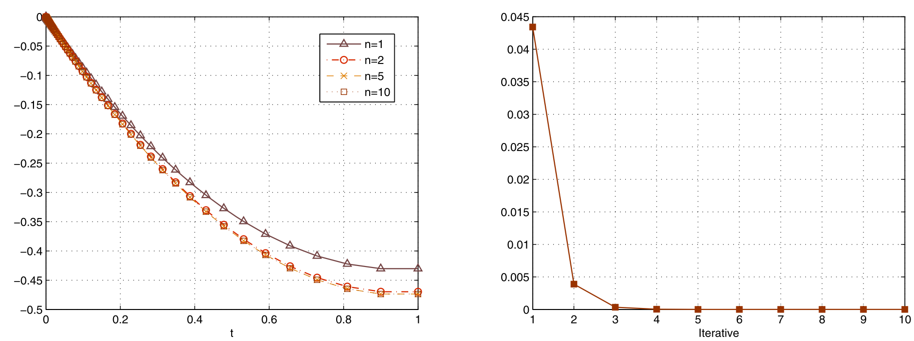


Figure 1 Numerical solution (left) and the maximum error (right) in Example 4.2.

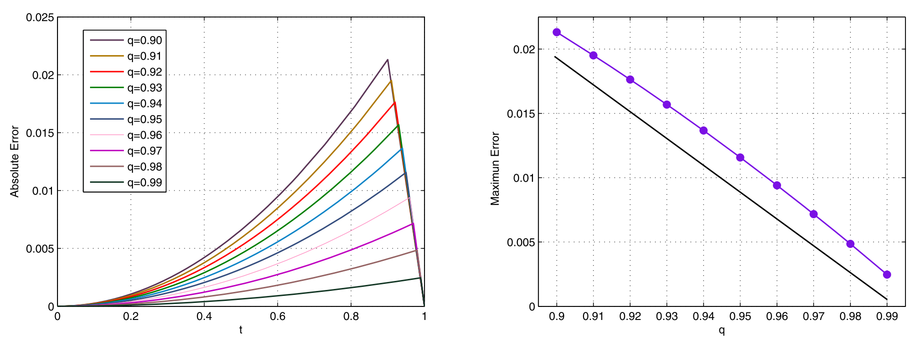


Figure 2 The absolute error (left) and the maximum error (right) in Example 4.2.

Table 2 Comparison of the error between q-approximate solution u_{15} and classical ODE

q	0.900	0.925	0.950	0.975
$\|u_{15} - u\|_{q,\infty}$	2.13158E–02	1.15705E–02	7.16421E–03	2.46256E–03

solution for the different values of n on the left side and it presents the maximum error on the right side with $q = 0.9$.

In Figure 2, the absolute errors with several values of q and the maximum errors are plotted with $n = 20$. Furthermore, we also compute the error between q-approximate solution, u_{15}, and the exact solution obtained by the classical ODE; the results are shown in Table 2.

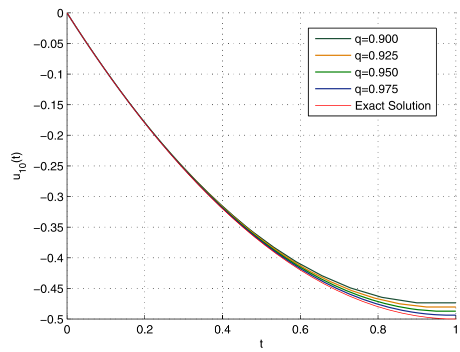
Finally, Figure 3 presents the q-approximate solution with several values of q and exact solution of the classical ODE in Example 4.2.

Example 4.3 ([13]) Consider the BVP

$$\begin{cases} D_q^3 u = L(\cos t + \tan^{-1} u), \\ u(0) = D_q u(0) = 0 \text{ and } u(1) = \alpha u(q^k), \end{cases}$$

with $f(t,u) = L(\cos t + \tan^{-1} u)$, $n = 3$, and $\eta = q^k$.

Figure 3 Comparison between q-approximate solution and classical ODE.



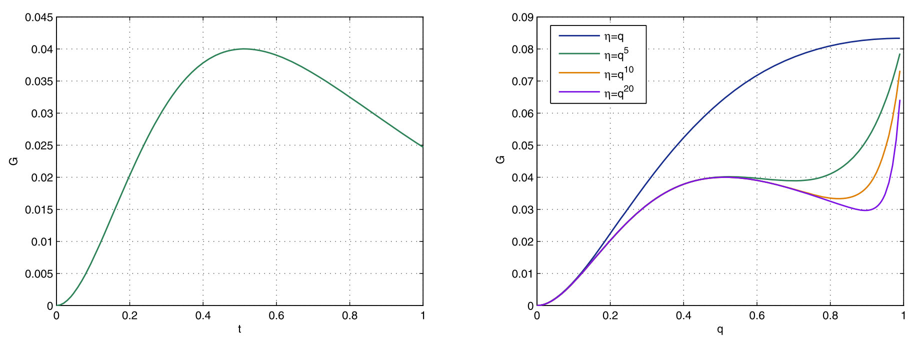


Figure 4 Plots of the function $\mathcal{G}$ with $\alpha = 0$ (left) and $\alpha = 1$ (right).

We find that

$$\left|f(t,u) - f(t,v)\right| \le L\left|\tan^{-1} u - \tan^{-1} v\right| \le L|u-v|.$$

We shall show that the BVP has a unique solution. We have $\Lambda_q = L$ and $C_1 = 1/[3]_q$ so that

$$\begin{aligned}\mathcal{G}_q &= \max\left\{\frac{|\alpha|q^{2k}(1-q^k)}{|1-\alpha q^{2k}|(1+q)(1+q+q^2)}, \frac{|\gamma|^3(1+q)q^2}{(1+q+q^2)^4}\right\}\\ &= \frac{|\gamma|^3(1+q)q^2}{(1+q+q^2)^4}.\end{aligned}$$

Here $\gamma = (1-q^{3k})/(1-q^{2k})$.

We can roughly estimate that

$$\mathcal{G}_q < \frac{1+q^k}{(1+q+q^2)^3}.$$

So, this problem has a unique solution provided $L < \frac{(1+q+q^2)^3}{1+q^k}$.

Some examples of the function $\mathcal{G}_q$ are shown in Figure 4 with $\alpha = 0$ on the left and $\alpha = 1$ on the right. In these cases, it is observed that we have the maximum values of $\mathcal{G}_q < 0.04$ and $\mathcal{G}_q < 0.09$ when $\alpha = 0$ and $\alpha = 1$, respectively. By using Theorem 3.5, we obtain the

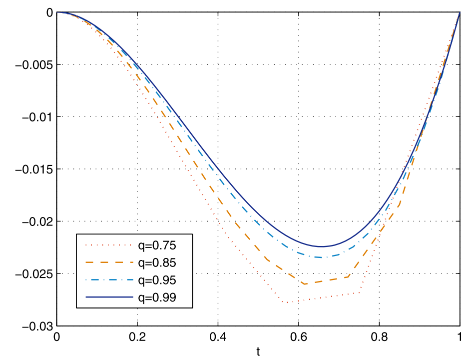


Figure 5 ***q*-approximate solution in Example 4.3 with $L = 1$ and $\eta = 0$.**

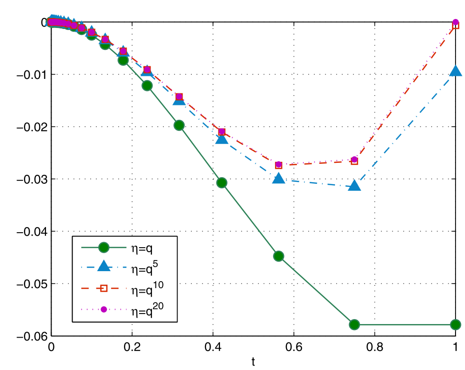


Figure 6 ***q*-approximate solution in Example 4.3 with $L = 1$ and $q = 0.75$.**

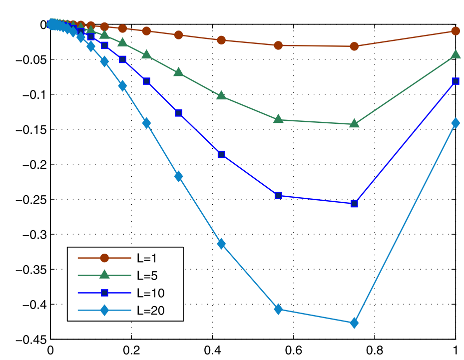


Figure 7 ***q*-approximate solution in Example 4.3 with $\eta = q^5$ and $q = 0.75$.**

existence and uniqueness of problem 4.3, however, we cannot obtain its exact solution. Hence, we will apply the numerical scheme to obtain its numerical solution.

Figure 5 presents the q-approximate solution with several values of q for $L = 1$ and $\eta = 0$. Moreover, the q-approximate solution with several values of η and L are displayed in Figure 6 and Figure 7, respectively.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The first author contributed conception and design of the work, drafted and critical revised the article. The second and the third author contributed in data analysis and interpretation of the numerical experiment. All authors read and approved the final manuscript. Overall percentage of contribution: Phothi (50%), Suebcharoen (25%), Wongsaijai (25%).

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