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Nonlinear state-dependent feedback control strategy in the SIR epidemic model with resource limitation

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Abstract

In present work, in order to avoid the spread of disease, the impulse control strategy is implemented to keep the density of infections at a low level. The SIR epidemic model with resource limitation including a nonlinear impulsive function and a state-dependent feedback control scheme is proposed and analyzed. Based on the qualitative properties of the corresponding continuous system, the existence and stability of positive order- k ($k \in \mathbf{Z}^+$) periodic solution are investigated. By using the Poincaré map and the geometric method, some sufficient conditions for the existence and stability of positive order-1 or order-2 periodic solution are obtained. Moreover, the sufficient conditions which guarantee the nonexistence of order- k ($k \geq 3$) periodic solution are given. Some numerical simulations are carried out to illustrate the feasibility of our main results.

Keywords: SIR model; nonlinear state-dependent impulsive function; periodic solution; orbital stability; chaotic solution

1 Introduction

Infectious diseases have had a tremendous influence on human life and have brought huge panic and disaster to mankind once out of control from antiquity to the present. Every year millions of human beings suffer from or die of various infectious diseases. For example, measles, dengue, tuberculosis, cholera, Ebola and avian influenza have had a tremendous influence on human health during the last few years. Therefore, epidemiological modeling of infectious disease transmission has an increasing influence on the theory and practice of disease management and control. It is well known that one of the most important concerns in the analysis of mathematical modeling of the spread of infectious diseases is the efficacy of vaccination programmes. In Central and South America [1, 2] and UK [3], vaccination strategies have a positive effect on the prevention of diseases such as measles, tetanus, diphtheria, pertussis, and tuberculosis. Long-term clinical data show that the vaccination strategies lead to infectious disease eradication if the proportion of the successfully vaccinated individuals is larger than a certain critical value, for example, approximately equal to 95% for measles. The effectiveness of vaccination has been widely studied and verified for the influenza vaccine [4], the human papilloma virus (HPV) vaccine [5], the chicken pox vaccine [6], and others. Generally, there are two types of vaccination strategies: continuous vaccination strategy (CVS) and pulse vaccination strategy (PVS). For certain kinds of

infectious diseases, PVS is more affordable and easier to implement than CVS. Recently, PVS has gained prominent achievement as a result of its highly successful application in the control of poliomyelitis and measles throughout Central and South America. In view of this, epidemiological models with PVS have been set up and investigated in many literature works (see, e.g., [7–13] and the references therein). Particularly, Agur *et al.* [7] first proposed a mathematical model with PVS, which consists of periodical repetitions of impulsive vaccinations of all the age cohorts in a population, which has been confirmed as an important and effective strategy for the elimination of infectious diseases.

In a real world application, however, the eradication of a disease is sometimes difficult both practically and economically in a short time. In order to prevent and control the spread of an infectious disease, impulsive vaccination and pulse treatment are important and effective methods. So far, it is necessary to keep the density of infections at a low level to avoid the spread of the disease. In many practical problems, impulse control strategy often occurs at state-dependent time, and it is more reasonable to take a strategy of state-dependent feedback control to model the issues of real world phenomena. Recently, the state-dependent feedback control, which is modeled by the impulsive semi-dynamic systems, has become a hot topic and has been applied in other fields and science [14–18], and it suggests that the control tactics should only be applied once the states of model reach a prescribed given threshold.

Following this idea, many epidemic models with state-dependent feedback control (including PVS and pulse treatment strategy (PTS)) have been proposed and analyzed by a number of authors in recent years. The SIR model with state-dependent PVS and PTS has been studied by Tang *et al.* [14] and later by Nie *et al.* [19]. Further, Nie *et al.* [20] proposed the SIRS model with state-dependent PVS and PTS, and analyzed the existence and stability of the periodic solution using the Poincaré map and the method of qualitative analysis. In these epidemic models with state-dependent pulse control strategies, it is usually assumed that the pulse vaccination rate p of the susceptibles and the pulse treatment rate q of the infectives are constants, which implies that the medical resources such as drugs, vaccines, hospital beds are very sufficient for infectious disease. In reality, however, every community or country has an appropriate or limited capacity for treatment, especially for emerging infectious diseases, and so understanding resource limitation is critical to effective management.

To the best of our knowledge, no work has been done for the effects of resource limitation on the SIR model with state-dependent PTS. In order to investigate the effect of limited vaccine and treatment availability on the spread of infectious disease, a saturation phenomenon of limited medical resources is considered. That is, we will study the dynamic behavior of the SIR epidemic model with state-dependent nonlinear PTS. This paper is structured as follows. In Section 2, the SIR epidemic model with resource limitation including state-dependent feedback control strategies and nonlinear impulsive function is constructed, some basic definitions, preliminaries and lemmas are given. In Section 3, we discuss the SIR model with the nonlinear impulsive vaccination as state-dependent feedback control, the existence and stability of positive periodic solution of this model. Finally, some numerical simulations are given to illustrate our results and suitability of state feedback control.

2 Model formulation and preliminaries

Recently, Zhou *et al.* [21] introduced a new continually differentiable treatment function $h(I) = \alpha I/(\omega + I)$ to characterize the saturation phenomenon of the limited medical resources and carefully investigated the dynamics of the following SIR model:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \frac{\beta S(t)I(t)}{1+kI(t)} - dS(t), \\ \frac{dI(t)}{dt} = \frac{\beta S(t)I(t)}{1+kI(t)} - (d + \gamma + \epsilon)I(t) - \frac{\alpha I(t)}{\omega+I(t)}, \\ \frac{dR(t)}{dt} = \gamma I(t) + \frac{\alpha I(t)}{\omega+I(t)} - dR(t), \end{cases} \tag{2.1}$$

where $S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible, infective and recovered individuals at time t , respectively. Λ is the recruitment rate of population, γ is the natural recovery rate, and ϵ is the disease-related mortality. The incidence rate $\beta S(t)I(t)/(1 + kI(t))$ of saturated type reflects the ‘psychological’ effect or inhibition effect, α represents the maximal medical resources supplied per unit time and ω is half-saturation constant. All parameters shown in model (2.1) are nonnegative constants.

We assume, throughout this paper, that $\epsilon = 0$. That is to say, the disease-related mortality is so very small that can be ignored. Motivated by the previous works [14, 15, 19, 20, 22], we propose a state-dependent nonlinear pulse vaccination for the susceptible at control threshold value. That is, when the number of the infected individuals reaches the higher hazardous threshold value RL at time $t_i(RL)$ at the i th time, vaccination is taken and the number of susceptible and recovered individuals turns very suddenly to a great degree to $(1 - p(t))S(t_i(RL))$ and $R(t_i(RL)) + p(t)S(t_i(RL))$, respectively, where $p(t) \in (0, 1)$ is the vaccination rate.

Typically, the previous works always assume that the vaccination rate $p(t)$ is a constant $p \in (0, 1)$ proportional to the number of susceptibles. But there is a limit to how fast the medical team can find and handle each susceptible for some emerging infectious diseases, that is, vaccination is often restricted by limited medical resources. The vaccination rate $p(t)$ always has some saturation effect and can be expressed as a saturation function as follows [23]:

$$p(t) = \frac{p_{\max} S(t)}{S(t) + \theta}, \quad 0 \leq p_{\max} < 1,$$

where p_{\max} is the maximum pulse vaccination proportion and θ is the half-saturation constant. Based on model (2.1) and the assumption $\epsilon = 0$, we consider the following model with the continuous treatment control strategy and the nonlinear impulsive vaccination as state-dependent feedback control.

$$\begin{cases} \left. \begin{aligned} \frac{dS(t)}{dt} &= \Lambda - \frac{\beta S(t)I(t)}{1+kI(t)} - dS(t) \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{1+kI(t)} - (d + \gamma)I(t) - \frac{\alpha I(t)}{\omega+I(t)} \\ \frac{dR(t)}{dt} &= \gamma I(t) + \frac{\alpha I(t)}{\omega+I(t)} - dR(t) \end{aligned} \right\} \begin{aligned} &I(t) < RL, \quad \text{or} \\ &I(t) = RL, \quad S(t) < G(RL), \end{aligned} \\ \left. \begin{aligned} S(t^+) &= S(t) - \frac{p_{\max} S^2(t)}{S(t) + \theta} \\ I(t^+) &= I(t) \\ R(t^+) &= R(t) + \frac{p_{\max} S^2(t)}{S(t) + \theta} \end{aligned} \right\} \begin{aligned} &I(t) = RL \quad \text{and} \quad S(t) > G(RL), \end{aligned} \end{cases} \tag{2.2}$$

Table 1 Parameter definitions, values and source of model (2.2)

Parameter	Definition	Value	Source
Λ	Recruitment rate of the population ($year^{-1}$)	16	[21]
β	Probability of transmission per contact ($year^{-1}$)	0.002~0.008	[21]
k	The parameter measuring the psychological or inhibitory effect	0.01	[21]
d	Nature death rate of the population ($year^{-1}$)	0.1	[21]
γ	Natural recovery rate of the infective individuals ($year^{-1}$)	0.01	[21]
α	The maximal medical resources supplied per unite time	6	[21]
ω	The half-saturation constant	Assumption	-
RL, RL'	The number of infectives such that control actions must be taken in order to avoid economic and social damages	Assumption	-
p_{max}	The maximal vaccination proportion	[0, 1)	[21, 23]
θ	The half-saturation constant	Assumption	-

where the initial condition $(S(0), I(0), R(0)) \in \mathbb{R}_+^3 = \{(x_1, x_2, x_3) | x_i \geq 0\}$ and $I(0) < RL$. Let $G(RL) = (1 + kRL)/\beta(d + \gamma + \alpha/(\omega + RL))$, in which $G(RL)$ as a function depends on the threshold RL , it is determined by the intersection of the horizontal isocline $\beta S/(1 + kI(t)) - (d + \gamma) - \alpha/(\omega + I) = 0$ and the line $I(t) = RL$, the parameter definitions of model (2.2) are summarized in Table 1.

Let the total population number of model (2.2) without impulsive effect be $N(t) = S(t) + I(t) + R(t)$, which satisfies

$$dN(t)/dt = \Lambda - dN(t). \tag{2.3}$$

It is clear that $N(t) = \Lambda/d$ is a solution of Eq. (2.3), that is, $S(t) + I(t) + R(t) = \Lambda/d$, and for any $N(t_0) \geq 0$, we have

$$\lim_{t \rightarrow \infty} N(t) = \frac{1}{d}(\Lambda - (\Lambda - dN(t_0)) \exp(-d(t - t_0))) = \frac{\Lambda}{d}.$$

Thus, the plane $S(t) + I(t) + R(t) = \Lambda/d$ is an invariant manifold of model (2.2) without impulsive effect. By the biological background of model (2.2), we only consider model (2.2) in the biological meaning region $\Omega = \{(S, I, R) | 0 \leq S + I + R \leq \Lambda/d\}$. Obviously, due to Lakshmikantham *et al.* [24] and Bainov and Simeonov [25], the global existence and uniqueness of solutions of model (2.2) are guaranteed by the smoothness properties of the right sides of model (2.2).

It is then seen that the initial value problems for models (2.2) are biologically well posed in the sense that the trajectories of model (2.2) with the initial condition $(S(t_0), I(t_0), R(t_0)) \in \mathbb{R}_+^3$ are positivity preserving.

Lemma 2.1 *Suppose that $(S(t), I(t), R(t))$ is a solution of model (2.2) with the initial condition $(S(t_0), I(t_0), R(t_0)) \in \mathbb{R}_+^3$. Then $(S(t), I(t), R(t)) \in \mathbb{R}_+^3$ for all $t \geq 0$.*

Proof For any initial value $(S(t_0), I(t_0), R(t_0)) \in \mathbb{R}_+^3$, we discuss the following two possibilities given by the number of possible contacts between the solution $(S(t), I(t), R(t))$ and the control line $I = RL$.

- (a) The solution intersects with $I = RL$ infinitely many times, at time instances t_k , $k = 1, 2, \dots$, and $t_k \rightarrow \infty$. In this case, if the conclusion of Lemma 2.1 is false, we

then obtain that there exists a positive integer n and $t^* \in (t_{n-1}, t_n)$ such that $\min\{S(t^*), I(t^*), R(t^*)\} = 0$, and $S(t) > 0, I(t) > 0, R(t) > 0$ for all $t \in [t_0, t^*)$.

The first possibility is that $S(t^*) = 0, I(t^*) > 0, R(t^*) > 0$. For this case, it follows from the first and fourth equations of model (2.2) that

$$S(t^*) > \prod_{i=1}^{n-1} \left(1 - \frac{p_{\max} S(t_i)}{S(t_i) + \theta}\right) S(t_0) \exp\left(-\int_{t_0}^{t^*} \left(\frac{\beta I(\tau)}{1 + kI(\tau) + d}\right) d\tau\right) > 0,$$

which contradicts the fact $S(t^*) = 0$.

The second possibility is that $S(t^*) > 0, I(t^*) = 0, R(t^*) > 0$. For this case, it follows from the second and fifth equations of model (2.2) that

$$I(t^*) = I(t_i) \exp\left(-\int_{t_0}^{t^*} \left(d + \gamma + \varepsilon + \frac{\alpha}{\omega + I(\tau)}\right) d\tau\right) > 0,$$

which contradicts the fact $I(t^*) = 0$, where $I(t_1) = \dots = I(t_{n-1}) = RL, i = 0, 1, \dots, n - 1$.

The third possibility is that $S(t^*) > 0, I(t^*) > 0, R(t^*) = 0$. For this case, it follows from the third and sixth equations of model (2.2) that

$$R(t^*) > R(t_0) \exp(-d(t^* - t_0)) + \sum_{i=1}^{n-1} \left(\frac{p_{\max} S^2(t_i)}{S(t_i) + \theta}\right) \exp(-d(t^* - t_i)) > 0,$$

which contradicts $R(t^*) = 0$.

- (b) The solution intersects with $I = RL$ finitely many times. In this case, since the endemic equilibrium (S_*, I_*, R_*) is globally asymptotically stable, so $S(t) > 0, I(t) > 0$, and $R(t) > 0$ for all $t \geq 0$. □

In the following, we start by introducing some definitions related to impulsive semi-dynamic systems, which are used in this work. We consider the following generalized planar impulsive semi-dynamic systems with state-dependent feedback control.

$$\begin{cases} \frac{dx}{dt} = f(x, y), & \frac{dy}{dt} = g(x, y), & (x, y) \notin \mathcal{M}, \\ x^+ = x + \xi(x, y), & y^+ = y + \eta(x, y), & (x, y) \in \mathcal{M}, \end{cases} \tag{2.4}$$

where $(x, y) \in \mathbb{R}^2$, we denote $x^+ = x(t^+)$ and $y^+ = y(t^+)$ for simplicity. f, g, ξ and η are continuous functions mapping \mathbb{R}^2 into \mathbb{R} , $\mathcal{M} \subset \mathbb{R}^2$ denotes the impulsive set. According to the denotations in [26–28], for any point $z(x, y) \in \mathcal{M}$, the map or impulsive function $\mathcal{I} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined as $\mathcal{I}(z) = z^+ = (x^+, y^+) \in \mathbb{R}^2, x^+ = x + \xi(x, y), y^+ = y + \eta(x, y)$, and z^+ is called an impulsive point of z . Let $\mathcal{N} = \mathcal{I}(\mathcal{M})$ be the phase set (i.e., for any $z(x, y) \in \mathcal{M}, \mathcal{I}(z) = z^+ = (x^+, y^+) \in \mathcal{N}$), $\mathcal{M} \cap \mathcal{N} = \emptyset$. Model (2.4) is generally known as a planar impulsive semi-dynamic model.

Let (X, Π, \mathbb{R}_+) or (X, Π) be a semi-dynamic model, where $X = R_+^2$ is a metric space, and \mathbb{R}_+ is the set of all nonnegative reals. For any $z \in X$, the function $\Pi_z : \mathbb{R}_+ \rightarrow X$ defined by $\Pi_z(t) = \Pi(z, t)$ is clearly continuous such that $\Pi(z, 0) = z$ for all $z \in X$, and $\Pi(\Pi(z, t), s) = \Pi(z, t + s)$ for all $z \in X$ and $t, s \in \mathbb{R}_+$. The set $C^+(z) = \{\Pi(x, t) | t \in \mathbb{R}\}$ is called the positive orbit of Z . For any set $\mathcal{M} \subset X$, let $\mathcal{M}^+(z) = C^+(z) \cap \mathcal{M} - z$, where $G(z, t) = \{w \in X | \Pi(w, t) =$

z) is the attainable set of z at $t \in \mathbb{R}_+$. Based on the above notations, we need the following definitions and lemma.

Definition 2.1 *An impulsive semi-dynamic model $(X, \Pi; \mathcal{M}, \mathcal{I})$ consists of a continuous semi-dynamic model (X, Π) together with a nonempty closed subset \mathcal{M} (or impulsive set) of \mathbb{R}^2 and a continuous function $\mathcal{I} : \mathcal{M} \rightarrow X$ such that the following property holds: No point $z \in X$ is a limit point of $\mathcal{M}(z); \{t | G(z, t) \cap \mathcal{M} \neq \emptyset\}$ is a closed subset of \mathbb{R} .*

Throughout the paper, we denote the points of discontinuity of Π_z by z_n^+ and define a function $\Phi : X \rightarrow \mathbb{R}_+ \cup \infty$ for any $z \in X$. If $\mathcal{M}^+(z) = \emptyset$, we set $\Phi(z) = \infty$; otherwise, $\mathcal{M}^+(z) \neq \emptyset$, and we set $\Phi(z) = s$, where $\Pi(z, t) \notin \mathcal{M}$ for $0 < t < s$ but $\Pi(z, s) \in \mathcal{M}$.

Definition 2.2 *A trajectory Π_z in $(X, \Pi, \mathcal{M}, \mathcal{I})$ is said to be periodic of period T_k and order k if there exist nonnegative integers $m \geq 0$ and $k \geq 1$ such that k is the smallest integer for which $z_m^+ = z_{m+k}^+$ and $T_k = \sum_{i=m}^{m+k-1} \Phi(z_i) = \sum_{i=m}^{m+k-1} s_i$.*

For more details on the concepts and properties of impulsive semi-dynamic model (2.4), see [24–26, 28]. For simplicity, we denote a periodic trajectory of period T_k and order- k by an *order- k periodic solution*. The local stability of an order- k periodic solution can be determined by using the following analogue of Poincaré criterion [25].

Lemma 2.2 (Analogue of Poincaré criterion [25]) *Let $\varphi(x, y)$ be a sufficiently smooth function with $\text{grad}\varphi(x, y) \neq 0$, and we denote $\varphi(x, y) \neq 0$ as $(x, y) \notin \mathcal{M}$ and $\varphi(x, y) = 0$ as $(x, y) \in \mathcal{M}$. The order- k periodic solution $(\phi(t), \psi(t))$ of model (2.4) is orbitally asymptotically stable and enjoys the property of asymptotic phase if the Floquet multiplier μ satisfies the condition $|\mu| < 1$, where*

$$\mu = \prod_{k=1}^n \Delta_k \exp \left\{ \int_0^T \left(\frac{\partial f}{\partial x}(\phi(t), \psi(t)) + \frac{\partial g}{\partial y}(\phi(t), \psi(t)) \right) dt \right\},$$

$$\Delta_k = \frac{\left(\frac{\partial \eta}{\partial y} \frac{\partial \varphi}{\partial x} - \frac{\partial \eta}{\partial x} \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial x} \right) f_+ + \left(\frac{\partial \xi}{\partial x} \frac{\partial \varphi}{\partial y} - \frac{\partial \xi}{\partial y} \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \right) g_+}{\frac{\partial \varphi}{\partial x} f_+ + \frac{\partial \varphi}{\partial y} g_+},$$

and $f, g, \partial \xi / \partial x, \partial \xi / \partial y, \partial \eta / \partial x, \partial \eta / \partial y, \partial \varphi / \partial x$ and $\partial \varphi / \partial y$ are calculated at the point $(\phi(\tau_k), \psi(\tau_k)), f_+ = f(\phi(\tau_k^+), \psi(\tau_k^+)), g_+ = g(\phi(\tau_k^+), \psi(\tau_k^+))$ and τ_k ($k \in \mathbb{N}$) is the time of the k th jump, $1 \leq k \leq n$, n being the total number of impulsive perturbations in $[0, T]$.

It follows from Zhou and Fan [21] that the disease-free equilibrium of the form $E_0 = (\Lambda/d, 0, 0)$ is globally asymptotically stable if $\mathcal{R}_0 < 1$, and the endemic equilibria $E^*(S_*, I_*, R_*)$ of model (2.1) are globally asymptotically stable if $\mathcal{R}_0 > 1$ and $\alpha \leq \omega^2(\beta + dk) + \omega^2 k(d + \gamma + \varepsilon) + \omega(2d + \alpha k)$, where

$$\mathcal{R}_0 = \frac{\beta \Lambda}{d(d + \gamma + \frac{\varepsilon}{\omega})}, \quad S_* = \frac{\Lambda(1 + kI_*)}{d + (\beta + dk)I_*}, \quad I_* = \frac{-B + \sqrt{B^2 - 4AC}}{2A},$$

$$R_* = \frac{1}{d} \left(rI_* + \frac{\alpha I_*}{\omega + I_*} \right)$$

and

$$\begin{aligned}
 A &= (\beta + dk)(d + \gamma), \\
 B &= -\beta\Lambda + \alpha(\beta + dk) + (d + \gamma)[d + \omega(\beta + dk)] \\
 &= d\left(d + \gamma + \frac{\alpha}{\omega}\right)(1 - \mathcal{R}_0) + (\beta + dk)(\alpha + \omega(d + \gamma)) - \frac{d\alpha}{\omega}, \\
 C &= \omega d(d + \gamma) + \alpha d - \beta\Lambda\omega = \omega d\left(d + \gamma + \frac{\alpha}{\omega}\right)(1 - \mathcal{R}_0).
 \end{aligned}$$

For model (2.2), we assume that the conditions $\mathcal{R}_0 > 1$ and $\alpha \leq \omega^2(\beta + dk) + \omega^2k(d + \gamma) + \omega(2d + \alpha k)$ hold. That is to say, model (2.2) without impulsive effects has a unique globally asymptotically stable endemic equilibrium (S_*, I_*, R_*) .

Note that the removed individuals $R(t)$ have no effect on the dynamic behaviors of susceptibles and infectives of model (2.2). Therefore, we consider the reduced model

$$\left. \begin{aligned}
 \left. \begin{aligned}
 \frac{dS(t)}{dt} &= \Lambda - \frac{\beta S(t)I(t)}{1+kI(t)} - dS(t) \\
 \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{1+kI(t)} - (d + \gamma)I(t) - \frac{\omega I(t)}{\omega + I(t)}
 \end{aligned} \right\} & \begin{aligned}
 I(t) &< RL, \quad \text{or} \\
 I(t) &= RL, \quad S < G(RL),
 \end{aligned} \\
 \left. \begin{aligned}
 S(t^+) &= S(t) - \frac{p_{\max} S^2(t)}{S(t) + \theta} \\
 I(t^+) &= I(t)
 \end{aligned} \right\} & I(t) = RL \quad \text{and} \quad S > G(RL).
 \end{aligned} \right\} \tag{2.5}$$

In model (2.5), we have the impulsive set $\mathcal{M} = \{(S, I) \in \mathbb{R}_+^2 \mid S \geq G(RL) \text{ and } I = RL\}$, and for any $(S, I) \in \mathcal{M}$, we have the continuous function $\mathcal{I}(S, RL) = (S - p_{\max} S^2 / (S + \theta), RL) \in \mathbb{R}_+^2$. It follows that the phase set is $\mathcal{N} = \mathcal{I}(\mathcal{M}) = \{(S^+, I^+) \in \mathbb{R}_+^2 \mid G(RL) - p_{\max} G^2(RL) / (G(RL) + \theta) < S^+ < G(RL), I^+ = RL\}$. It is clear that the total population size of model (2.5) tends to a constant Λ/d as t tends to infinity, by a biological point of view, we focus on the region $\Omega = \{(S, I) \mid S > 0, RL \geq I > 0, S + I \leq \Lambda/d\}$ to investigate model (2.5).

In order to address the dynamic behavior of model (2.5), the Poincaré map is constructed by the impulsive points on a phase set. Define two sections as follows:

$$\Sigma_{p_{\max}} = \{(S, I) \mid 0 < S < G(RL), I = RL\}, \Sigma_{RL} = \{(S, I) \mid S \geq G(RL), I = RL\}.$$

Choose section $\Sigma_{p_{\max}}$ as a Poincaré section. Suppose that point $P_n^+ = (S_n^+, RL)$ lies in section $\Sigma_{p_{\max}}$, and the trajectory $\Pi_{P_n^+} = \{(S(t), I(t)) \in \Omega \mid G(RL) > S(t_n^+) = S_n^+, I(t_n^+) = RL, t \geq t_n^+\}$ initiating from P_n^+ will reach section Σ_{RL} at point $P_{n+1} = (S_{n+1}, RL)$ in a finite time, where S_{n+1} can be only determined by S_n^+ , which can be expressed by $S_{n+1} = f(S_n^+)$. Clearly, a function f is continuously differentiable according to the Cauchy-Lipschitz theorem, and f is decreasing on $(0, G(RL))$ by the geometrical construction of the phase space of model (2.5). We assume that $p_{\max} \in (p^*, 1)$, in which

$$p^* = \mathcal{P}(RL) = \left(1 - \frac{dG(RL)}{\Lambda - dRL}\right) \left(1 + \frac{d\theta}{\Lambda - dRL}\right) \tag{2.6}$$

depends on the threshold RL as a function $\mathcal{P}(RL)$ is a sufficient condition for any point lying in section Σ_{RL} to map to section $\Sigma_{p_{\max}}$ after once impulsive effect. One time state-dependent feedback action is implemented at point P_{n+1} such that it jumps to point $P_{n+1}^+ =$

(S_{n+1}^+, RL) with $S_{n+1}^+ = (1 - p_{\max} S_{n+1} / (S_{n+1} + \theta)) S_{n+1}$ on section $\Sigma_{p_{\max}}$, where point P_{n+1}^+ is the impulsive point of P_{n+1} after one time impulsive effect. Therefore, we defined a Poincaré map of section $\Sigma_{p_{\max}}$ as follows:

$$S_{n+1}^+ = f(S_n^+) + \frac{p_{\max} f^2(S_n^+)}{f(S_n^+) + \theta} =: \mathcal{F}_1(S_n^+, p_{\max}, \theta), \quad p_{\max} \in (p^*, 1). \tag{2.7}$$

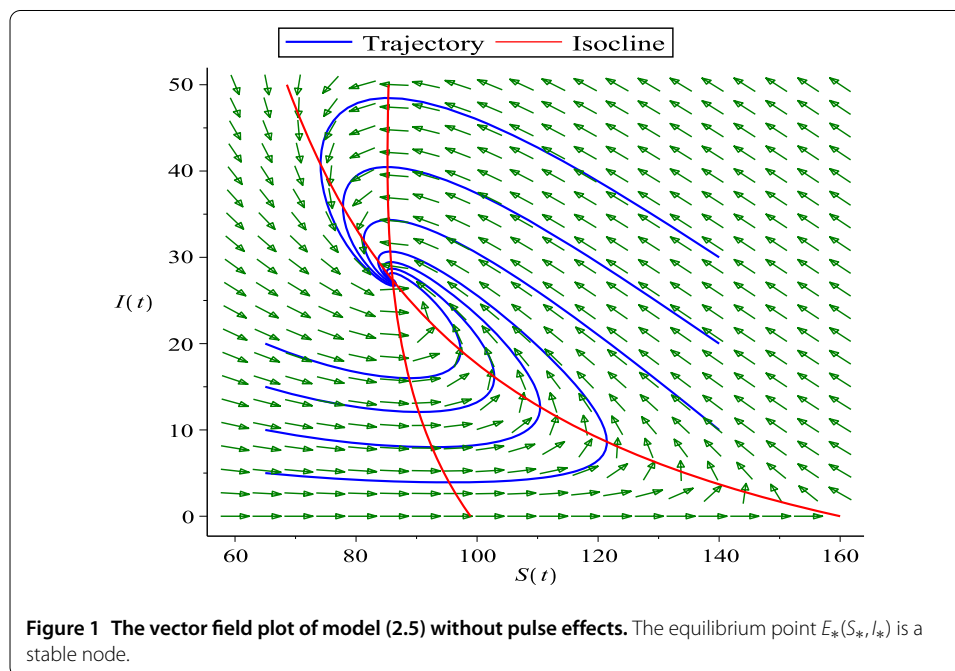
It is easy to see that the function $\mathcal{F}_1 : \Sigma_{p_{\max}} \rightarrow \Sigma_{p_{\max}}$ is monotone decreasing in the section $\Sigma_{p_{\max}}$. Thus, we will discuss the existence and stability of positive periodic solutions of model (2.5) using the Poincaré map (2.7).

3 Main results

From the assumption conditions of model (2.2), we know that the endemic equilibria $E^*(S_*, I_*)$ is a globally asymptotically stable node if $\mathcal{R}_0 > 1$ (see Figure 1). From the vector field of model (2.5) without impulsive effect, it is easy to see that the trajectory with the initial condition $(S_0, I_0) \in \Sigma_{p_{\max}}$ will reach section Σ_{RL} infinitely many times when $RL \leq I_*$. However, if $I_* < RL$, then the trajectory with the initial condition $(S_0, I_0) \in \Sigma_{p_{\max}}$ does not reach Σ_{RL} or tend to the endemic equilibrium (S_*, I_*) after reaching Σ_{RL} finitely many times. So, next, we will discuss the existence and stability of positive periodic solution of model (2.5) for two cases: (1) $RL \leq I_*$; (2) $I_* < RL$, respectively.

3.1 The case of $RL \leq I_*$

Theorem 3.1 *For the case $RL \leq I_*$, if $p_{\max} \in (p^*, 1)$, then model (2.5) has a positive order-1 periodic solution.*



Proof For any $p_{\max} \in (p^*, 1)$, we can choose small enough positive constants ε_1 and δ such that

$$\varepsilon_1 + \delta < \frac{p_{\max} G^2(RL)}{G(RL) + \theta}. \tag{3.1}$$

In view of the vector field of model (2.5), suppose that the trajectory $\Pi_{P_0^+}$ starting from the initial point $P_0^+(G(RL) - \varepsilon_1, RL) \in \Sigma_{p_{\max}}$ will reach section Σ_{RL} at the point $P_1(G(RL) + \mu_1, RL)$ in a finite time. Suppose that $\mu_1 \leq \delta$, then point P_1 maps to point $P_1^+(G(RL) - \varepsilon_2, RL)$ after one time impulsive effect. From the third equation of model (2.5) and condition (3.1), we have

$$\begin{aligned} G(RL) - \varepsilon_2 &= \left(1 - \frac{p_{\max}(G(RL) + \mu_1)}{G(RL) + \mu_1 + \theta}\right)(G(RL) + \mu_1) \\ &< G(RL) + \mu_1 - \frac{p_{\max} G^2(RL)}{G(RL) + \theta} < G(RL) + \mu_1 - (\varepsilon_1 + \delta) < G(RL) - \varepsilon_1. \end{aligned}$$

It implies that $\varepsilon_1 < \varepsilon_2$ and point P_1^+ is on the left of point P_0^+ . Thus, from (2.7), we have $G(RL) - \varepsilon_2 = \mathcal{F}_1(G(RL) - \varepsilon_1, p_{\max}, \theta)$

$$\mathcal{F}_1(G(RL) - \varepsilon_1, p_{\max}, \theta) - (G(RL) - \varepsilon_1) = \varepsilon_1 - \varepsilon_2 < 0. \tag{3.2}$$

Conversely, $\mu_1 > \delta$. There is a positive constant $\varepsilon_1^* < \varepsilon_1$ such that the trajectory $\Pi_{\widehat{P}_0^+}$ starting from the initial point $\widehat{P}_0^+(G(RL) - \varepsilon_1^*, RL)$ will reach section Σ_{RL} at point $\widehat{P}_1(G(RL) + \delta, RL)$ in a finite time, and then point \widehat{P}_1 maps to point $\widehat{P}_1^+(G(RL) - \varepsilon_2^*, RL)$. From the third equation of model (2.5) and (3.1), we have

$$\begin{aligned} G(RL) - \varepsilon_2^* &= \left(1 - \frac{p_{\max}(G(RL) + \delta)}{G(RL) + \delta + \theta}\right)(G(RL) + \delta) \\ &< G(RL) + \delta - \frac{p_{\max} G(RL)^2}{G(RL) + \theta} < G(RL) + \delta - (\varepsilon_1^* + \delta) < G(RL) - \varepsilon_1^*. \end{aligned}$$

It implies that $\varepsilon_1^* < \varepsilon_2^*$ and point \widehat{P}_1^+ is on the left of point \widehat{P}_0^+ . It follows from (2.7) that we have $G(RL) - \varepsilon_2^* = \mathcal{F}_1(G(RL) - \varepsilon_1^*, p_{\max}, \theta)$ and

$$\mathcal{F}_1(G(RL) - \varepsilon_1^*, p_{\max}, \theta) - (G(RL) - \varepsilon_1^*) = \varepsilon_1^* - \varepsilon_2^* < 0. \tag{3.3}$$

On the other hand, for any $p_{\max} \in (p^*, 1)$, we can choose small enough positive constant η such that

$$\eta < \left(1 - \frac{p_{\max} G(RL)}{G(RL) + \theta}\right) G(RL). \tag{3.4}$$

Suppose that the trajectory $\Pi_{Q_0^+}$ from the initial point $Q_0^+(\eta, RL)$ reaches section Σ_{RL} at point $Q_1(S_1, RL)$ in a finite time, next jumps to point $Q_1(S_1^+, RL)$ on section $\Sigma_{p_{\max}}$ due to the impulsive effects. From the third equation of model (2.5) and $p_{\max} \in (p^*, 1)$, we have $\eta < S_1^+ < G(RL)$. It implies that point Q_1^+ is on the right of point Q_0^+ . Therefore, from (2.7) we have $S_1^+ = \mathcal{F}_1(\eta, p_{\max}, \theta)$ and

$$\mathcal{F}_1(\eta, p_{\max}, \theta) - \eta = S_1^+ - \eta > 0. \tag{3.5}$$

By (3.2), (3.3) and (3.5), it follows that the Poincaré map (2.7) has a fixed point, that is, model (2.5) has a fixed point, thus, model (2.5) has a positive order-1 periodic solution. This completes the proof of this theorem. \square

Next, we state and prove our result on the existence and stability of positive order- k ($k = 1, 2$) periodic solutions of model (2.5).

Theorem 3.2 *For case $RL \leq I_*$, if $p_{\max} \in (p^*, 1)$, then model (2.5) has a positive order-1 or order-2 periodic solution, which is orbitally asymptotically stable. Further, model (2.5) has no order- k solution ($k \geq 3, k \in \mathbb{N}$).*

Proof If $RL \leq I_*$, then any trajectory of model (2.5) initiating from section $\Sigma_{p_{\max}}$ will reach section Σ_{RL} and experience infinitely many impulses by the geometrical construction of the phase space of model (2.5). Under the condition $p_{\max} \in (p^*, 1)$, for any two points $P_i^+(S_i^+, RL)$ and $P_j^+(S_j^+, RL)$ satisfying $0 < S_i^+ < S_j^+ < G(RL)$, the trajectories $\Pi_{S_i^+}$ and $\Pi_{S_j^+}$ will reach section Σ_{RL} at points $P_{i+1}(S_{i+1}, RL)$ and $P_{j+1}(S_{j+1}, RL)$, respectively. Further, the trajectories $\Pi_{S_i^+}$ and $\Pi_{S_j^+}$ are mapped to points $P_{i+1}(S_{i+1}^+, RL)$ and $P_{j+1}(S_{j+1}^+, RL)$ due to impulsive effect, respectively. Therefore, if the conditions $0 < S_i^+ < S_j^+ < G(RL)$ and $p^* < p < 1$ hold, then it follows from the monotonicity of function \mathcal{F}_1 that

$$0 < S_{j+1}^+ < S_{i+1}^+ < G(RL). \tag{3.6}$$

Suppose that the trajectory of model (2.5) with the initial point $P_0^+(S_0^+, RL)$ on section $\Sigma_{p_{\max}}$ will reach section Σ_{RL} at point $P_1(S_1, RL)$ in a finite time, and point P_1 is mapped to point $P_1^+(S_1^+, RL)$ on section $\Sigma_{p_{\max}}$ after once impulsive effects, where $S_1^+ < G(RL)$ (due to the fact $p_{\max} \in (p^*, 1)$). Thus, there exist the following three cases:

- (i) If $S_0^+ = S_1^+$, model (2.5) has a positive order-1 periodic solution;
- (ii) If $S_0^+ \neq S_1^+$, without loss of generality, suppose that $S_0^+ < S_1^+$, it follows from (3.6) that $S_2^+ < S_1^+$. Furthermore, if $S_2^+ = S_0^+$, then model (2.5) has a positive order-2 periodic solution;
- (iii) If $S_0^+ \neq S_1^+ \neq S_2^+ \neq \dots \neq S_k^+$ ($k \geq 3$) and $S_0^+ = S_k^+$, then model (2.5) has a positive order- k periodic solution. In fact, this is impossible. If $S_1^+ < S_0^+$, then from disjointness of any two trajectories and (2.7), we have $S_1^+ < S_2^+$ and then $S_1^+ < S_2^+ < S_0^+$ or $S_1^+ < S_0^+ < S_2^+$. If $S_0^+ < S_1^+$, we have $S_2^+ < S_1^+$ and then $S_0^+ < S_2^+ < S_1^+$ or $S_2^+ < S_0^+ < S_1^+$. Therefore, there are four relations among S_0^+, S_1^+ and S_2^+ given by

$$\begin{aligned} \text{(a)} \quad & S_1^+ < S_2^+ < S_0^+, & \text{(b)} \quad & S_1^+ < S_0^+ < S_2^+, \\ \text{(c)} \quad & S_0^+ < S_2^+ < S_1^+, & \text{(d)} \quad & S_2^+ < S_0^+ < S_1^+. \end{aligned}$$

Now, we consider each case, respectively.

- (a) If $S_1^+ < S_2^+ < S_0^+$, it follows that $S_1^+ < S_3^+ < S_2^+$ by (3.6). Repeating the above process, we have

$$0 < S_1^+ < S_3^+ < \dots < S_{2k+1}^+ < \dots < S_2^+ < S_0^+ < G(RL). \tag{3.7}$$

Similar to case (a), we have

(b) If $S_1^+ < S_0^+ < S_2^+$, then

$$0 < \dots < S_{2k+1}^+ < \dots < S_3^+ < S_1^+ < S_0^+ < S_2^+ < \dots < S_{2k}^+ < \dots < G(RL). \tag{3.8}$$

(c) If $S_0^+ < S_2^+ < S_1^+$, then

$$0 < S_0^+ < S_2^+ < \dots < S_{2k}^+ < \dots < S_3^+ < S_1^+ < G(RL). \tag{3.9}$$

(d) If $S_2^+ < S_0^+ < S_1^+$, then

$$0 < \dots < S_{2k}^+ < \dots < S_2^+ < S_0^+ < S_1^+ < S_3^+ < \dots < S_{2k+1}^+ < \dots < G(RL). \tag{3.10}$$

If there exists an order- k periodic solution ($k \geq 3, k \in \mathbb{N}$) in model (2.5), then $S_0^+ \neq S_1^+ \neq S_2^+ \neq \dots \neq S_{k-1}^+$, and $S_0^+ = S_k^+$, which contradicts (3.7)-(3.10). So we conclude that model (2.5) has no period- k ($k \geq 3, k \in \mathbb{N}$) solution with $p_{\max} \in (p^*, 1)$.

Further, for $k \in \mathbb{N}$, the sequences $\{S_{2k}\}$ and $\{S_{2k+1}\}$ are convergent. Therefore, there exist S_1^* and S_2^* such that $\lim_{k \rightarrow \infty} S_{2k}^+ = S_1^*$, $\lim_{k \rightarrow \infty} S_{2k+1}^+ = S_2^*$. Using the Poincaré (2.7), we have $S_1^* = \mathcal{F}_1(S_2^*, p_{\max}, \theta)$, $S_2^* = \mathcal{F}_1(S_1^*, p_{\max}, \theta)$. Therefore, model (2.5) has an orbitally asymptotically stable positive periodic solution. Due to the vector field of (2.5), the positive periodic solution is order-1 in the cases of (a) and (c) and it is an order-2 periodic solution in the cases of (b) and (d). □

3.2 The case of $I_* < RL$

For $I_* < RL$, any trajectory starting from $(S_0, I_0) \in \Sigma_{p_{\max}}$ does not reach Σ_{RL} or reaches Σ_{RL} finitely (or infinitely) many times, which depends on the initial conditions and the control parameter p_{\max} . From the vector field construction of the phase space of model (2.5), we suppose there exists a trajectory $\Pi(C, t_0)$ starting from the initial point $C(S_C, RL) \in \Sigma_{p_{\max}}$ such that the line $I = RL$ tangents to this trajectory at point $D(G(RL), RL)$. If the trajectory of model (2.5) with the initial condition (S_0, I_0) ($I_0 < RL$) does not reach section Σ_{RL} , then model (2.5) has no periodic solutions. Therefore, we always assume that there exist trajectories of model (2.5) reaching section Σ_{RL} infinitely many times. For this reason, the condition $\Lambda/d - RL - p_{\max}((\Lambda/d - RL)^2/(\Lambda/d - RL + \theta)) < S_C$, that is, $p^{**} < p_{\max} < 1$, where

$$p^{**} = \left(1 - \frac{dS_C}{\Lambda - dRL}\right) \left(1 + \frac{d\theta}{\Lambda - dRL}\right), \tag{3.11}$$

in which is a sufficient condition for a trajectory $\Pi(C_0, t_0)$ of model (2.5) from the initial point $C_0(S_0, RL)$ ($0 < S_0 < S_C$) intersects with section Σ_{RL} infinitely many times due to impulsive effects. However, if $0 < p_{\max} < \tilde{p}^{**}$, where

$$\tilde{p}^{**} = \left(1 - \frac{S_C}{G(RL)}\right) \left(1 + \frac{\theta}{G(RL)}\right), \tag{3.12}$$

then the trajectory $\Pi(C_0, t_0)$ from the initial point $C_0(S_0, RL)$ ($0 < S_0 < G(RL)$) does not reach Σ_{RL} or tends to the endemic equilibrium (S_*, I_*) after reaching Σ_{RL} finitely many times.

Theorem 3.3 For the case $I_* < RL$, if $p^{**} < p_{\max} < 1$, then model (2.5) has a positive order-1 or order-2 periodic solution, which is orbitally asymptotically stable. Further, if $0 < p_{\max} < \tilde{p}^{**}$, then model (2.5) has no positive order- k ($k \geq 1$) periodic solution.

The proof of Theorem 3.2 is similar to the proof of Theorem 3.3, we therefore omit it here.

4 Numerical simulation

In this section, the following numerical results are provided to illustrate the theoretical results and the feasibility of state-dependent control strategy. Let $\Lambda = 16$, $\alpha = 10$, $\beta = 0.004$, $k = 0.01$, $d = 0.1$, $\omega = 35$ and $\gamma = 0.01$ by Table 1. Now we consider the following SIR epidemic model with the continuous treatment control strategy and nonlinear impulsive vaccination as state-dependent feedback control:

$$\left. \begin{aligned} \frac{dS(t)}{dt} &= 16 - \frac{0.004S(t)I(t)}{1+0.01I(t)} - 0.1S(t) \\ \frac{dI(t)}{dt} &= \frac{0.004S(t)I(t)}{1+0.01I(t)} - (0.1 + 0.01)I(t) - \frac{10I(t)}{35+I(t)} \\ \frac{dR(t)}{dt} &= 0.01I(t) + \frac{10I(t)}{35+I(t)} - 0.1R(t) \end{aligned} \right\} \begin{aligned} &I(t) < RL, \quad \text{or} \\ &I(t) = RL, \quad S(t) < G(RL), \end{aligned} \tag{4.1}$$

$$\left. \begin{aligned} S(t^+) &= S(t) - \frac{p_{\max}S^2(t)}{S(t)+\theta} \\ I(t^+) &= I(t) \\ R(t^+) &= R(t) + \frac{p_{\max}S^2(t)}{S(t)+\theta} \end{aligned} \right\} \begin{aligned} &I(t) = RL \quad \text{and} \quad S(t) > G(RL). \end{aligned}$$

Firstly, by directly calculating, we have $\mathcal{R}_0 \approx 1.62 > 1$ and $\alpha = 10 < \omega^2(\beta + dk) + \omega^2k(d + \gamma) + \omega(2d + \alpha k) \approx 17.97$. It is easy to know that model (4.1) without pulse effects has a globally asymptotically stable endemic equilibrium $(S_*, I_*, R_*) = (86.09, 27.33, 46.58)$, which is illustrated in Figures 2(a)-2(c) by the red lines.

Secondly, we choose the control parameters to be $\theta = 0.8$ and $RL = 20 < I_* = 27.33$, then $G(RL) = (1 + kRL)/\beta(d + \gamma + \alpha/(\omega + RL)) \approx 87.55$, it follows from (2.6) that we get

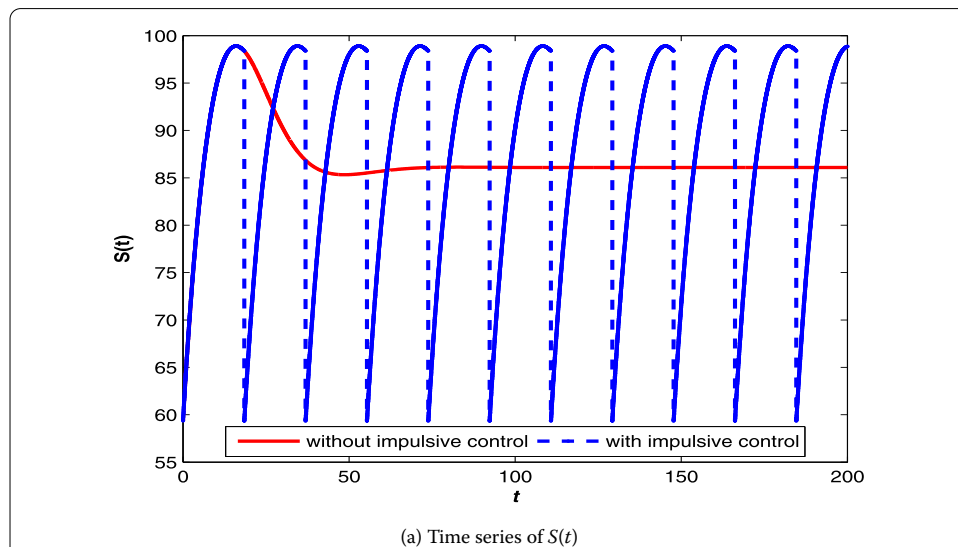
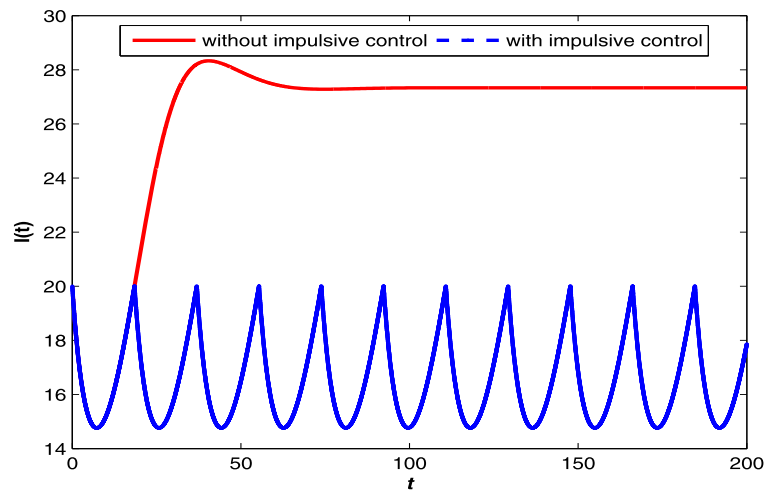
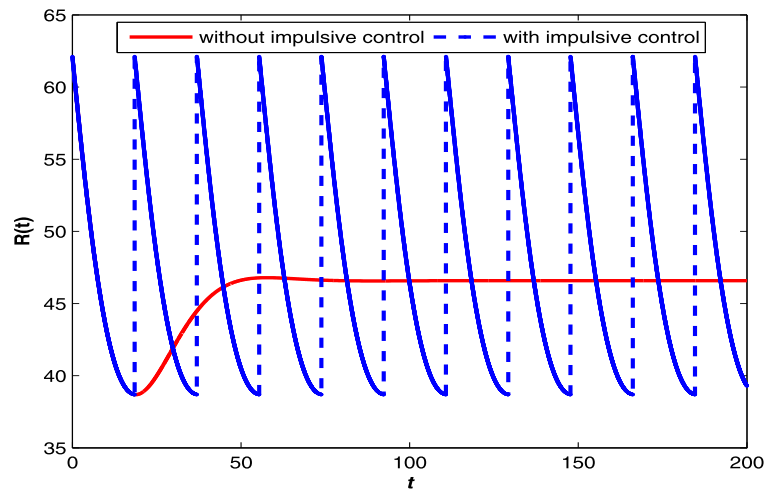


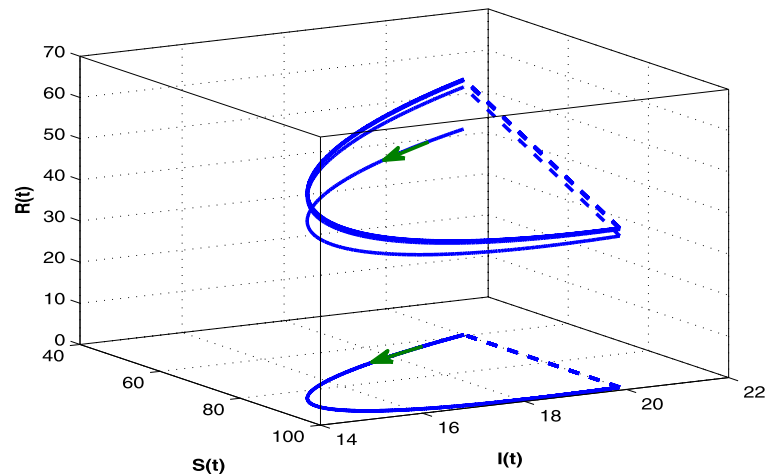
Figure 2 The trajectory of model (4.1) with/without $RL = 20 < I_* = 27.33$ and $p_{\max} = 0.4 > p^* \approx 0.38$. The initial value is $(S_0, I_0, R_0) = (59.34, 20, 62.11)$.



(b) Time series of $I(t)$

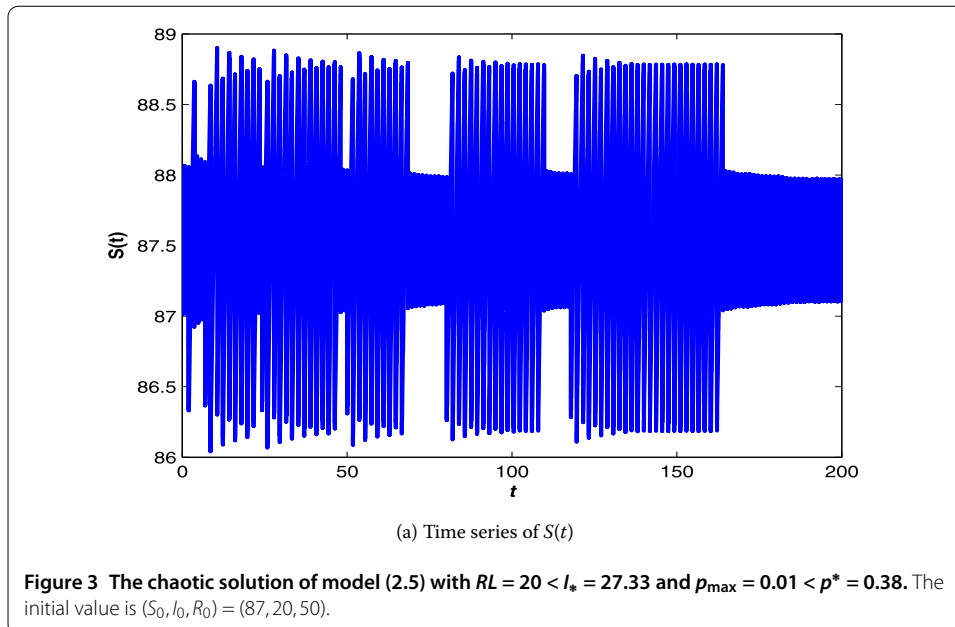


(c) Time series of $R(t)$



(d) Phase portrait and its projection on (S, I)

Figure 2 Continued

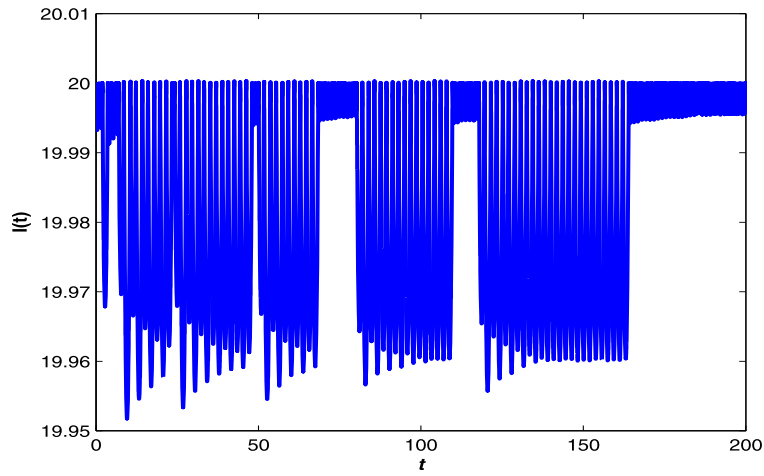


$p^* \approx 0.38$. Let $p_{\max} = 0.4 > p^*$, thus from Theorems 3.1 and 3.2 we know that model (4.1) has a positive order-1 periodic solution, which is shown in Figures 2(a)-2(c) by the blue lines. Further, the order-1 periodic solution is orbitally asymptotically stable and has the asymptotic phase property (see Figure 2(d)). Figures 2(a)-2(c) show the control strategy $\Delta S(t) = p_{\max} S^2(t)/(S(t) + \theta) = 0.4 S^2(t)/(S(t) + 0.8)$ of state-dependent pulse vaccination as an effective technique in controlling and preventing of disease. The phase portrait and time series of a chaotic solution at $p_{\max} = 0.01 < p^*$ are shown in Figure 3.

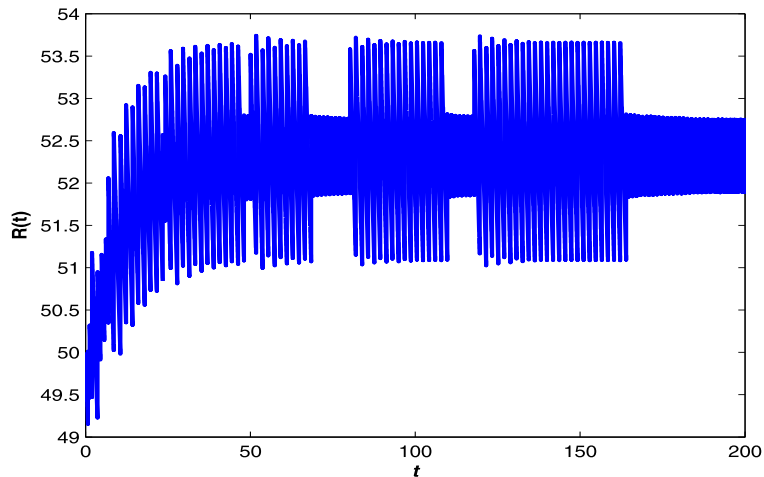
Finally, we choose $RL = 28 > I_* = 27.33$, $\theta = 0.8$ and p_{\max} to be 0.2, 0.3, 0.4, 0.62, 0.7 and 0.8, respectively. Numerical simulation shows that model (4.1) has a trajectory $\Pi(C, t_0)$ starting from the initial point $C(S_C, RL) = (51.55, 28)$ which is tangent to the line $I = RL = 28$ at point $D(G(RL), RL) = (86, 28)$. It follows from (3.11) and (3.12) that $p^{**} \approx 0.61$ and $\tilde{p}^{**} \approx 0.41$. By Theorem 3.3, it is easy to see that model (4.1) has an orbitally asymptotically stable positive order-1 periodic solution when $p_{\max} (= 0.62, 0.7, 0.8) > p^{**} \approx 0.61$, but trajectories may be free from pulse effect or experience finitely many impulses and then tend to the endemic equilibrium (S_*, I_*) when $p_{\max} (= 0.2, 0.3, 0.4) < \tilde{p}^{**} \approx 0.41$, which is shown in Figure 4.

5 Discussion

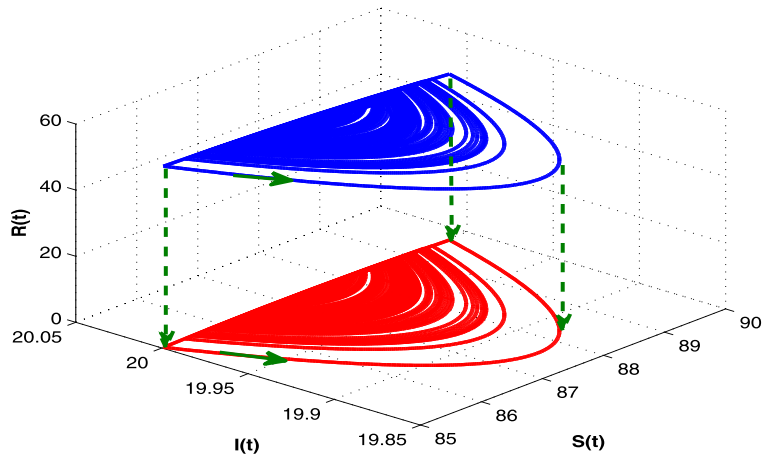
In order to control the infected individuals, the SIR epidemic model with resource limitation including nonlinear impulsive function and state-dependent feedback control is studied both theoretically and numerically. From Theorems 3.1 and 3.2, we obtain sufficient conditions for the existence and stability of positive order-1 or order-2 periodic solution. It is assumed that $\Lambda = 16$, $\alpha = 10$, $\beta = 0.004$, $k = 0.01$, $d = 0.1$, $\omega = 35$, $\gamma = 0.01$, $\theta = 0.8$, and we choose the economic threshold $RL = 20 < I^*$. It is easy to know that the conditions of Theorems 3.1 and 3.2 are satisfied, then the solution of model (4.1) initiating from $(S_0, I_0, R_0) = (59.34, 20, 62.11)$ tends to the orbitally asymptotically stable positive order-1 periodic solution (see Figure 2). However, the numerical simulations show that the dy-



(b) Time series of $I(t)$

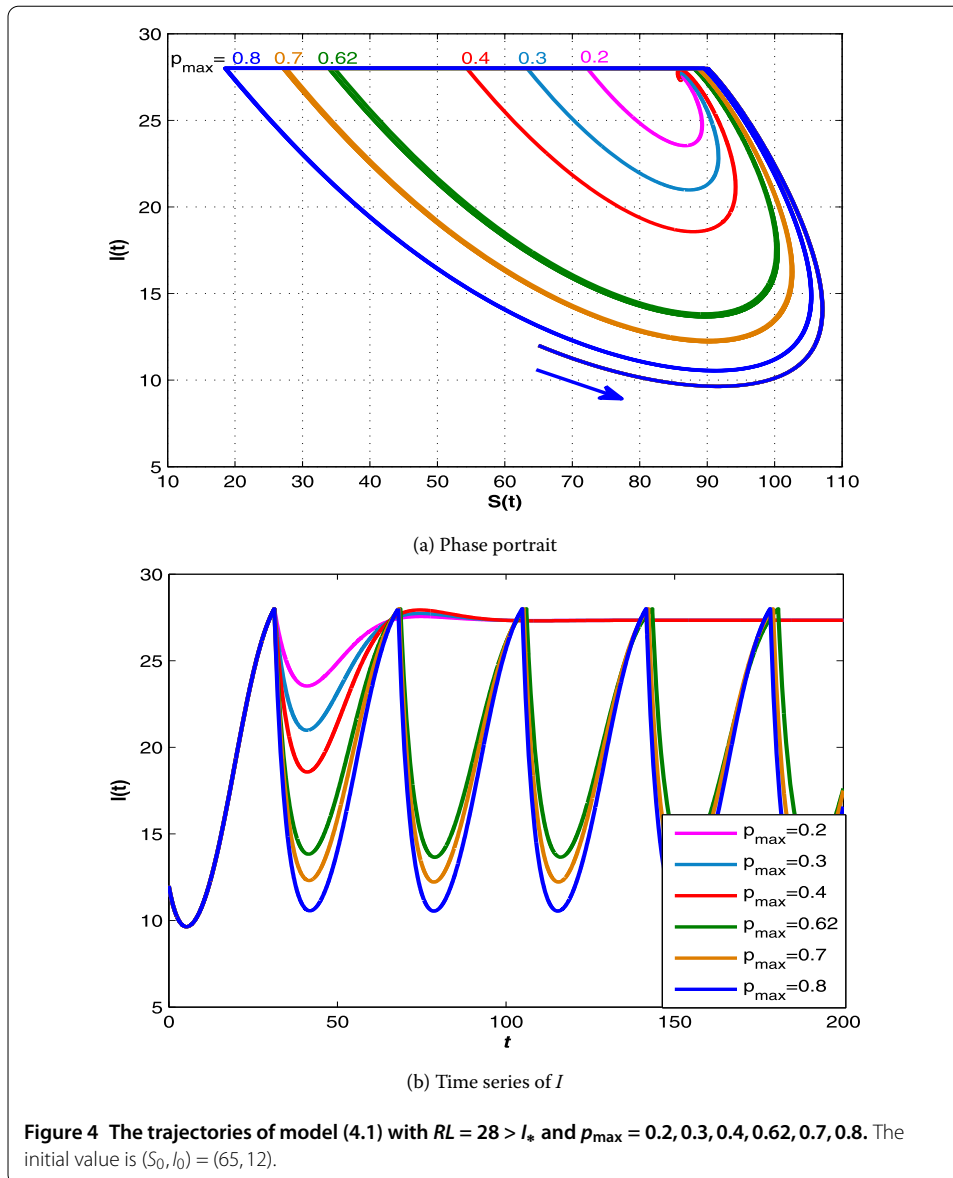


(c) Time series of $R(t)$



(d) Phase portrait and its projection on (S, I)

Figure 3 Continued



dynamic behavior of model (4.1) will become more complex if the conditions of Theorem 3.2 are unsatisfied (see Figure 3). Further, we choose the economic threshold $RL = 28 > I_*$. From Theorem 3.3, the solution of model (4.1) initiating from $(S_0, I_0) = (65, 12)$ tends to a stable positive order-1 periodic solution when $p_{\max} > p^{**}$. On the other hand, the solution of model (4.1) initiating from $(S_0, I_0) = (65, 12)$ tends to the endemic equilibrium (S_*, I_*) when $p_{\max} < \tilde{p}^{**}$ (see Figure 4). Our main results imply that we can choose proper control parameters to maintain the density of infections at a low level for preventing the spread of the disease. At the same time, some numerical simulations also show that model (2.5) has richer dynamic behaviors because of the effects of state-dependent impulse control strategies.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

ZLH carried out the main part of this article, LFN corrected the manuscript, JGL and ZZ brought forward some suggestions on this article. All authors have read and approved the final manuscript.

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