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Existence of solution for stochastic differential equations driven by G-Lévy process with discontinuous coefficients

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Abstract

The existence theory for the vector-valued stochastic differential equations driven by G-Brownian motion and pure jump G-Lévy process (G-SDEs) of the type
$dY_t = f(t, Y_t)\, dt + g_{j,k}(t, Y_t)\, d\langle B^j, B^k \rangle_t + \sigma_i(t, Y_t)\, dB_t^i + \int_{R_0^d} K(t, Y_t, z) L(dt, dz)$, $t \in [0, T]$, with first two and last discontinuous coefficients, is established. It is shown that the G-SDEs have more than one solution if the coefficients f, g, K are discontinuous functions. The upper and lower solution method is used.

MSC: 60H05; 60H10; 60H20

Keywords: stochastic differential equations; G-Lévy process; upper and lower solution; discontinuous coefficients

1 Introduction

In recent years much effort has been made to develop the theory of sublinear expectations connected with the volatility uncertainty and the so-called G-Brownian motion. G-Brownian motion was introduced by Shige Peng in [1, 2] as a way to incorporate the unknown volatility into financial models. Its theory is tightly associated with the uncertainty problems involving an undominated family of probability measures. Soon other connections have been discovered, not only in the field of financial mathematics, but also in the theory of path-dependent partial differential equations or backward stochastic differential equations. Thus G-Brownian motion and connected G-expectation are attractive mathematical objects.

Returning, however, to the original problem of volatility uncertainty in the financial models, one feels that G-Brownian motion is not sufficient to model the financial world, as both G- and the standard Brownian motion share the same property, which makes them often unsuitable for modeling, namely, the continuity of paths. Therefore, it is not surprising that Hu and Peng [3] introduced the process with jumps, which they called G-Lévy process. Then Ren [4] introduced the representation of the sublinear expectation as an upper-expectation. In [5], the author concentrated on establishing the integration theory for G-Lévy process with finite activity, introduced the integral w.r.t. the jump measure associated with the pure jump G-Lévy process and gave the Itô formula for general G-Itô Lévy process.

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Under the integration theory for G-Lévy process, Paczka [5] established the existence and uniqueness of solutions for the following stochastic differential equation driven by G-Brownian motion and pure jump G-Lévy process with Lipschitz continuous coefficients:

$$
\begin{aligned}
Y_t = Y_0 &+ \int_0^t f(s, Y_s)\,ds + \int_0^t g_{j,k}(s, Y_s)\,d\langle B^j, B^k \rangle_s \\
&+ \int_0^t \sigma_i(s, Y_s)\,dB_s^i + \int_0^t \int_{R_0^d} K(s, Y_s, z) L(ds, dz),
\end{aligned}
\tag{1.1}
$$

where $Y_0 \in R^n$, $(\langle B^j, B^k \rangle_t)_{t \geq 0}$ is the mutual variation process of the G-Brownian motion $(B_t)_{t \geq 0}$, $L(t, z)$ is pure jump G-Lévy process. For each $x \in R^n$, the coefficients $f(t, x), g_{j,k}(t, x), \sigma_i(t, x)$ are in the space $\hat{M}_G^2(0, T; R^n)$, $K(t, x, z) \in \hat{H}_G^2([0, T] \times R_0^d; R^n)$ (which will be introduced in Section 2). A process Y_t belonging to $\hat{M}_G^2(0, T; R^n)$ and satisfying G-SDE (1.1) is said to be its solution.

Motivated by the importance of discontinuous functions, Faizullah and Piao [6] established the existence of solutions for the stochastic differential equations driven by G-Brownian motion with a discontinuous drift coefficient. Then Faizullah [7] developed the existence theory when the coefficient f or the coefficients f and g simultaneously are discontinuous functions. Motivated by the aforementioned works, in this paper, we consider equation (1.1) and assume that $f(t, x)$, $g(t, x)$ and $K(t, x, z)$ are discontinuous for all $x \in R^n$.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, the existence of solutions for G-SDE (1.1) with simultaneous discontinuous coefficients f, g and K is developed.

2 Preliminaries

In this section, we introduce some notations and preliminary results in G-framework which are needed in the sequence. More details can be found in [5, 8–13].

Definition 2.1 Let Ω be a given set, and let $\mathcal{H}$ be a linear space of real-valued functions defined on Ω. Moreover, if $X_i \in \mathcal{H}$, $i = 1, 2, \ldots, d$, then $\varphi(X_1, \ldots, X_d) \in \mathcal{H}$ for all $\varphi \in C_{b,\mathrm{lip}}(R^d)$, where $C_{b,\mathrm{lip}}(R^d)$ is the space of all bounded real-valued Lipschitz continuous functions. A sublinear expectation $\mathbb{E}$ is a functional $\mathbb{E} : \mathcal{H} \to R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have:

(i) Monotonicity: $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \geq Y$;
(ii) Constant preserving: $\mathbb{E}[C] = C$ for $C \in R$;
(iii) Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;
(iv) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H}, \mathbb{E})$. We often call $Y = (Y_1, \ldots, Y_d)$, $Y_i \in \mathcal{H}$ a d-dimensional random vector in $(\Omega, \mathcal{H}, \mathbb{E})$.

Definition 2.2 In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, an n-dimensional random vector $Y = (Y_1, \ldots, Y_n)$ is said to be independent from an m-dimensional random vector $X = (X_1, \ldots, X_m)$ if for each $\varphi \in C_{b,\mathrm{lip}}(R^{m+n})$,

$$
\mathbb{E}\big[\varphi(X, Y)\big] = \mathbb{E}\big[\mathbb{E}\big[\varphi(x, Y)\big]_{x=X}\big].
$$

Definition 2.3 Let X_1, X_2 be two n-dimensional random vectors defined on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$\mathbb{E}_1\big[\varphi(X_1)\big] = \mathbb{E}_2\big[\varphi(X_2)\big], \quad \forall \varphi \in C_{b,\mathrm{lip}}\big(R^n\big).$$

$\bar{X}$ is said to be an independent copy of X if $\bar{X}$ is identically distributed with X and independent of X.

Definition 2.4 (*G*-Lévy process) Let $X = (X_t)_{t \geq 0}$ be a d-dimensional càdlàg process on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We say that X is a Lévy process if:

 (i) $X_0 = 0$,
 (ii) for each $s, t \geq 0$, the increment $X_{t+s} - X_s$ is independent of $(X_{t_1}, \ldots, X_{t_n})$ for every $n \in N$ and every partition $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq s$,
 (iii) the distribution of the increment $X_{t+s} - X_s$, $s, t \geq 0$ is stationary, i.e., does not depend on s.

Moreover, we say that a Lévy process X is a *G*-Lévy process if it satisfies additionally the following conditions:

 (iv) there is a $2d$-dimensional Lévy process $(X_t^c, X_t^d)_{t \geq 0}$ such that for each $t \geq 0$, $X_t = X_t^c + X_t^d$,
 (v) processes X_t^c and X_t^d satisfy the following conditions:

$$\lim_{t \downarrow 0} \mathbb{E}\big[|X_t^c|^3\big]t^{-1} = 0; \qquad \mathbb{E}\big[|X_t^d|\big] < Ct \quad \text{for all } t \geq 0.$$

Peng and Hu noticed in their paper that each G-Lévy process might be characterized by a non-local operator G_X.

Theorem 2.1 ([3]) *Let X be a G-Lévy process in R^d. For every $f \in C_b^3(R^d)$ such that $f(0) = 0$, we put*

$$G_X\big[f(\cdot)\big] := \lim_{\delta \downarrow 0} \mathbb{E}\big[f(X_\delta)\big]\delta^{-1}.$$

The above limit exists. Moreover, G_X has the following Levy-Khintchine representation:

$$G_X\big[f(\cdot)\big] = \sup_{(v,p,Q) \in \mathcal{U}} \left\{ \int_{R_0^d} f(z)v(dz) + \langle Df(0), p \rangle + \frac{1}{2}\operatorname{tr}\big[D^2 f(0)QQ^T\big] \right\},$$

where $R_0^d := R^d \backslash \{0\}$, $\mathcal{U}$ is a subset $\mathcal{U} \subset \mathcal{V} \times R^d \times R^{d \times d}$ and $\mathcal{V}$ is a set of all Borel measures on $(R_0^d, \mathcal{B}(R_0^d))$. We know additionally that $\mathcal{U}$ has the property

$$\sup_{(v,p,Q) \in \mathcal{U}} \left\{ \int_{R_0^d} |z|v(dz) + |p| + \operatorname{tr}\big[QQ^T\big] \right\} < \infty. \tag{2.1}$$

Theorem 2.2 ([3]) *Let X be a d-dimensional G-Lévy process. For each $\phi \in C_{b,\mathrm{lip}}(R^d)$, define $u(t,x) := \mathbb{E}[\phi(x + X_t)]$. Then u is the unique viscosity solution of the following integro-*

PDE:

$$
\begin{aligned}
0 &= \partial_t u(t,x) - G_X\big[u(t,x+\cdot) - u(t,x)\big] \\
&= \partial_t u(t,x) - \sup_{(v,p,Q)\in\mathcal{U}}\left\{\int_{R_0^d}\big[u(t,x+z) - u(t,x)\big]v(dz)\right. \\
&\quad \left. + \big\langle Du(t,x),p\big\rangle + \frac{1}{2}\,\mathrm{tr}\big[D^2 u(t,x)QQ^T\big]\right\},
\end{aligned}
\tag{2.2}
$$

with the initial condition $u(0,x) = \phi(x)$.

Theorem 2.3 *Let $\mathcal{U}$ satisfy (2.1). Consider the canonical space $\Omega := \mathbb{D}_0(R^+, R^d)$ of all càdlàg functions taking values in R^d equipped with the Skorohod topology. Then there exists a sublinear expectation $\hat{\mathbb{E}}$ on $\mathbb{D}_0(R^+, R^d)$ such that the canonical process $(X_t)_{t\geq 0}$ is a G-Lévy process satisfying Levy-Khintchine representation with the same set $\mathcal{U}$.*

The proof might be found in [3]. We will give, however, the construction of $\hat{\mathbb{E}}$ as it is important to understand it.

We denote $\Omega_T := \{w_{\cdot \wedge T} : w \in \Omega\}$. Put

$$
\begin{aligned}
\mathrm{Lip}(\Omega_T) := \big\{&\xi \in L^0(\Omega_T) : \xi = \phi(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}), \\
&\phi \in C_{b,\mathrm{lip}}(R^{d\times n}), 0 \le t_1 < \cdots < t_n < T\big\},
\end{aligned}
$$

where $X_t(w) = w_t$ is the canonical process on the space $\mathbb{D}_0(R^+, R^d)$ and $L^0(\Omega)$ is the space of all random variables, which are measurable to the filtration generated by the canonical process. We also set

$$
\mathrm{Lip}(\Omega) := \bigcup_{T=1}^{\infty} \mathrm{Lip}(\Omega_T).
$$

Firstly, consider the random variable $\xi = \phi(X_{t+s} - X_s), \phi \in C_{b,\mathrm{lip}}(R^d)$. We define

$$
\hat{\mathbb{E}}[\xi] := u(s,0),
$$

where u is a unique viscosity solution of integro-PDE (2.2) with the initial condition $u(0,x) = \phi(x)$. For general

$$
\xi = \phi(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}), \quad \phi \in C_{b,\mathrm{lip}}(R^{d\times n}),
$$

we set $\hat{\mathbb{E}}[\xi] := \phi_n$, where ϕ_n is obtained via the following iterated procedure:

$$
\begin{aligned}
\phi_1(x_1,\ldots,x_{n-1}) &= \hat{\mathbb{E}}\big[\phi(x_1,\ldots,x_{n-1}, X_{t_n} - X_{t_{n-1}})\big], \\
\phi_2(x_1,\ldots,x_{n-2}) &= \hat{\mathbb{E}}\big[\phi_1(x_1,\ldots,x_{n-2}, X_{t_{n-1}} - X_{t_{n-2}})\big], \\
&\vdots \\
\phi_{n-1}(x_1) &= \hat{\mathbb{E}}\big[\phi_{n-2}(x_1, X_{t_2} - X_{t_1})\big], \\
\phi_n &= \hat{\mathbb{E}}\big[\phi_{n-1}(X_{t_1})\big].
\end{aligned}
$$

Lastly, we extend the definition of $\hat{\mathbb{E}}$ on the completion of $\text{Lip}(\Omega_T)$ (respectively $\text{Lip}(\Omega)$) under the norm $\| \cdot \|_p^p = \hat{\mathbb{E}}[|\cdot|^p]$, $p \geq 1$. We denote such a completion by $L_G^p(\Omega_T)$ (or resp. $L_G^p(\Omega)$).

Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of Ω. It was proved in [4] that there exists a weakly compact probability measure family $\mathcal{P}$ defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{\mathbb{E}}[X] = \sup_{p \in \mathcal{P}} E_P[X], \quad \forall X \in L_G^1(\Omega),$$

where E_P is the linear expectation with respect to P.

Definition 2.5 We define the capacity c associated with $\hat{\mathbb{E}}$ by putting

$$c(A) := \sup_{p \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

We will say that a set $A \in \mathcal{B}(\Omega)$ is polar if $c(A) = 0$. We say that a property holds quasi-surely (q.s.) if it holds outside a polar set.

Remark 2.1 The condition (v) in Definition 2.4 implies that X^c is a d-dimensional generalized G-Brownian motion and the pure jump part X^d is of finite variation (see [3]). Moreover, X^c is just the d-dimensional G-Brownian motion B_t when $p = 0$ in (2.1). In this paper, we always let $p = 0$, i.e., the G-Lévy process X consists of G-Brownian motion B_t and the pure jump part.

Let $M_G^{0,p}(0, T)$ be the collection of processes of the following form: for a given partition $\{t_0, \ldots, t_N\} = \pi_T$ of $[0, T]$,

$$\eta_t = \sum_{i-0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1})}(t),$$

where $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, \ldots, N - 1$, $p \geq 1$. For each $p \geq 1$, denote by $M_G^p(0, T)$ the completion of $M_G^{0,p}(0, T)$ under the norm $\|\eta\|_{M_G^p} := (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p \, dt])^{\frac{1}{p}}$.

For each $\eta \in M_G^p(0, T)$, $p \geq 2$, the G-Itô integral $\{\int_0^t \eta_s \, dB_s^i\}_{t \in [0, T]}$ is well defined. For each $\eta_s^{j,k} \in M_G^p(0, T)$, $p \geq 1$, the integral $\{\int_0^t \eta_s^{j,k} \, d\langle B^j, B^k \rangle_s\}_{t \in [0, T]}$ is well defined. $i, j, k = 1, \ldots, d$. See Peng [12] and Li et al. [13].

Lemma 2.1 ([7]) *Let* $\eta_t^{j,k}, \zeta_t^{j,k} \in M_G^1(0, T)$. *If* $\eta_t^{j,k} \leq \zeta_t^{j,k}$ *for* $t \in [0, T]$, *then*

$$\int_0^T \eta_t^{j,k} \, d\langle B^j, B^k \rangle_t \leq \int_0^T \zeta_t^{j,k} \, d\langle B^j, B^k \rangle_t.$$

Assume that the G-Lévy process X has finite activity, i.e.,

$$\lambda := \sup_{v \in \mathcal{V}} v(R_0^d) < \infty.$$

Let X_{u-} denote the left limit of X at point u, $\Delta X_u = X_u - X_{u-}$, then we can define a random measure $L(\cdot, \cdot)$ associated with the G-Lévy process X by putting

$$L(]s, t], A) = \sum_{s < u \leq t} \mathbb{I}_A(\Delta X_u), \quad \text{q.s.}$$

for any $0 < s < t < \infty$ and $A \in \mathcal{B}(R_0^d)$. The random measure is well defined and may be used to define the pathwise integral.

Let $H_G^S([0, T] \times R_0^d)$ be a space of all elementary random fields on $[0, T] \times R_0^d$ of the form

$$K(u, z)(w) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} F_{k,l}(w) \mathbb{I}_{]t_k, t_{k+1}]}(u) \psi_l(z), \quad n, m \in \mathbb{N},$$

where $0 \leq t_1 < \cdots < t_n \leq T$ is the partition of $[0, T]$, $\{\psi_l\}_{l=1}^m \subset C_{b,\text{lip}}(R^d)$ are functions with disjoint supports s.t. $\psi_l(0) = 0$ and $F_{k,l} = \phi_{k,l}(X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}})$, $\phi_{k,l} \in C_{b,\text{lip}}(R^{d \times k})$. We introduce the norm on this space

$$\|K\|_{H_G^p([0,T] \times R_0^d)} := \hat{\mathbb{E}} \left[\int_0^T \sup_{v \in \mathcal{V}} \int_{R_0^d} |K(u, z)|^p v(dz) \, du \right]^{\frac{1}{p}}, \quad p = 1, 2.$$

Definition 2.6 Let $0 \leq s < t \leq T$. The Itô integral of $K \in H_G^S([0, T] \times R_0^d)$ w.r.t. jump measure L is defined as

$$\int_s^t \int_{R_0^d} K(u, z) L(du, dz) := \sum_{s < u \leq t} K(u, \Delta X_u), \quad \text{q.s.}$$

Lemma 2.2 *For every $K \in H_G^S([0, T] \times R_0^d)$, we have that $\int_0^T \int_{R_0^d} K(u, z) L(du, dz)$ is an element of $L_G^2(\Omega_T)$.*

Let $H_G^p([0, T] \times R_0^d)$ denote the topological completion of $H_G^S([0, T] \times R_0^d)$ under the norm $\| \cdot \|_{H_G^p([0,T] \times R_0^d)}$, $p = 1, 2$. Then Itô integral can be continuously extended to the whole space $H_G^p([0, T] \times R_0^d)$, $p = 1, 2$. Moreover, by Lemma 2.2 we know that the extended operator takes value in $L_G^2(\Omega_T)$, $p = 1, 2$.

Lemma 2.3 *Let $K^1(t, z), K^2(t, z) \in H_G^2([0, T] \times R_0^d)$. If $K^1(t, z) \leq K^2(t, z)$ for $t \in [0, T]$, then*

$$\int_0^T \int_{R_0^d} K^1(t, z) L(dt, dz) \leq \int_0^T \int_{R_0^d} K^2(t, z) L(dt, dz).$$

Proof Let $K^1(t, z), K^2(t, z) \in H_G^S([0, T] \times R_0^d)$, by Definition 2.6, the following holds:

$$\int_0^T \int_{R_0^d} K^1(t, z) L(dt, dz) = \sum_{0 < u \leq T} K^1(u, \Delta X_u)$$

$$\leq \sum_{0 < u \leq T} K^2(u, \Delta X_u) = \int_0^T \int_{R_0^d} K^2(t, z) L(dt, dz).$$

For $K(t, z) \in H_G^p([0, T] \times R_0^d)$, the inequality still holds under a regular argument. □

To consider the solution of *G*-SDEs, let us introduce the new norm on the integrands: for a process η, define

$$\|\eta\|_{\hat{M}^p_G(0,T)}^p := \int_0^T \hat{\mathbb{E}}\big[|\eta_t|^p\big]\,dt, \quad p \geq 1.$$

The completion of the space under this norm will be denoted as $\hat{M}^p_G(0,T)$. Note that

$$\hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p\,dt\right] \leq \int_0^T \hat{\mathbb{E}}\big[|\eta_t|^p\big]\,dt,$$

thus appropriate integrals will be always well defined.

Similarly, we need to adjust the space of integrands for the jump measure. Let $\hat{H}^2_G([0, T] \times R_0^d)$ denote the completion of all $H^S_G([0,T] \times R_0^d)$ under the norm

$$\|K\|_{\hat{H}^2_G([0,T]\times R_0^d)}^2 := \int_0^T \hat{\mathbb{E}}\left[\sup_{v\in\mathcal{V}} \int_{R_0^d} K^2(u,z)v(dz)\right]du.$$

We consider the following *G*-SDE driven by *d*-dimensional *G*-Brownian motion *B* and the pure jump *G*-Lévy process *L* (in this paper we always use Einstein's convention):

$$dY_t = f(t,Y_t)\,dt + g_{j,k}(t,Y_t)\,d\langle B^j, B^k\rangle_t + \sigma_i(t,Y_t)\,dB_t^i$$
$$+ \int_{R_0^d} K(t,Y_t,z)L(dt,dz). \tag{2.3}$$

Let $\hat{M}^2_G(0,T;R^n)$ denote the space of R^n-valued process and for each element belong to $\hat{M}^2_G(0,T)$. We can define the space $\hat{H}^2_G([0,T] \times R_0^d;R^n)$ in a similar way.

Theorem 2.4 ([5]) *Suppose that $f(t,x)$, $g_{j,k}(t,x)$, $\sigma_i(t,x)$, $K(t,x)$ are Lipschitz continuous w.r.t. x uniformly. For each $x \in R^n$, $f(\cdot,x), g_{j,k}(\cdot,x), \sigma_i(\cdot,x) \in \hat{M}^2_G(0,T;R^n)$, $K(\cdot,x,\cdot) \in \hat{H}^2_G([0,T] \times R_0^d;R^n)$. Then G-SDE (2.3) with the initial condition $Y_0 \in R^n$ has a unique solution $Y_t \in \hat{M}^2_G(0,T;R^n)$.*

3 Existence of solution for *G*-SDEs with discontinuous coefficients

Definition 3.1 If, for any $0 \leq s \leq t$, the process $U_t \in \hat{M}^2_G(0,T;R^n)$ satisfies the following inequality:

$$U_t \geq U_s + \int_s^t f(u,U_u)\,du + \int_s^t g_{j,k}(u,U_u)\,d\langle B^j, B^k\rangle_u$$
$$+ \int_s^t \sigma_i(u,U_u)\,dB_u^i + \int_s^t \int_{R_0^d} K(u,U_u,z)L(du,dz), \tag{3.1}$$

q.s., then it is said to be an upper solution of *G*-SDE (2.3) on the interval $[0,T]$.

Definition 3.2 If, for any $0 \le s \le t$, the process $L_t \in \hat{M}_G^2(0, T; R^n)$ satisfies the following inequality:

$$L_t \le L_s + \int_s^t f(u, L_u)\,du + \int_s^t g_{j,k}(u, L_u)\,d\langle B^j, B^k\rangle_u$$
$$+ \int_s^t \sigma_i(u, L_u)\,dB_u^i + \int_s^t \int_{R_0^d} K(u, L_u, z)L(du, dz), \tag{3.2}$$

q.s., then it is said to be a lower solution of G-SDE (2.3) on the interval $[0, T]$.

Suppose that U_t and L_t are the respective upper and lower solutions of the G-SDE

$$dY_t = f(t, w)\,dt + g_{j,k}(t, w)\,d\langle B^j, B^k\rangle_t + \sigma_i(t, Y_t)\,dB_t^i$$
$$+ \int_{R_0^d} K(t, w, z)L(dt, dz), \quad t \in [0, T], w \in \Omega, \tag{3.3}$$

where $f(\cdot, w), g_{j,k}(\cdot, w) \in \hat{M}_G^2(0, T; R^n)$, $K(\cdot, w, \cdot) \in \hat{H}_G^2([0, T] \times R_0^d; R^n)$ for $w \in \Omega$, $\sigma_i(\cdot, x) \in \hat{M}_G^2(0, T; R^n)$ for each $x \in R^n$ and $\sigma_i(t, x)$ is Lipschitz continuous in x. Define two functions $p, q : [0, T] \times R^n \times \Omega \to R^n$ by

$$p(t, x, w) = \max\{L_t(w), \min\{U_t(w), x\}\},$$
$$q(t, x, w) = p(t, x, w) - x, \tag{3.4}$$

and consider the following G-SDE:

$$dY_t = \tilde{f}(t, Y_t)\,dt + \tilde{g}_{j,k}(t, Y_t)\,d\langle B^j, B^k\rangle_t + \tilde{\sigma}_i(t, Y_t)\,dB_t^i$$
$$+ \int_{R_0^d} \tilde{K}(t, Y_t, z)L(dt, dz), \quad t \in [0, T] \tag{3.5}$$

with a given constant initial condition $Y_0 \in R^n$, where

$$\tilde{f}(t, x, w) = f(t, w) + q(t, x, w),$$
$$\tilde{g}_{j,k}(t, x, w) = g_{j,k}(t, w) + q(t, x, w),$$
$$\tilde{\sigma}_i(t, x, w) = \sigma_i(t, p(t, x, w)), \tag{3.6}$$
$$\tilde{K}(t, x, z) = K(t, w, z) + q(t, x, w).$$

It is clear that $\tilde{f}, \tilde{g}_{j,k}, \tilde{\sigma}_i, \tilde{K}$ are Lipschitz continuous in x, thus the conditions of Theorem 2.4 are satisfied. Then G-SDE (3.5) has a unique solution $Y_t \in \hat{M}_G^2(0, T; R^n)$.

Lemma 3.1 *Suppose that U_t and L_t are the respective upper and lower solutions of G-SDE (3.3) satisfying $L_t \le U_t$ for $t \in [0, T]$. Then U_t and L_t are the upper and lower solutions of G-SDE (3.5), respectively.*

Proof For $0 \le s \le t$, we have

$$
U_s + \int_s^t \tilde{f}(u, U_u)\, du + \int_s^t \tilde{g}_{j,k}(u, U_u)\, d\langle B^j, B^k \rangle_u
$$

$$
+ \int_s^t \tilde{\sigma}_i(u, U_u)\, dB_u + \int_s^t \int_{R_0^d} \tilde{K}(u, U_u, u) L(du, dz)
$$

$$
= U_s + \int_s^t \left[f(u, w) + q(u, U_u, w) \right] du + \int_s^t \left[g_{j,k}(u, w) + q(u, U_u, w) \right] d\langle B^j, B^k \rangle_u
$$

$$
+ \int_s^t \sigma_i \big(u, p(u, U_u, w)\big)\, dB_u^i + \int_s^t \int_{R_0^d} \left[K(u, w, z) + q(u, U_u, w) \right] L(du, dz)
$$

$$
= U_s + \int_s^t f(u, w)\, du + \int_s^t g_{j,k}(u, w)\, d\langle B^j, B^k \rangle_u
$$

$$
+ \int_s^t \sigma_i(u, U_u)\, dB_u^i + \int_s^t \int_{R_0^d} K(u, w, z) L(du, dz)
$$

$$
\le U_t, \tag{3.7}
$$

where we have used $p(t, U_t, w) = U_t$ and $q(t, U_t, w) = 0$. Therefore U_t is an upper solution of *G*-SDE (3.5). One can show that L_t is a lower solution of *G*-SDE (3.5) in a similar way as above. □

By Lemma 3.1 we know that if U_t and L_t are upper and lower solutions of *G*-SDE (3.3), then they are the respective upper and lower solutions for *G*-SDE (3.5). Suppose that Y_t is the solution of *G*-SDE (3.5) such that

$$
L_t < Y_t < U_t, \quad t \in [0, T], \text{q.s.} \tag{3.8}
$$

Since $p(t, Y_t, w) = Y_t$, $q(t, Y_t, w) = 0$, then

$$
dY_t = \tilde{f}(t, Y_t)\, dt + \tilde{g}_{j,k}(t, Y_t)\, d\langle B^j, B^k \rangle_t + \tilde{\sigma}_i(t, Y_t)\, dB_t^i + \int_{R_0^d} \tilde{K}(t, Y_t, z) L(dt, dz)
$$

$$
= \left[f(t, w) + q(t, Y_t, w) \right] dt + \left[g_{j,k}(t, w) + q(u, Y_t, w) \right] d\langle B^j, B^k \rangle_t
$$

$$
+ \sigma_i \big(t, p(t, Y_t, w)\big)\, dB_t^i + \int_{R_0^d} \left[K(t, w, z) + q(t, Y_t, w) \right] L(dt, dz)
$$

$$
= f(t, w)\, dt + g_{j,k}(t, w) d\langle B^j, B^k \rangle_t + \sigma_i(t, Y_t)\, dB_t^i + \int_{R_0^d} K(t, w, z) L(dt, dz), \tag{3.9}
$$

which implies that Y_t is a solution of *G*-SDE (3.3). Thus, if we can show that any solution Y_t of problem (3.5) does satisfy inequality (3.8), then Y_t is also the solution of *G*-SDE (3.3).

Theorem 3.1 *Suppose that*
  (i) *$f(\cdot, w), g_{j,k}(\cdot, w) \in \hat{M}_G^2(0, T; R^n)$, $K(\cdot, w, \cdot) \in \hat{H}_G^2([0, T] \times R_0^d; R^n)$ for $w \in \Omega$,*
    $\sigma_i(\cdot, x) \in \hat{M}_G^2(0, T; R^n)$ for each $x \in R^n$ and $\sigma(t, x)$ is Lipschitz continuous in x;
  (ii) *the respective upper and lower solutions U_t and L_t of G-SDE (3.3) satisfy $L_t < U_t$ for*
    $t \in [0, T]$;

(iii) $Y_0 \in R^n$ is a given initial value with $\hat{\mathbb{E}}[|Y_0|^2] < \infty$ and $L_0 < Y_0 < U_0$.

Then there exists a unique solution $Y_t \in \hat{M}_G^2(0, T; R^n)$ of G-SDE (3.3) such that $L_t < Y_t < U_t$ for $t \in [0, T]$, q.s.

Proof We only need to prove that the solution Y_t of G-SDE (3.5) does satisfy inequality (3.8). Assume that there exists an arbitrary interval $(t_1, t_2) \subset [0, T]$ such that $Y_{t_1} = L_{t_1}$ and $Y_t < L_t$ for $t \in (t_1, t_2)$, then we have

$$
\begin{aligned}
Y_t - L_t &\geq \int_{t_1}^t \tilde{f}(u, Y_u) \, du + \int_{t_1}^t \tilde{g}_{j,k}(u, Y_u) \, d\langle B^j, B^k \rangle_u \\
&\quad + \int_{t_1}^t \tilde{\sigma}_i(u, Y_u) \, dB_u^i + \int_{t_1}^t \int_{R_0^d} \tilde{K}(u, Y_u, z) L(du, dz) \\
&\quad - \int_{t_1}^t \tilde{f}(u, L_u) \, du - \int_{t_1}^t \tilde{g}_{j,k}(u, L_u) \, d\langle B^j, B^k \rangle_u \\
&\quad - \int_{t_1}^t \tilde{\sigma}_i(u, L_u) \, dB_u^i - \int_{t_1}^t \int_{R_0^d} \tilde{K}(u, L_u, z) L(du, dz) \\
&= \int_{t_1}^t \left[f(u, w) + q(u, Y_u, w) \right] du + \int_{t_1}^t \left[g_{j,k}(u, w) + q(u, Y_u, w) \right] d\langle B^j, B^k \rangle_u \\
&\quad + \int_{t_1}^t \sigma_i \big(u, p(u, Y_u, w) \big) \, dB_u^i \\
&\quad + \int_{t_1}^t \int_{R_0^d} \left[K(u, z) + q(u, Y_u, w) \right] L(du, dz) - \int_{t_1}^t \left[f(u, w) + q(u, L_u, w) \right] du \\
&\quad - \int_{t_1}^t \left[g_{j,k}(u, w) + q(u, L_u, w) \right] d\langle B^j, B^k \rangle_u - \int_{t_1}^t \sigma_i \big(u, p(u, L_u, w) \big) \, dB_u^i \\
&\quad - \int_{t_1}^t \int_{R_0^d} \left[K(u, z) + q(u, L_u, w) \right] L(du, dz).
\end{aligned}
\tag{3.10}
$$

Since $Y_t \leq L_t \leq U_t$ for $t \in (t_1, t_2)$, then $p(t, L_t, w) = L_t$ and $p(t, Y_t, w) = L_t$. Also $q(t, L_t, w) = 0$ and $q(t, Y_t, w) = L_t - Y_t$. Thus

$$
\begin{aligned}
Y_t - L_t &\geq \int_{t_1}^t (L_u - Y_u) \, du + \int_{t_1}^t (L_u - Y_u) \, d\langle B^j, B^k \rangle_u \\
&\quad + \int_{t_1}^t \int_{R_0^d} (L_u - Y_u) L(du, dz) > 0,
\end{aligned}
\tag{3.11}
$$

which yields a contradiction. Thus $Y_t \geq L_t$ for $t \in [0, T]$. By using similar arguments as above, one can show that $Y_t \leq U_t$ for $t \in [0, T]$. Thus the proof is finished. □

Now we consider the following G-SDE:

$$
\begin{aligned}
dY_t &= f(t, Y_t) \, dt + g_{j,k}(t, Y_t) \, d\langle B^j, B^k \rangle_t \\
&\quad + \sigma_i(t, Y_t) \, dB_t^i + \int_{R_0^d} K(t, Y_t, z) L(dt, dz),
\end{aligned}
\tag{3.12}
$$

where $f(t,x)$, $g_{j,k}(t,x)$ and $K(t,x,z)$ do not need to be Lipschitz continuous with respect to x, only $\sigma_i(t,x)$ is Lipschitz continuous in x.

Theorem 3.2 *Suppose that*

(i) *for each* $x \in R^n$, $f(\cdot,x), g_{j,k}(\cdot,x), \sigma_i(\cdot,x) \in \hat{M}_G^2(0,T;R^n)$, $K(\cdot,x,\cdot) \in \hat{H}_G^2([0,T] \times R_0^d; R^n)$;

(ii) $\sigma_i(t,x)$ *is Lipschitz continuous in* x, $f(t,x)$, $g_{j,k}(t,x)$ *and* $K(t,x,z)$ *are increasing in* x;

(iii) U_t *and* L_t *are the respective upper and lower solutions of G-SDE* (3.12). *Moreover,*
$K(\cdot,U_t,\cdot), K(\cdot,L_t,\cdot) \in \hat{H}_G^2([0,T] \times R_0^d; R^n)$ *and* $L_t \leq U_t$ *for* $t \in [0,T]$.

Then there exists at least one solution $Y_t \in \hat{M}_G^2(0,T;R^n)$ *of G-SDE* (3.12) *such that* $L_t \leq Y_t \leq U_t$ *for* $t \in [0,T]$, *q.s.*

Proof Denote the order interval $[L,U]$ in $\hat{M}_G^2(0,T;R^n)$ by $\mathcal{H}$, that is, $\mathcal{H} = \{Y : Y \in \hat{M}_G^2(0,T;R^n)$ and $L_t \leq Y_t \leq U_t\}$ for $t \in [0,T]$, which is closed and bounded. By using the monotone convergence theorem in [5], one can prove the convergence of a monotone sequence that belongs to $\mathcal{H}$ in $\hat{M}_G^2(0,T;R^n)$. Thus $\mathcal{H}$ is a regularly ordered metric space with the norm of $\hat{M}_G^2(0,T;R^n)$.

Since $f(t,x), g_{j,k}(t,x)$ and $K(t,x,z)$ are increasing in x, it is easy to see that for any process $V \in \mathcal{H}$, U_t and L_t are the respective upper and lower solutions for the G-SDE

$$dY_t = f(t,V_t)\,dt + g_{j,k}(t,V_t)\,d\langle B^j, B^k \rangle_t + \sigma_i(t,Y_t)\,dB_t^i + \int_{R_0^d} K(t,V_t,z)L(dt,dz). \qquad (3.13)$$

Hence, by Theorem 3.1, for any $Y_0 \in R^n$ with $\hat{\mathbb{E}}[|Y_0|^2] < \infty$, and $L_0 \leq Y_0 \leq U_0$, G-SDE (3.13) has a unique solution $Y_t \in \hat{M}_G^2(0,T;R^n)$ such that $L_t \leq Y_t \leq U_t$ for $t \in [0,T]$, q.s.

Define an operator $F : \mathcal{H} \to \mathcal{H}$ by $F(V) = Y$, where Y is the unique solution of G-SDE (3.13). For all $t \in [0,T]$, let $V_t^1, V_t^2 \in \mathcal{H}$ and $V_t^1 \leq V_t^2$ and define $Y_t^1 = F(V_t^1)$, $Y_t^2 = F(V_t^2)$. Since f, g, K are increasing functions, then

$$\begin{aligned} Y_t^1 &= F\big(V_t^1\big) \\ &= Y_0 + \int_0^t f\big(u, V_u^1\big)\,du + \int_0^t g_{j,k}\big(t, V_u^1\big)\,d\langle B^j, B^k \rangle_u + \int_0^t \sigma_i\big(t, Y_t^1\big)\,dB_t^i \\ &\quad + \int_0^t \int_{R_0^d} K\big(t, V_t^1, z\big)L(dt,dz) \\ &\leq Y_0 + \int_0^t f\big(u, V_u^2\big)\,du + \int_0^t g_{j,k}\big(t, V_u^2\big)\,d\langle B^j, B^k \rangle_u + \int_0^t \sigma_i\big(t, Y_t^1\big)\,dB_t^i \\ &\quad + \int_0^t \int_{R_0^d} K\big(t, V_t^2, z\big)L(dt,dz), \end{aligned} \qquad (3.14)$$

which implies that Y_t^1 is a lower solution of the G-SDE

$$\begin{aligned} Y_t &= Y_0 + \int_0^t f\big(u, V_u^2\big)\,du + \int_0^t g_{j,k}\big(t, V_u^2\big)\,d\langle B^j, B^k \rangle_u + \int_0^t \sigma_i(t, Y_t)\,dB_t^i \\ &\quad + \int_0^t \int_{R_0^d} K\big(t, V_t^2, z\big)L(dt,dz). \end{aligned} \qquad (3.15)$$

However, this problem has an upper solution U_t. Then the solution of G-SDE (3.15) Y_t^2 satisfies $Y_t^1 \leq Y_t^2 \leq U_t$. Hence F is an increasing mapping and, by Theorem 3.3, it has a

fixed point $Y^* = F(Y^*) \in \mathcal{H}$ such that $L_t \leq Y_t^* \leq U_t$, q.s. and

$$Y_t^* = Y_0 + \int_0^t f\left(u, Y_t^*\right) du + \int_0^t g_{j,k}\left(t, Y_t^*\right) d\langle B^j, B^k \rangle_u + \int_0^t \sigma_i\left(t, Y_t^*\right) dB_t^i$$

$$+ \int_0^t \int_{R_0^d} K\left(t, Y_t^*, z\right) L(dt, dz). \tag{3.16}$$

Thus the proof is finished. $\qquad\square$

Example 3.1 Consider the following scalar stochastic differential equation:

$$dY_t = H(Y_t) \, dt + \{Y_t\} \, d\langle B \rangle_t + dB_t + \int_{R_0^d} \{Y_t\} L(dt, dz), \tag{3.17}$$

where the Heaviside function $H : R \to R$ is defined by

$$H(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x \geq 0. \end{cases} \tag{3.18}$$

This is an important function in science, and it is considered to be a fundamental function in engineering. The fractional part function $\{x\} : R \to [0, 1)$ has discontinuities at the integers and is defined by

$$\{x\} = x - [x], \quad x \in R,$$

where $[x]$ is the floor function. The importance of this function is clear from the sawtooth waves which are used in music and computer graphics.

Let $U_t = U_0 + \int_0^t du + \int_0^t d\langle B \rangle_u + \int_0^t dB_u + \int_0^t \int_{R_0} L(du, dz)$ for $t \in [0, T]$. Then we have

$$U_t = U_0 + \int_0^t du + \int_0^t d\langle B \rangle_u + \int_0^t dB_u + \int_0^t \int_{R_0} L(du, dz)$$

$$= U_s + \int_s^t du + \int_s^t d\langle B \rangle_u + \int_s^t dB_u + \int_s^t \int_{R_0} L(du, dz)$$

$$\geq U_s + \int_s^t H(U_u) \, du + \int_s^t \{U_u\} d\langle B \rangle_u + \int_s^t dB_u + \int_s^t \int_{R_0} \{U_u\} L(du, dz), \tag{3.19}$$

where $U_s = U_0 + \int_0^s du + \int_0^s d\langle B \rangle_u + \int_0^s dB_u + \int_0^s \int_{R_0} L(du, dz)$ for $0 \leq s \leq t \leq T$. This implies that U_t is the upper solution of equation (3.17). In a similar way, one can show that $L_t = L_0 + \int_0^t dB_u$ is a lower solution of equation (3.17). Then, by Theorem 3.2, there exists at least one solution for equation (3.17).

For the following definition and theorem, see [14].

Definition 3.3 An ordered metric space M is called regularly (resp. fully regularly) ordered if each monotone and order (resp. metrically) bounded ordinary sequence of M converges.

Theorem 3.3 *If* $[a, b]$ *is a non-empty order interval in a regularly ordered metric space, then each increasing mapping* $F : [a, b] \rightarrow [a, b]$ *has the least and the greatest fixed points.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both of the authors jointly worked on deriving the results and approved the final manuscript.

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