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A Lazer-Leach-type condition for singular differential equations with a deviating argument at resonance

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Abstract

In this paper, we establish a Lazer-Leach-type condition depending on the delay for the existence of positive periodic solutions for singular differential equations with a deviating argument at resonance. The proof of the main result is based on the phase-plane analysis and topological degree methods.

MSC: 34C25; 34D40

Keywords: periodic solutions; deviating argument; singular; at resonance; Lazer-Leach-type condition

1 Introduction

In this paper, we deal with the following delay differential equations at resonance:

$$x'' + \frac{1}{4}n^2 x + g\big(x(t), x(t-\tau)\big) = p(t), \tag{1.1}$$

where $g : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, $\tau \geq 0$ is a constant, $p(t)$ is continuous and 2π-periodic, and the function g has a singularity of repulsive type at the origin for its first variable, that is, $\lim_{s_1 \to \infty} g(s_1, s_2) = -\infty$.

The periodic problems of singular differential equations had attracted the attentions of many researchers during more than the last two decades because of their background in applied science [1–17]. A landmark work on mathematical treatment of the differential equations with singularities, as we all know, is done by Lazer and Solimini [11]. From then on, some classical mathematical tools were used successfully to study these singular equations, such as Mawhin's continuation theorem in the coincidence degree theory [17], the method of upper and lower solutions [8], some fixed point theorems in cones [2], the Poincaré-Birkhoff theorem [14], the phase-plane analysis and topological degree methods [12], and so on.

Wang and Ma [14] first studied the resonant singular equation

$$x'' + \frac{1}{4}n^2 x + g(x) = p(t), \tag{1.2}$$

where g has a singularity and satisfies

$$\lim_{x\to\infty} g(x) = g(+\infty).$$

They obtained the existence of 2π-periodic solutions of (1.2) under the following so-called Lazer-Leach-type condition:

$$4g(+\infty) - \int_0^{2\pi} p(t)\left|\sin\left(\theta + \frac{nt}{2}\right)\right| dt \neq 0 \quad \text{for all } \theta \in \mathbb{R}. \tag{1.3}$$

Note that they perfectly answered the open problem raised by del Pino and Manasevich [4].

After that, Wang [18] discussed the periodic problem of the resonant Liénard equation with constant delay but without singularity and established some Lazer-Leach-type conditions depending on the delay. A natural and delicate idea is that the delay may affect the existence of periodic solutions not only for the equations in [18], but also for many other kinds of equations. Based on this, in the present paper, we consider equation (1.1) and look for Lazer-Leach-type conditions. The main difficulty to overcome is the coexistence of the singularity and delay. A possible way for us is to use the phase-plane analysis and topological degree methods, also used in [12], and give the following fundamental hypotheses.

(H_1) The variables of g are separable, that is, there exist two functions g_0 and g_1 such that $g(x(t), x(t-\tau)) = g_0(x(t)) + g_1(x(t-\tau))$. Moreover, g_0 is bounded on $[1, +\infty)$, $\lim_{x\to+\infty} g_0(x) = 0$, and $\lim_{x\to+\infty} g_1(x) = g(+\infty)$ is finite.

(H_2) For all $\theta \in \mathbb{R}$,

$$4\cos\frac{n\tau}{2} g(+\infty) \neq \int_0^{2\pi} p(t)\left|\sin\left(\frac{n}{2}t + \theta\right)\right| dt. \tag{1.4}$$

(H_3) Put $G_0(x) = \int_1^x g_0(s)\,ds$. It satisfies

$$\lim_{x\to 0^+} g_0(x) = -\infty, \qquad \lim_{x\to 0^+} G_0(x) = +\infty. \tag{1.5}$$

Furthermore, there exist $0 < \varepsilon_0 \leq 1$, $0 < l \leq 1$, and $0 < L < \infty$ such that

$$\frac{-g_0(x)}{G_0^{1+l}(x)} > L \quad \text{for all } x \in (0, \varepsilon_0). \tag{1.6}$$

We now state our main theorem.

Theorem 1.1 *Let* (H_1), (H_2), *and* (H_3) *hold. Then* (1.1) *has at least one periodic solution.*

Remark 1.2 (H_3) is a universal strong singularity condition. We give some examples to show that (H_3) can be satisfied.

- $g_0(x) = -\frac{1}{x}$. Then $G_0(x) = -\ln x$, and (1.5) holds. Let $\varepsilon_0 = e^{-2}$ and $l = L = 1$. By using a short calculation we can obtain that (1.6) holds.
- $g_0(x) = -\frac{1}{x^3}$. Then $G_0(x) = \frac{1}{2x^2} - \frac{1}{2}$, and (1.5) holds. Let $\varepsilon_0 = L = 1$ and $l = \frac{1}{2}$. We can also check that (1.6) holds.

- $g_{0,1}(x) + g_{0,2}(x)$, where the functions $g_{0,i}(x)$ $(i = 1,2)$ satisfy (1.5) and (1.6). In fact, it is easy to see that (1.5) holds for $g_{0,1}(x) + g_{0,2}(x)$ and $G_{0,1}(x) + G_{0,2}(x)$, where $G_{0,i}(x) = \int_1^x g_{0,i}(s)\,ds$ $(i = 1,2)$. On the other hand, we assume that there exist $0 < \varepsilon_{0,i} \le 1$, $0 < l_i \le 1$, and $0 < L_i \le \infty$ such that

  $$\frac{-g_{0,i}(x)}{G_{0,i}^{1+l}(x)} > L_i \quad \text{for all } x \in (0, \varepsilon_{0,i}).$$

  Without loss of generality, we assume also that $G_{0,i}(x) > 1$ and $g_{0,i}(x) < -1$ for all $x \in (0, \varepsilon_{0,i})$. Set $\varepsilon_0 = \min\{\varepsilon_{0,1}, \varepsilon_{0,2}\}$. For all $x \in (0, \varepsilon_0)$, if $G_{0,1}(x) > G_{0,2}(x)$, then

  $$\frac{-g_{0,1}(x) - g_{0,2}(x)}{[G_{0,1}(x) + G_{0,2}(x)]^{1+l}} > \frac{-g_{0,1}(x)}{[2G_{0,1}(x)]^{1+l}} > \frac{L_1}{2^{1+l}};$$

  otherwise,

  $$\frac{-g_{0,1}(x) - g_{0,2}(x)}{[G_{0,1}(x) + G_{0,2}(x)]^{1+l}} > \frac{-g_{0,2}(x)}{[2G_{0,2}(x)]^{1+l}} > \frac{L_2}{2^{1+l}}.$$

  Hence (1.6) holds.

Remark 1.3 When $\tau = 0$, condition (1.4) degenerates to condition (1.3). Therefore Theorem 1.1 generalizes the result in [14]. Moreover, the delay τ may affect the existence of periodic solutions.

This paper is structured into three sections. Section 2 is devoted to the proof of a useful lemma. In Section 3, we state some lemmas to prove the main theorem.

2 Preliminary lemma

To use the phase-plane analysis and topological degree methods, we embed (1.1) into a family of equations with one parameter $\lambda \in [0,1]$,

$$x'' + \frac{1}{4}n^2 x + (1-\lambda)\left(-1 - \frac{1}{x^3}\right) + \lambda g\bigl(x(t), x(t-\tau)\bigr) = \lambda p(t). \tag{2.1}$$

Now, we give the following fundamental lemma.

Lemma 2.1 *Suppose that there exist three positive constants M_0, M_1, and M_2 such that, for any 2π-periodic solution $x(t)$ of* (2.1),

$$M_0 < x(t) < M_1 \quad \textit{for all } t \in \mathbb{R}$$

and

$$\|x'\|_\infty \triangleq \max_{t\in[0,2\pi]} |x'(t)| < M_2.$$

Then Eq. (1.1) *has at least one 2π-periodic solution.*

Since the proof is similar as that in [12], we omit it.

Remark 2.2 In fact, Lemma 2.1 is also valid if we embed (1.1) into the following family of equations with one parameter $\lambda \in [0,1]$:

$$x'' + \frac{1}{4}n^2 x + (1-\lambda)\left(1 - \frac{1}{x^3}\right) + \lambda g\big(x(t), x(t-\tau)\big) = \lambda p(t). \tag{2.2}$$

3 The proof of the main theorem

Condition (H_2) reduces to

(H_2') for all $\theta \in \mathbb{R}$,

$$4\cos\frac{n\tau}{2} g(+\infty) < \int_0^{2\pi} p(t)\left|\sin\left(\frac{n}{2}t + \theta\right)\right| dt$$

or

(H_2'') for all $\theta \in \mathbb{R}$,

$$4\cos\frac{n\tau}{2} g(+\infty) > \int_0^{2\pi} p(t)\left|\sin\left(\frac{n}{2}t + \theta\right)\right| dt.$$

In this section, we always assume that (H_2') holds. The argument for (H_2'') is similar.

We first suppose that the sequence $\{(x_k, y_k)\}_{k=1}^{\infty}$ satisfies

$$x_k' = y_k, \qquad y_k' = -\frac{1}{4}n^2 x_k - g\big(x_k, x_k(t-\tau), \lambda_k\big) \tag{3.1}$$

with $\|x_k\|_\infty + \|y_k\|_\infty \to \infty$ as $k \to \infty$, where

$$g\big(x_k, x_k(t-\tau), \lambda_k\big) = (1-\lambda_k)\left(-1 - \frac{1}{x_k^3}\right) + \lambda_k g_0(x_k) + \lambda_k g_1\big(x_k(t-\tau)\big) - \lambda_k p(t).$$

In this position, we only consider $\lambda_k \to \lambda_0 \in [0,1]$. Even if λ_k has no limit, we can consider its convergent subsequence $\{\lambda_{k_i}\}$ and the corresponding solution sequence $\{(x_{k_i}, y_{k_i})\}$ for (3.1) because the sequence $\{\lambda_k\}$ is bounded. For simplicity, we omit these discussions. It is easy to see that $\|x_k\|_\infty + \|y_k\|_\infty \to \infty$ is equivalent to $\|x_k\|_\infty \to \infty$ and $\|y_k\|_\infty \to \infty$ as $k \to \infty$. Define

$$g_0(x_k, \lambda_k) = -(1-\lambda_k)\frac{1}{x_k^3} + \lambda_k g_0(x_k)$$

and

$$g_1\big(x_k(t-\tau), \lambda_k\big) = -(1-\lambda_k) + \lambda_k g_1\big(x_k(t-\tau)\big).$$

Obviously, $g_0(x_k, \lambda_k)$ satisfies condition (H_3).

Take the transformation

$$x_k = 1 + r_k \cos\theta_k, \qquad y_k = \frac{n}{2} r_k \sin\theta_k.$$

Then system (3.1) is equivalent to the following system:

$$\begin{cases} \frac{d\theta_k}{dt} = -\frac{n}{2} - \frac{2}{nr_k} g(1 + r_k \cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k)\cos\theta_k + \frac{n}{2r_k}\cos\theta_k, \\ \frac{dr_k}{dt} = -\frac{n}{2}\sin\theta_k - \frac{2}{n} g(1 + r_k\cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k)\sin\theta_k. \end{cases} \tag{3.2}$$

Lemma 3.1 *Assume that* (H_1) *and* (H_3) *hold. For k large enough, we have*

$$\theta_k'(t) < 0 \quad \forall t \in \mathbb{R}.$$

Proof By (H_3) and the definition of $g(x_k, x_k(t-\tau), \lambda_k)$ we have that there exists $0 < \delta < 1$ such that

$$g\bigl(x_k, x_k(t-\tau), \lambda_k\bigr) - \frac{n^2}{4} < 0.$$

Noticing that $\cos\theta_k < 0$ for $0 < x_k \le \delta < 1$, we get, for $0 < x_k \le \delta$,

$$\frac{d\theta_k}{dt} \le -\frac{n}{2}. \tag{3.3}$$

Since $\|x_k\|_\infty \to \infty$ and $\|y_k\|_\infty \to \infty$ as $k \to \infty$, we have that if $x_k > \delta$, then

$$r_k \to \infty \quad \text{as } k \to \infty.$$

Meanwhile, if $x_k > \delta$, by (H_1), then we get that there exists $M > 0$ such that

$$\bigl|g\bigl(x_k, x_k(t-\tau), \lambda_k\bigr)\bigr| < M.$$

Hence, for k large enough, if $x_k > \delta$, then

$$\left| -\frac{2}{nr_k} g\bigl(1 + r_k\cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k\bigr)\cos\theta_k + \frac{n}{2r_k}\cos\theta_k \right| < \frac{n}{4},$$

and then

$$\frac{d\theta_k}{dt} \le -\frac{n}{2} + \frac{n}{4} = -\frac{n}{4}. \tag{3.4}$$

From (3.3) and (3.4) we obtain the conclusion of Lemma 3.1. □

From Lemma 3.1 we conclude that, for k large enough, the solution $(x_k(t), x_k(t))$ of (3.1) makes clockwise rotations around the point $(1, 0)$. Without loss of generality, we take the initial point $(x_k(t_0^{m,k}), y_k(t_0^{m,k}))$ of the mth rotation that satisfies

$$x_k\bigl(t_0^{m,k}\bigr) = 1, \qquad y_k\bigl(t_0^{m,k}\bigr) = x_k'\bigl(t_0^{m,k}\bigr) > 0,$$

and

$$\theta_k\bigl(t_0^{m,k}\bigr) = \frac{\pi}{2} - 2(m-1)\pi,$$

where $m = 1, 2, \ldots$.

Let $(x_k(t), y_k(t))$ take exactly one rotation from $t_0^{m,k}$. Then there are two points where the curve $(x_k(t), y_k(t))$ meets the line $x_k = 1$ and two points where the curve $(x_k(t), y_k(t))$ meets the x_k-axis. We denote them by $(x_k(t_1^{m,k}), 0)$, $(1, y_k(t_2^{m,k}))$, $(x_k(t_3^{m,k}), 0)$, and $(1, y_k(t_4^{m,k}))$, where $x_k(t_1^{m,k}) > 1$, $y_k(t_2^{m,k}) < 0$, $0 < x_k(t_3^{m,k}) < 1$, and $y_k(t_4^{m,k}) = y_k(t_0^{m+1,k}) > 0$.

Define

$$\mathcal{R}_k^2(t) = \big(x_k(t) - 1\big)^2 + \frac{4}{n^2} y_k^2(t) + \int_{t_0^{1,k}}^t \frac{8}{n^2} y_k(s) \left(g\big(x_k(s), x_k(s-\tau), \lambda_k\big) + \frac{1}{4} n^2 \right) ds. \tag{3.5}$$

Since $\frac{d\mathcal{R}_k^2}{dt} \equiv 0$, we get $\mathcal{R}_k^2(t) = \mathcal{R}_k^2$, where $\mathcal{R}_k > 0$. Obviously, we also get that

$$\mathcal{R}_k^2(t) = x_k^2(t) + \frac{4}{n^2} y_k^2(t) + \int_{t_0^{1,k}}^t \frac{8}{n^2} y_k(s) g\big(x_k(s), x_k(s-\tau), \lambda_k\big)\, ds - 1$$

and define

$$G_k(t) = \int_{t_0^{1,k}}^t \frac{8}{n^2} y_k(s) \left(g\big(x_k(s), x_k(s-\tau), \lambda_k\big) + \frac{1}{4} n^2 \right) ds. \tag{3.6}$$

Therefore we have $\mathcal{R}_k^2 = r_k^2(t) + G_k(t)$.

Lemma 3.2 *Assume that* (H$_1$) *and* (H$_3$) *hold. Then*

$$\lim_{k\to\infty} x_k\big(t_1^{m,k}\big) = +\infty \quad \textit{and} \quad \lim_{k\to\infty} x_k\big(t_3^{m,k}\big) = 0$$

for $m = 1, 2, \ldots$.

Proof Without loss of generality, we assume that there exist a positive integer m^* such that

$$x_k\big(t_1^{m^*,k}\big) = \max_{t\in\mathbb{R}} x_k(t).$$

Noticing that $\lim_{k\to\infty} \max_{t\in\mathbb{R}} x_k(t) \to \infty$, we get $\lim_{k\to\infty} x_k(t_1^{m^*,k}) \to \infty$. We will prove that $\lim_{k\to\infty} x_k(t_3^{m^*,k}) = 0$ and $\lim_{k\to\infty} x_k(t_1^{m^*+1,k}) = +\infty$, and the others are similar.

We first prove that $\lim_{k\to\infty} x_k(t_3^{m^*,k}) = 0$. Assume by contradiction that there exists a constant $0 < c < 1$ such that $x_k(t_3^{m^*,k}) > c$ for $k \in \mathbb{N}^+$. Thus, if $t \in [t_1^{m^*,k}, t_3^{m^*,k}]$, then $g(x_k, x_k(t-\tau), \lambda_k)$ is bounded. Hence, from the second equality of (3.2) we obtain $r_k(t_3^{m^*,k}) = r_k(t_1^{m^*,k}) + O(1)$. Therefore, $r_k(t_3^{m^*,k}) \to \infty$, which contradicts the fact $0 < x_k(t_3^{m^*,k}) < 1$ and $x_k'(t_3^{m^*,k}) = 0$. Consequently, $\lim_{k\to\infty} x_k(t_3^{m^*,k}) = 0$.

Next, we prove that $\lim_{k\to\infty} x_k(t_1^{m^*+1,k}) = +\infty$. Since $g(x_k, x_k(t-\tau), \lambda_k)$ is bounded for $t \in [t_1^{m^*,k}, t_2^{m^*,k}]$, we have $\lim_{k\to\infty} r_k(t_2^{m^*,k}) \to \infty$. It follows from $x_k(t_2^{m^*,k}) = 1$ that $\lim_{k\to\infty} y_k(t_2^{m^*,k}) \to \infty$. By (3.6) we obtain

$$\begin{aligned} G_k\big(t_0^{m^*+1,k}\big) - G_k\big(t_2^{m^*,k}\big) &= G_k\big(t_4^{m^*,k}\big) - G_k\big(t_2^{m^*,k}\big) \\ &= \int_{t_2^{m^*,k}}^{t_4^{m^*,k}} \frac{8}{n^2} y_k(s) \left(g\big(x_k(s), x_k(s-\tau), \lambda_k\big) + \frac{1}{4} n^2 \right) ds \end{aligned}$$

$$= \int_{t_2^{m^*,k}}^{t_4^{m^*,k}} \frac{8\lambda_k}{n^2} y_k(s) g_1\bigl(x_k(s-\tau)\bigr)\,ds$$
$$= o\bigl(r_k^2\bigl(t_2^{m^*,k}\bigr)\bigr).$$

It follows from the relation $\mathcal{R}_k^2 = r_k^2(t) + G_k(t)$ that

$$r_k^2\bigl(t_0^{m^*+1,k}\bigr) = r_k^2\bigl(t_2^{m^*,k}\bigr) - \bigl(G_k\bigl(t_0^{m^*+1}\bigr) - G_k\bigl(t_2^{m^*,k}\bigr)\bigr) = r_k^2\bigl(t_2^{m^*,k}\bigr) + o\bigl(r_k^2\bigl(t_2^{m^*,k}\bigr)\bigr).$$

Hence, $r_k(t_0^{m^*+1,k}) \to \infty$. By using the second equality of (3.2) again we obtain $r_k(t_1^{m^*+1,k}) \to \infty$. Equivalently, $\lim_{k\to\infty} x_k(t_1^{m^*+1,k}) = +\infty$. □

Denote by τ_k^m the required time for the solution $(x_k(t), y_k(t))$ to complete the mth rotation around the point $(0,1)$. Then,

$$\tau_k^m = \bigl(t_2^{m,k} - t_0^{m,k}\bigr) + \bigl(t_4^{m,k} - t_2^{m,k}\bigr).$$

Lemma 3.3 *Assume that* (H_1) *and* (H_3) *hold. Then, for k large enough,*

$$\tau_k^m = \frac{2\pi}{n} + o(1).$$

Proof We first compute $t_2^{m,k} - t_0^{m,k}$. Since $g(x_k, x_k(t-\tau), \lambda_k)$ is bounded on the interval $[t_0^{m,k}, t_2^{m,k}]$, we get from the first equality of (3.2) that

$$\frac{dt}{d\theta_k} = -\frac{2}{n} + o(1),$$

which implies $t_2^{m,k} - t_0^{m,k} = \frac{2\pi}{n} + o(1)$.

Next, we estimate $(t_4^{m,k} - t_2^{m,k})$. By (3.5) we have, for $t \in [t_2^{m,k}, t_3^{m,k}]$,

$$\frac{dt}{dx_k} = \frac{1}{y_k} = -\frac{2}{n}\frac{1}{\sqrt{\mathcal{R}_k^2 - (x_k(t)-1)^2 - G_k(t)}}.$$

Furthermore,

$$-\frac{2}{n}\,dx_k = \sqrt{\mathcal{R}_k^2 - \bigl(x_k(t)-1\bigr)^2 - G_k(t)}\,dt. \tag{3.7}$$

By (H_3) and Remark 1.2 we have that there exist $0 < \varepsilon_0 \le 1$, $0 < l \le 1$, and $0 < L < \infty$ such that

$$\frac{-g_0(x_k, \lambda_k)}{G_0^{1+l}(x_k, \lambda_k)} > L \quad \text{for all } x \in (0, \varepsilon_0).$$

For above l, we can choose $t_-^{m,k} \in [t_2^{m,k}, t_3^{m,k}]$ such that

$$G_k\bigl(t_-^{m,k}\bigr) = \mathcal{R}_k^{\frac{2}{1+\frac{l}{2}}},$$
$$\mathcal{R}_k^{\frac{2}{1+\frac{l}{2}}} \le G_k(t) \le \mathcal{R}_k^2 \quad \text{for } t \in \bigl[t_-^{m,k}, t_3^{m,k}\bigr],$$

and

$$G_k(t) \le \mathcal{R}_k^{\frac{2}{1+\frac{l}{2}}} \quad \text{for } t \in \left[t_2^{m,k}, t_-^{m,k}\right].$$

Integrating over $[t_2^{m,k}, t_-^{m,k}]$ for (3.7), we have

$$\int_{x_k(t_2^{m,k})}^{x_k(t_-^{m,k})} -\frac{2}{n}\,dx_k = \int_{t_2^{m,k}}^{t_-^{m,k}} \sqrt{\mathcal{R}_k^2 - \left(x_k(t)-1\right)^2 - G_k(t)}\,dt.$$

Noticing that $x_k(t_2^{m,k}) \to 1$ and $x_k(t_-^{m,k}) \to 0$, we get

$$\int_{x_k(t_2^{m,k})}^{x_k(t_-^{m,k})} -\frac{2}{n}\,dx_k = \frac{2}{n}\left(1 - x_k\left(t_-^{m,k}\right)\right) = \frac{2}{n} + o(1).$$

On the other hand, we get

$$\begin{aligned}
&\int_{t_2^{m,k}}^{t_-^{m,k}} \sqrt{\mathcal{R}_k^2 - \left(x_k(t)-1\right)^2 - G_k(t)}\,dt \\
&\quad = \left(t_-^{m,k} - t_2^{m,k}\right)\int_0^1 \sqrt{\mathcal{R}_k^2 - \left(x_k\left(t_2^{m,k} + s\left(t_-^{m,k} - t_2^{m,k}\right)\right) - 1\right)^2 - G_k\left(t_2^{m,k} + s\left(t_-^{m,k} - t_2^{m,k}\right)\right)}\,ds \\
&\quad = \left(t_-^{m,k} - t_2^{m,k}\right)\mathcal{R}_k \int_0^1 \sqrt{1 - \frac{(x_k(t_2^{m,k} + s(t_-^{m,k} - t_2^{m,k})) - 1)^2}{\mathcal{R}_k^2} - \frac{G_k(t_2^{m,k} + s(t_-^{m,k} - t_2^{m,k}))}{\mathcal{R}_k^2}}\,ds.
\end{aligned}$$

Thus we have

$$\frac{\frac{2}{n} + o(1)}{(t_-^{m,k} - t_2^{m,k})\mathcal{R}_k} = \int_0^1 \sqrt{1 - \frac{(x_k(t_2^{m,k} + s(t_-^{m,k} - t_2^{m,k})) - 1)^2}{\mathcal{R}_k^2} - \frac{G_k(t_2^{m,k} + s(t_-^{m,k} - t_2^{m,k}))}{\mathcal{R}_k^2}}\,ds.$$

Passing to limits and using the Lebesgue dominated convergence theorem, we get

$$\lim_{k\to\infty} \mathcal{R}_k\left(t_-^{m,k} - t_2^{m,k}\right) = \frac{2}{n},$$

which implies

$$t_-^{m,k} - t_2^{m,k} = \frac{2}{n\mathcal{R}_k} + o\left(\frac{1}{\mathcal{R}_k}\right). \tag{3.8}$$

By (3.1) we get, for $t \in [t_-^{m,k}, t_3^{m,k}]$,

$$\begin{aligned}
\frac{dt}{dy_k} &= \frac{1}{-\frac{1}{4}n^2 x_k - g(x_k, x_k(t-\tau), \lambda_k)} \\
&= \frac{1}{-\frac{1}{4}n^2 x_k - g_0(x_k, \lambda_k) - g_1(x_k(t-\tau), \lambda_k) + \lambda_k p(t)} \\
&= \frac{1}{-g_0(x_k, \lambda_k) + O(1)}.
\end{aligned}$$

Since $x_k(t) \to 0$ as $k \to \infty$ for $t \in [t_-^{m,k}, t_3^{m,k}]$, we obtain, for k large enough, $[x_k(t_3^{m,k}), x_k(t_-^{m,k})] \subset (0, \varepsilon_0)$. Then, for $t \in [t_-^{m,k}, t_3^{m,k}]$,

$$\frac{1}{-g_0(x_k,\lambda_k)} < \frac{1}{L}\frac{1}{G_0^{1+l}(x_k,\lambda_k)},$$

which yields that, for $t \in [t_-^{m,k}, t_3^{m,k}]$,

$$\begin{aligned}\frac{dt}{dy_k} &\le \frac{1}{L}\frac{1}{G_0^{1+l}(x_k,\lambda_k)} + o\left(\frac{1}{G_0^{1+l}(x_k,\lambda_k)}\right) \\ &= \frac{1}{L}\frac{1}{G_k^{1+l}(t)} + o\left(\frac{1}{G_k^{1+l}(t)}\right) \\ &\le \frac{1}{L}\frac{1}{\mathcal{R}_k^{\frac{2(1+l)}{1+\frac{l}{2}}}} + o\left(\frac{1}{\mathcal{R}_k^{\frac{2(1+l)}{1+\frac{l}{2}}}}\right).\end{aligned}$$

Integrating over $[t_-^{m,k}, t_3^{m,k}]$ and noticing that $y_k(t) \le \frac{n}{2}\mathcal{R}_k$, we have

$$t_3^{m,k} - t_-^{m,k} \le \frac{n}{2L\mathcal{R}_k^{\frac{2(1+l)}{1+\frac{l}{2}}-1}} + o\left(\frac{1}{\mathcal{R}_k^{\frac{2(1+l)}{1+\frac{l}{2}}-1}}\right),$$

which implies that

$$t_3^{m,k} - t_-^{m,k} = o\left(\frac{1}{\mathcal{R}_k}\right). \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$t_3^{m,k} - t_2^{m,k} = \frac{2}{n\mathcal{R}_k} + o\left(\frac{1}{\mathcal{R}_k}\right). \tag{3.10}$$

Using a similar argument, we can prove that $t_4^{m,k} - t_3^{m,k} = \frac{2}{n\mathcal{R}_k} + o(\frac{1}{\mathcal{R}_k})$. It follows that

$$t_4^{m,k} - t_2^{m,k} = \frac{4}{n\mathcal{R}_k} + o\left(\frac{1}{\mathcal{R}_k}\right). \tag{3.11}$$

Then $t_4^{m,k} - t_2^{m,k} = o(1)$. This completes the proof. □

Since (x_k, y_k) is 2π-periodic, by Lemma 3.3 we know that, for k sufficiently large, (x_k, y_k) makes exactly n clockwise revolutions around the point $(1, 0)$ as t varies from 0 to 2π.

We use the following lemma to estimate the maxima.

Lemma 3.4 *Assume that* (H_1), (H_2'), *and* (H_3) *hold. There exist two positive constants* M_1 *and* M_2 *such that, for any* 2π*-periodic solution* $x(t)$ *of Eq.* (2.1),

$$\|x\|_\infty < M_1, \qquad \|x'\|_\infty < M_2. \tag{3.12}$$

Proof Assume by contradiction that there exists a sequence $\{(x_k, y_k)\}_{k=1}^{\infty}$ satisfying the system (3.1) with $\|x_k\|_\infty + \|y_k\|_\infty \to \infty$ as $k \to \infty$. It follows that $\|x_k\|_\infty \to \infty$ and $\|y_k\|_\infty \to \infty$.

Using the transformation

$$x_k = 1 + r_k \cos\theta_k, \qquad y_k = \frac{n}{2} r_k \sin\theta_k,$$

we have system (3.2).

Without loss of generality, we take the initial point $(x_k(t_0^{m,k}), y_k(t_0^{m,k}))$ of the mth rotation satisfying

$$x_k\bigl(t_0^{m,k}\bigr) = 1, \qquad y_k\bigl(t_0^{m,k}\bigr) = x_k'\bigl(t_0^{m,k}\bigr) > 0,$$

and

$$\theta_k\bigl(t_0^{m,k}\bigr) = \frac{\pi}{2} - 2(m-1)\pi$$

for $m = 1, 2, \ldots, n$.

Let $(x_k(t), y_k(t))$ take exactly one rotation from $t_0^{m,k}$. Denote by τ_k the required time for the solution $(x_k(t), y_k(t))$ to complete n rotations around the point $(0, 1)$. Then,

$$\tau_k = \sum_{m=1}^{n} \tau_k^m = \sum_{m=1}^{n} \bigl(t_4^{m,k} - t_0^{m,k}\bigr) = \sum_{m=1}^{n} \bigl(t_2^{m,k} - t_0^{m,k}\bigr) + \sum_{m=1}^{n} \bigl(t_4^{m,k} - t_2^{m,k}\bigr).$$

We further estimate $\sum_{m=1}^{n}(t_2^{m,k} - t_0^{m,k})$. By the second equality of (3.2) we get, for $t \in (t_0^{m,k}, t_2^{m,k})$,

$$r_k(t) = r_k\bigl(t_0^m\bigr) + O(1).$$

Hence, for all $t \in [t_0^{m,k}, t_2^{m,k}]$, $1 \le m \le n$,

$$r_k(t) = \mathcal{R}_k + o(\mathcal{R}_k). \tag{3.13}$$

Furthermore, $\frac{1}{r_k(t)} = \frac{1}{\mathcal{R}_k} + o(\frac{1}{\mathcal{R}_k})$ for $t \in [t_0^{m,k}, t_2^{m,k}]$.

By the first equality of (3.2), we obtain, for $t \in [t_0^{m,k}, t_2^{m,k}]$,

$$\begin{aligned} \frac{dt}{d\theta_k} &= -\frac{2}{n} \\ &\quad \cdot \frac{1}{1 + \frac{4}{n^2 \mathcal{R}_k} g(1 + r_k\cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k)\cos\theta_k + \frac{1}{\mathcal{R}_k}\cos\theta_k + o(\frac{1}{\mathcal{R}_k})} \\ &= -\frac{2}{n} + \frac{8}{n^3 \mathcal{R}_k} g\bigl(1 + r_k\cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k\bigr)\cos\theta_k \\ &\quad + \frac{2}{n\mathcal{R}_k}\cos\theta_k + o\left(\frac{1}{\mathcal{R}_k}\right). \end{aligned}$$

Integrating over $[-\frac{\pi}{2} - 2(m-1)\pi, \frac{\pi}{2} - 2(m-1)\pi]$, we get

$$
\begin{aligned}
& t_2^{m,k} - t_0^{m,k} \\
&\quad = \int_{-\frac{\pi}{2}-2(m-1)\pi}^{\frac{\pi}{2}-2(m-1)\pi} \left[\frac{2}{n} - \frac{8}{n^3 \mathcal{R}_k} g\bigl(1 + r_k \cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k\bigr)\cos\theta_k \right. \\
&\qquad \left. - \frac{2}{n\mathcal{R}_k}\cos\theta_k + o\left(\frac{1}{\mathcal{R}_k}\right)\right] d\theta_k \\
&\quad = \frac{2\pi}{n} - \frac{4}{n\mathcal{R}_k} - \int_{-\frac{\pi}{2}-2(m-1)\pi}^{\frac{\pi}{2}-2(m-1)\pi} \frac{8}{n^3\mathcal{R}_k} \\
&\qquad \cdot g\bigl(1 + r_k \cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k\bigr)\cos\theta_k \, d\theta_k + o\left(\frac{1}{\mathcal{R}_k}\right).
\end{aligned}
$$

Recalling that $\lambda_k \to \lambda_0 \in [0,1]$, we distinguish two cases.

Case 1: $\lambda_0 = 0$. In this case, recalling that $g_0(x_k) + g_1(x_k(t-\tau)) - p(t)$ is bounded for $\theta_k \in (-\frac{\pi}{2} - 2(m-1)\pi, \frac{\pi}{2} - 2(m-1)\pi)$, we have, for k large enough,

$$
\bigl|\lambda_k g_0(x_k) + \lambda_k g_1\bigl(x_k(t-\tau)\bigr) - \lambda_k p(t)\bigr| < \frac{1}{3}.
$$

Then it follows from the definition of $g(1 + r_k\cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k)$ that, for k large enough (such that $\lambda_k < \frac{1}{3}$) and for $\theta_k \in (-\frac{\pi}{2} - 2(m-1)\pi, \frac{\pi}{2} - 2(m-1)\pi)$,

$$
g\bigl(1 + r_k\cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k\bigr) < -(1-\lambda_k) + \frac{1}{3} < -\frac{1}{3}.
$$

Hence,

$$
t_2^{m,k} - t_0^{m,k} > \frac{2\pi}{n} - \frac{4}{n\mathcal{R}_k} + \frac{16}{3n^3\mathcal{R}_k} + o\left(\frac{1}{\mathcal{R}_k}\right),
$$

and then

$$
\sum_{m=1}^{n} \bigl(t_2^{m,k} - t_0^{m,k}\bigr) > 2\pi - \frac{4}{\mathcal{R}_k} + \frac{16}{3n^2\mathcal{R}_k} + o\left(\frac{1}{\mathcal{R}_k}\right).
$$

Therefore, by (3.11) we have, for k large enough,

$$
\tau_k = \sum_{m=1}^{n}\bigl(t_2^{m,k} - t_0^{m,k}\bigr) + \sum_{m=1}^{n}\bigl(t_4^{m,k} - t_2^{m,k}\bigr) > 2\pi,
$$

which is a contradiction. Consequently, (3.12) holds.

Case 2: $\lambda_0 > 0$. Then there exists a constant $l_0 > 0$ such that, for k large enough, $\lambda_k \geq l_0$. From (3.11) and the first equality of (3.2) we have, for $\theta_k \in (-\frac{\pi}{2} - 2(m-1)\pi, \frac{\pi}{2} - 2(m-1)\pi)$,

$$
t(\theta_k) = t_0^{1,k} + \frac{2}{n}\left(\frac{\pi}{2} - \theta_k\right) - \frac{2(m-1)\pi}{n} + o(1).
$$

Then, from the definition of $g(1 + r_k \cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k)$ we obtain

$$\begin{aligned}
&\int_{-\frac{\pi}{2}-2(m-1)\pi}^{\frac{\pi}{2}-2(m-1)\pi} g\bigl(1 + r_k \cos\theta_k, 1 + r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k\bigr)\cos\theta_k \, d\theta_k \\
&\quad\le \lambda_k \int_{-\frac{\pi}{2}-2(m-1)\pi}^{\frac{\pi}{2}-2(m-1)\pi} \bigl(g_1\bigl(1 + r_k(t-\tau)\cos\theta_k(t-\tau)\bigr) - p(t)\bigr)\cos\theta_k \, d\theta_k \\
&\quad= -\frac{n\lambda_k}{2}\int_{t_0^{1,k}+\frac{2m\pi}{n}+o(1)}^{t_0^{1,k}+\frac{2(m-1)\pi}{n}+o(1)} \bigl(g_1\bigl(x_k(t-\tau)\bigr) - p(t)\bigr)\cdot\left|\cos\left(\frac{\pi}{2} - \frac{n}{2}t\right)\right| dt \\
&\quad= \frac{n\lambda_k}{2}\int_{\frac{2(m-1)\pi}{n}}^{\frac{2m\pi}{n}} \bigl(g_1\bigl(x_k(t-\tau)\bigr) - p(t)\bigr)\cdot\left|\cos\left(\frac{\pi}{2} - \frac{n}{2}t\right)\right| dt + o(1).
\end{aligned}$$

Denote

$$I_1 = \left[\frac{2(m-1)\pi}{n}, \frac{2m\pi}{n}\right] \cap \bigl\{t : x_k(t-\tau) \ge 1\bigr\}$$

and

$$I_0 = \left[\frac{2(m-1)\pi}{n}, \frac{2m\pi}{n}\right] \cap \bigl\{t : x_k(t-\tau) < 1\bigr\}.$$

Then $\operatorname{mes}(I_0) = o(1)$. Denote again

$$I_1' = \left[\frac{2(m-1)\pi}{n} - \tau, \frac{2m\pi}{n} - \tau\right] \cap \bigl\{t : x_k(t) \ge 1\bigr\}.$$

For $m = 2i - 1$, $i = 1, 2, \ldots$, we have

$$\begin{aligned}
&\int_{\frac{2(m-1)\pi}{n}}^{\frac{2m\pi}{n}} g_1\bigl(x_k(t-\tau)\bigr)\cdot\left|\cos\left(\frac{\pi}{2} - \frac{n}{2}t\right)\right| dt \\
&\quad= \int_{\frac{2(m-1)\pi}{n}}^{\frac{2m\pi}{n}} g_1\bigl(x_k(t-\tau)\bigr)\cdot\sin\left(\frac{n}{2}t\right) dt \\
&\quad= \int_{I_1} g_1\bigl(x_k(t-\tau)\bigr)\cdot\sin\left(\frac{n}{2}t\right) dt + \int_{I_0} g_1\bigl(x_k(t-\tau)\bigr)\cdot\sin\left(\frac{n}{2}t\right) dt \\
&\quad= \int_{I_1} g_1\bigl(1 + r_k(t-\tau)\cos\theta_k(t-\tau)\bigr)\cdot\sin\left(\frac{n}{2}t\right) dt + o(1) \\
&\quad= \int_{I_1} g_1\left(\mathcal{R}_k \sin\left(\frac{n}{2}t - \frac{n}{2}\tau\right) + O(1)\right)\cdot\sin\left(\frac{n}{2}t\right) dt + o(1) \\
&\quad= \int_{I_1'} g_1\left(\mathcal{R}_k \sin\left(\frac{n}{2}t\right) + O(1)\right)\cdot\sin\left(\frac{n}{2}t + \frac{n}{2}\tau\right) dt + o(1) \\
&\quad= \cos\left(\frac{n}{2}\tau\right)\int_{I_1'} g_1\left(\mathcal{R}_k \sin\left(\frac{n}{2}t\right) + O(1)\right)\cdot\sin\left(\frac{n}{2}t\right) dt \\
&\qquad+ \sin\left(\frac{n}{2}\tau\right)\int_{I_1'} g_1\left(\mathcal{R}_k \sin\left(\frac{n}{2}t\right) + O(1)\right)\cdot\cos\left(\frac{n}{2}t\right) dt + o(1).
\end{aligned}$$

Passing to limits, we get

$$\lim_{k\to\infty}\int_{\frac{2(m-1)\pi}{n}}^{\frac{2m\pi}{n}} g_1\big(x_k(t-\tau)\big)\cdot\left|\cos\left(\frac{\pi}{2}-\frac{n}{2}t\right)\right| dt = \frac{4}{n}\cos\left(\frac{n}{2}\tau\right)g(+\infty).$$

For $m = 2i$, $i = 1, 2, \dots$, we similarly get

$$\lim_{k\to\infty}\int_{\frac{2(m-1)\pi}{n}}^{\frac{2m\pi}{n}} g_1\big(x_k(t-\tau)\big)\cdot\left|\cos\left(\frac{\pi}{2}-\frac{n}{2}t\right)\right| dt = \frac{4}{n}\cos\left(\frac{n}{2}\tau\right)g(+\infty).$$

Therefore we have

$$\int_{\frac{2(m-1)\pi}{n}}^{\frac{2m\pi}{n}} g_1\big(x_k(t-\tau)\big)\cdot\left|\cos\left(\frac{\pi}{2}-\frac{n}{2}t\right)\right| dt = \frac{4}{n}\cos\left(\frac{n}{2}\tau\right)g(+\infty) + o(1).$$

Consequently, we get

$$\begin{aligned}
&\sum_{m=1}^{n}\int_{-\frac{\pi}{2}-2(m-1)\pi}^{\frac{\pi}{2}-2(m-1)\pi} g\big(1+r_k\cos\theta_k, 1+r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k\big)\cos\theta_k\, d\theta_k \\
&\quad\le \sum_{m=1}^{n}\frac{n\lambda_k}{2}\int_{\frac{2(m-1)\pi}{n}}^{\frac{2m\pi}{n}}\big(g_1\big(x_k(t-\tau)\big)-p(t)\big)\cdot\left|\cos\left(\frac{\pi}{2}-\frac{n}{2}t\right)\right| dt + o(1) \\
&\quad= \frac{n\lambda_k}{2}\left(4\cos\left(\frac{n}{2}\tau\right)g(+\infty) - \int_0^{2\pi} p(t)\left|\sin\left(\frac{n}{2}t+\theta\right)\right| dt\right) + o(1).
\end{aligned}$$

By (H_2') we obtain, for k large enough,

$$\sum_{m=1}^{n}\int_{-\frac{\pi}{2}-2(m-1)\pi}^{\frac{\pi}{2}-2(m-1)\pi} g\big(1+r_k\cos\theta_k, 1+r_k(t-\tau)\cos\theta_k(t-\tau), \lambda_k\big)\cos\theta_k\, d\theta_k < 0.$$

Consequently, for k large enough,

$$\sum_{m=1}^{n}\big(t_2^{m,k}-t_0^{m,k}\big) > 2\pi - \frac{4}{\mathcal{R}_k},$$

and then

$$\tau_k > 2\pi,$$

which also contradicts the 2π-periodicity of $(x_k(t), y_k(t))$. This completes the proof. □

Remark 3.5 Similarly, if (H_1), (H_2''), and (H_3) hold, then the result in Lemma 3.4 is valid. Under these conditions, to estimate the maxima, we can we embed (1.1) into (2.2).

Lemma 3.6 *Assume* (H_1), (H_2), *and* (H_3) *hold. Then there exists a positive constant* M_0 *such that, for any* 2π*-periodic solution* $x(t)$ *of Eq.* (2.1),

$$\min_{t\in\mathbb{R}} x(t) > M_0.$$

Proof Let $\mathcal{R}_k$ be defined by (3.5). We first prove that there exists $c_1 > 0$ such that, for all $k \in \mathbb{N}^+$,

$$x_k\bigl(t_3^{1,k}\bigr) > c_1. \tag{3.14}$$

Assume by contradiction that $x_k(t_3^{1,k}) \to 0$ (as $k \to \infty$). By Lemma 3.4 and Remark 3.5 there exist $M_1 > 0$ and $M_2 > 0$ such that

$$\|x_k\|_\infty < M_1, \qquad \|x_k'\|_\infty < M_2.$$

Then $\mathcal{R}_k(t) \equiv c < +\infty$. However, by (H_3),

$$\begin{aligned} r_k^2\bigl(t_3^{1,k}\bigr) - r_k^2\bigl(t_2^{1,k}\bigr) &= \frac{8}{n^2}\int_{t_2^{1,k}}^{t_3^{1,k}} y_k g\bigl(x_k, x_k(t-\tau), \lambda_k\bigr)\,dt \\ &= \frac{8\lambda_k}{n^2}\int_{t_3^{1,k}}^{t_2^{1,k}} y_k \frac{1}{x_k^3}\,dt - (1-\lambda_k)\frac{8}{n^2}\int_{t_3^{1,k}}^{t_2^{1,k}} y_k g_0(x_k)\,dt + O(1) \\ &= \frac{8\lambda_k}{n^2}\cdot\frac{1}{x_k^2(t_3^{1,k})} - (1-\lambda_k)\frac{8}{n^2}\int_{x(t_3^{1,k})}^{1} g_0(x_k)\,dx_k + O(1) \\ &\to +\infty, \end{aligned}$$

which is impossible. Therefore (3.14) holds. Similarly, we can obtain

$$x_k\bigl(t_3^{m,k}\bigr) > c_m$$

for $m = 1, 2, \ldots, n$. Consequently, we get the conclusion of Lemma 3.6. □

Proof of Theorem 1.1 The result is obtained directly by Lemma 2.1, Lemma 3.4, Remark 3.5, and Lemma 3.6. □

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