# RESEARCH

# Advances in Difference Equations a SpringerOpen Journal

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# Existence of solutions for impulsive fractional integrodifferential equations with mixed boundary conditions

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# Abstract

In this paper, by using some fixed point theorems and the measure of noncompactness, we discuss the existence of solutions for a boundary value problem of impulsive integrodifferential equations of fractional order  $\alpha \in (1, 2]$ . Our results improve and generalize some known results in (Zhou and Chu in Commun. Nonlinear Sci. Numer. Simul. 17:1142-1148, 2012; Bai *et al.* in Bound. Value Probl. 2016:63, 2016). Finally, an example is given to illustrate that our result is valuable.

MSC: 34A37; 34B15; 34A08

**Keywords:** impulsive fractional integrodifferential equations; boundary value problem; measure of noncompactness

# 1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and they have been emerging as an important area of investigation in the last few decades; see [3–7].

The theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical, population dynamics, ecology, chemical, biological systems, and engineering background. Therefore, it has been an object of intensive investigation in recent years, some basic results on impulsive differential equations have been obtained and applications to different areas have been considered by many authors, see [8–11]. However, the concept of solutions for impulsive fractional differential equations [4–13] has been argued extensively, while the concept presented in Refs. [4–8, 13] could be controversial and deserves a further argument and mending. In [9, 12, 14], Wang *et al.* and Shu *et al.* gave a new concept of some impulsive differential equations used in [6, 8, 13]. Furthermore, the theory of boundary value problems for nonlinear impulsive fractional differential equations is still in the initial stages and many aspects of this theory need to be explored, we refer the readers to [1, 2, 15– 17].



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In [1], Zhou discussed the existence of solutions for a nonlinear multi-point boundary value problem of integro-differential equations of fractional order as follows:

$$\begin{cases} {}^{c}D_{0^{+}}^{\alpha}u(t)=f(t,u(t),(Hu)(t),(Ku)(t)), \quad t\in[0,1],\alpha\in(1,2],\\ a_{1}u(0)-b_{1}u'(0)=d_{1}u(\xi_{1}), \qquad a_{2}u(1)+b_{2}u'(1)=d_{2}u(\xi_{2}), \end{cases}$$

where  ${}^{c}D_{0^{+}}^{\alpha}$  denotes the fractional Caputo derivative and

$$(Hu)(s) = \int_0^t g(t,s)u(s) \, ds, \qquad (Ku)(s) = \int_0^t h(t,s)u(s) \, ds$$

with respect to strong topology.

In [2], Bai studied the existence of solutions for an impulsive fractional differential equation with nonlocal conditions in a Banach space E

$$\begin{cases} {}^{c}D_{0^{+}}^{\alpha}u(t) = f(t,u(t)), & t \in J', \\ \Delta u(t_{k}) = I_{k}(u(t_{k})), & \Delta u'(t_{k}) = I_{k}^{*}(u(t_{k})), & k = 1,2,\dots,p, \\ u(0) + u'(0) = 0, & u(1) + u'(1) = 0, \end{cases}$$

by using the contraction mapping principle and Krasnoselskii's fixed point theorem.

Dong *et al.* [15] investigated the boundary value problem for *p*-Laplacian fractional differential equations

$$\begin{cases} D^{\alpha}(\phi_p(D^{\alpha}u(t))) = f(t,u(t)), & 0 < t < 1, \\ u(0) = u(1) = D^{\alpha}u(0) = D^{\alpha}u(1) = 0, \end{cases}$$

where  $1 < \alpha \le 2$  is a real number,  $D^{\alpha}$  is the conformable fractional derivative,  $\phi_p(s) = |s|^{p-2}s$ , p > 1,  $\phi_p^{-1} = \phi_q$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous. By the approximation method and fixed point theorems on cone, some existence and multiplicity results of positive solutions are obtained.

Bai *et al.* [16] investigated the boundary value problem of fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), & t \in (0, h), \\ t^{1-\alpha} u(t)|_{t=0} = u_0, \end{cases}$$

where  $f \in C([0, h] \times R, R)$ ,  $D_{0^+}^{\alpha} u(t)$  is the standard Riemann-Liouville fractional derivative,  $0 < \alpha < 1$ . The existence of the blow-up solution, that is to say,  $u \in C(0, h]$  and  $\lim_{t\to 0^+} u(t) = \infty$ , is obtained by the use of the lower and upper solution method.

In this paper, motivated by the above references, we investigate the existence of solutions to the following impulsive fractional integro-differential equations with mixed boundary conditions:

$$\begin{cases} {}^{c}D_{0^{+}}^{\alpha}u(t) = f(t, u(t), (Tu)(t), (Su)(t)), & t \in J', \\ \Delta u(t_{k}) = I_{k}(u(t_{k})), & \Delta u'(t_{k}) = I_{k}^{*}(u(t_{k})), & k = 1, 2, \dots, p, \\ au(0) + bu'(0) = \gamma_{1}(u), & au(1) + bu'(1) = \gamma_{2}(u), \end{cases}$$
(1.1)

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where  $f \in C(J \times E \times E \times E, E)$ ,  $I_k$ ,  $I_k^* \in C(E, E)$ , J = [0, 1],  $J' = J \setminus \{t_1, t_2, ..., t_p\}$ ,  ${}^cD_{0^+}^{\alpha}$  is the Caputo fractional derivative of order  $\alpha \in (1, 2]$ ,  $\{t_k\}$  satisfy  $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$ ,  $p \in N$ ,  $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ ,  $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$ ,  $u(t_k^+)$  and  $u(t_k^-)$  represent the right and left limits of u(t) at  $t = t_k$ , respectively. T and S are two linear operators defined by

$$(Tu)(t) = \int_0^t k(t,s)p_1(s,u(s)) \, ds, \qquad (Su)(t) = \int_0^a h(t,s)p_2(s,u(s)) \, ds,$$

where  $k \in C(D, R^+)$ ,  $h \in C(D_0, R^+)$ ,  $D = \{(t, s) \in R^2 : 0 \le s \le t \le a\}$ ,  $D_0 = \{(t, s) \in R^2 : 0 \le t, s \le a\}$  and  $p_i \in C(J \times E, E)$ ,  $\gamma_i : J \to E$  (*i* = 1, 2) is to be specified later.

The paper is organized as follows. In Section 2 we recall some basic known results. In Section 3 we discuss the existence theorem of solutions for problem (1.1). In Section 4, we provide an example to illustrate our result.

#### 2 Preliminaries

In this section, we introduce notations, definitions and preliminary results which will be used throughout this paper.

Let *E* be a real Banach space and denote by  $\Psi$  the family of all functions  $\psi : \mathbb{R}_+ \to \mathbb{R}_+$  satisfying the following conditions:

- ( $\Psi$ 1)  $\psi$  is nondecreasing;
- ( $\Psi$ 2)  $\sum_{n=1}^{\infty} \psi^n < \infty$  for all t > 0, where  $\psi^n$  is the *n*th iterate of  $\psi$ .

For each  $\psi \in \Psi$ , the following assertions hold:

- (1)  $\lim_{n\to\infty} \psi^n(t) = 0$  for all t > 0;
- (2)  $\psi(t) < t$  for all t > 0;
- (3)  $\psi(0) = 0$ .

Furthermore, we write B(x, r) to denote the closed ball centered at x with radius r and  $\overline{X}$ , *ConvX* to denote the closure and closed convex hull of X, respectively. Moreover, let  $m_E$  indicate the family of all nonempty bounded subsets of E and  $n_E$  indicate the family of all relatively compact sets.

Let  $J_0 = (0, t_1], J_1 = (t_1, t_2], \dots, J_{p-1} = (t_{p-1}, t_p], J_p = (t_p, 1]$  and C(J, E) denote the Banach space of all continuous *E*-valued functions on the interval *J*,  $PC(J, E) = \{u : J \to E | u \in C(J', E), u(t_k^+), u(t_k^-) \text{ exist and } u(t_k^-) = u(t_k), 1 \le k \le p\}$ . Obviously, PC(J, E) is a Banach space with the norm  $||u|| = \sup_{t \in J} ||u(t)||_E$ .

We use the following definition of the measure of noncompactness given in [18].

**Definition 2.1** A mapping  $\mu : m_E \to \mathbb{R}+$  is said to be a measure of noncompactness in *E* if it satisfies the following conditions:

- (1) The family ker  $\mu = \{X \in m_E : \mu(X) = 0\}$  is nonempty and ker  $\mu \subset n_E$ .
- (2)  $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$ .
- (3)  $\mu(\overline{X}) = \mu(X).$
- (4)  $\mu(ConvX) = \mu(X)$ .
- (5)  $\mu(\lambda X + (1 \lambda)Y) \le \lambda \mu(X) + (1 \lambda)\mu(Y)$  for  $\lambda \in [0, 1]$ .
- (6) If (X<sub>n</sub>) is a sequence of closed sets from m<sub>E</sub> such that X<sub>n+1</sub> ⊂ X<sub>n</sub> (n = 1, 2, ...) and if lim<sub>n→∞</sub> μ(X<sub>n</sub>) = 0, then the intersection set X<sub>∞</sub> = ⋂<sub>n=1</sub><sup>∞</sup> X<sub>n</sub> is nonempty.

**Lemma 2.1** ([19]) Let  $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$  be a nondecreasing and upper semicontinuous function. Then the following two conditions are equivalent:

(i) 
$$\lim_{n\to\infty} \varphi^n(t) = 0$$
 for any  $t > 0$ ;  
(ii)  $\varphi(t) < t$  for any  $t > 0$ .

For any nonempty bounded subset,  $X \in m_E$ . For  $x \in X$ ; T > 0 and  $\varepsilon > 0$ , let

$$\begin{split} &\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in J, |t - s| \le \varepsilon\},\\ &\omega(X,\varepsilon) = \sup\{w^T(x,\varepsilon) : x \in X\},\\ &\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X,\varepsilon), X(t) = \{x(t) : x \in X\},\\ &\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}, \end{split}$$

and

$$\mu(X) = \omega_0(X) + \lim_{t \to \infty} \sup \operatorname{diam} X(t).$$
(2.1)

In [18], Banaś has shown that the function  $\mu$  is a measure of noncompactness in the spaces PC(*J*, *E*).

For completeness, we recall the definition of the Caputo derivative of fractional order.

**Definition 2.2** The fractional integral of order  $\alpha$  of a function  $f : [0, \infty) \to \mathbb{R}$  is defined as

$$I_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds,\quad t>0,\alpha>0,$$

provided the right-hand side is point-wise defined on  $(0, \infty)$ , where  $\Gamma(\cdot)$  is the gamma function.

**Definition 2.3** The Riemann-Liouville derivative of order  $\alpha$  with the lower limit zero for a function  $f : [0, \infty) \to \mathbb{R}$  can be written as

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} \, ds, \quad t > 0, n-1 < \alpha < n.$$

**Definition 2.4** The Caputo fractional derivative of order  $\alpha$  for a function  $f : [0, \infty) \to \mathbb{R}$  can be written as

$${}^{c}D_{0^{+}}^{\alpha}f(t) = D_{0^{+}}^{\alpha}\left[f(t) - \sum_{k=0}^{n-1}\frac{t^{k}}{k!}f^{(k)}(0)\right], \quad t > 0, n-1 < \alpha < n,$$

where  $n = [\alpha] + 1$  and  $[\alpha]$  denotes the integer part of  $\alpha$ .

**Remark 2.1** In the case  $f(t) \in C^n[0, \infty)$ , then

$$^{c}D_{0^{+}}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-s)^{n-\alpha-1}f^{(n)}(s)\,ds = I_{0+}^{n-\alpha}f^{n}(t), \quad t > 0, n-1 < \alpha < n.$$

That is to say, Definition 2.4 is just the usual Caputo fractional derivative. In this paper, we consider an impulsive problem, so Definition 2.4 is appropriate.

Moreover, we need the following known results.

**Lemma 2.2** Let  $\alpha > 0$ , then the differential equation

 $^{c}D_{0^{+}}^{\alpha}u(t)=0$ 

has the solution  $u(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_n t^{n-1}$ ,  $c_i \in \mathbb{R}$ ,  $i = 0, 1, \dots, n$ ,  $n = [\alpha] + 1$ .

In view of Lemma 2.2, we have the following.

**Lemma 2.3** Let  $\alpha > 0$ , then

$$I_{0^{+}}^{\alpha} \left( {}^{c}D_{0+}^{\alpha}u(t) \right) = u(t) + c_{0} + c_{1}t + c_{2}t^{2} + \dots + c_{n}t^{n-1}$$

for some  $c_i \in \mathbb{R}$ , i = 0, 1, ..., n,  $n = [\alpha] + 1$ .

**Definition 2.5** ([20]) Let (E, d) be a metric space with *w*-distance *p* and  $f : E \to E$  be a given mapping. We say that *f* is a  $(\gamma, \psi, p)$ -contractive mapping if there exist two functions  $\gamma : E \times E \to [0, \infty)$  and  $\psi \in \Psi$  such that

$$\gamma(x,y)p(fx,fy) \le \psi(p(x,y))$$

for all  $x, y \in E$ .

In the following, we will show some fixed point theorems of Darbo type proved by Aghajani *et al.* and that  $(\gamma, \psi, p)$  is a contractive mapping, which plays a key role in the proof of our main results.

**Lemma 2.4** ([19]) Let  $\Omega$  be a nonempty, bounded, closed and convex subset of a Banach space *E*, and let  $T : \Omega \to \Omega$  be a continuous operator satisfying the inequality

$$\mu(TX) \le \phi(\mu(X)) \tag{2.2}$$

for any nonempty subset X of  $\Omega$ , where  $\mu$  is an arbitrary measure of noncompactness and  $\phi : \mathbb{R}_+ \to \mathbb{R}_+$  is a nondecreasing function such that  $\lim_{n\to\infty} \varphi^n(t) = 0$  for each  $t \ge 0$ . Then T has at least one fixed point in the set  $\Omega$ .

**Lemma 2.5** ([20]) Let p be a w-distance on a complete metric space (E, d), and let  $f : E \to E$  be a  $(\gamma, \psi, p)$ -contractive mapping. Suppose that the following conditions hold:

- (i) f is a  $\gamma$ -admissible mapping;
- (ii) there exists a point  $x_0 \in E$  such that  $\gamma(x_0, fx_0) \ge 1$ ;
- (iii) either f is continuous or, for any sequence  $\{x_n\}$  in E, if  $\gamma(x_n, x_{n+1}) \ge 1$  for all  $n \in \mathbb{N}$ and  $x_n \to x \in E$  as  $n \to \infty$ , then  $\gamma(x_n, x) \ge 1$  for all  $n \in \mathbb{N}$ . Then there exists a point  $u \in E$  such that fu = u. Moreover, if  $\gamma(u, u) \ge 1$ , then p(u, u) = 0.

### 3 Main results

In this section, we establish the existence theorems of solutions for problem (1.1). For convenience, we give some notations.

For  $B \subset PC(J, E)$ , let  $B(t) = \{u(t) : u \in B\}$  and denote  $B_R = \{u \in B : ||u|| \le R\}$ . Now, following [9, 21, 22], let us introduce the definition of a solution of problem (1.1).

**Definition 3.1** A function  $u \in PC(J, E)$  is said to be a solution of problem (1.1) if  $u(t) = u_k(t)$  for  $t \in (t_k, t_{k+1})$  and  $u_k \in C([0, t_{k+1}], E)$  satisfies the equation  ${}^cD_{0^+}^{\alpha}u(t) = f(t, u(t), (Tu)(t), (Su)(t))$  a.e. on  $(0, t_{k+1})$ , and the conditions  $\Delta u(t_k) = I_k(u(t_k)), \Delta u'(t_k) = I_k^*(u(t_k)), k = 1, 2, \dots, p$ , and  $au(0) + bu'(0) = \gamma_1(u), au(1) + bu'(1) = \gamma_2(u)$  hold.

By using a similar technique as in [2], Section 2, we obtain the following lemma.

**Lemma 3.1** Let  $\rho \in C(J, E)$  and  $\alpha \in (1, 2]$ , a function u given by

$$u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \rho(s) \, ds + (\frac{b}{a}-t) [\frac{1}{\Gamma(\alpha)} \int_{0}^{1} (1-s)^{\alpha-1} \rho(s) \, ds \\ + \frac{b}{a\Gamma(\alpha-1)} \int_{0}^{1} (1-s)^{\alpha-2} \rho(s) \, ds] + \frac{1}{a^{2}} [(a(1-t)+b)\gamma_{1}(u) + (at-b)\gamma_{2}(u)], \\ t \in [0,t_{1}]; \\ \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \rho(s) \, ds + (\frac{b}{a}-t) [\frac{1}{\Gamma(\alpha)} \int_{0}^{1} (1-s)^{\alpha-1} \rho(s) \, ds \\ + \frac{b}{a\Gamma(\alpha-1)} \int_{0}^{1} (1-s)^{\alpha-2} \rho(s) \, ds] + (\frac{b}{a}+1-t) \sum_{j=1}^{p} I_{j}^{*}(u(t_{j})) (\frac{b}{a}-t_{j}) \\ + (\frac{b}{a}+1-t) \sum_{j=1}^{p} I_{j}(u(t_{j})) - (t-t_{j}) \sum_{j=k+1}^{p} I_{j}^{*}(u(t_{j})) - \sum_{j=k+1}^{p} I_{j}(u(t_{j})) \\ + \frac{1}{a^{2}} [(a(1-t)+b)\gamma_{1}(u) + (at-b)\gamma_{2}(u)], \\ t \in (t_{k}, t_{k+1}], k = 1, 2, \dots, p-1; \\ \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \rho(s) \, ds + (\frac{b}{a}-t) [\frac{1}{\Gamma(\alpha)} \int_{0}^{1} (1-s)^{\alpha-1} \rho(s) \, ds \\ + \frac{b}{a\Gamma(\alpha-1)} \int_{0}^{1} (1-s)^{\alpha-2} \rho(s) \, ds] + (\frac{b}{a}+1-t) \sum_{j=1}^{p} I_{j}^{*}(u(t_{j})) (\frac{b}{a}-t_{j}) \\ + (\frac{b}{a}+1-t) \sum_{j=1}^{p} I_{j}(u(t_{j})) + \frac{1}{a^{2}} [(a(1-t)+b)\gamma_{1}(u) + (at-b)\gamma_{2}(u)], \\ t \in (t_{p}, t_{p+1}] \end{cases}$$

is a unique solution of the following impulsive problem:

$$\begin{cases} {}^{c}D_{0^{+}}^{\alpha}u(t) = \rho(t), \quad t \in J', \\ \Delta u(t_{k}) = I_{k}(u(t_{k})), \quad \Delta u'(t_{k}) = I_{k}^{*}(u(t_{k})), \quad k = 1, 2, \dots, p, \\ au(0) + bu'(0) = \gamma_{1}(u), \quad au(1) + bu'(1) = \gamma_{2}(u). \end{cases}$$

$$(3.2)$$

*Proof* With Lemma 2.3, a general solution *u* of the equation  ${}^{c}D_{0+}^{\alpha}u(t) = \rho(t)$  on each interval  $(t_k, t_{k+1}]$  (k = 0, 1, 2, ..., p) is given by

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(s) \, ds + a_k + b_k t, \quad \text{for } t \in (t_k, t_{k+1}], \tag{3.3}$$

where  $t_0 = 0$  and  $t_{p+1} = 1$ . Then we have

$$u'(t) = \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha - 2} \rho(s) \, ds + b_k, \quad \text{for } t \in (t_k, t_{k+1}].$$

We have

$$u(0) = a_0, \qquad u'(0) = b_0,$$
  
$$u(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \rho(s) \, ds + a_p + b_p,$$
  
$$u'(1) = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \rho(s) \, ds + b_p.$$

So, applying the boundary conditions (3.2), we have

$$aa_0 + bb_0 = \gamma_1(u),$$
 (3.4)

$$\frac{a}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds + \frac{b}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} ds + aa_p + ab_p + bb_p = \gamma_2(u).$$
(3.5)

Furthermore, in view of  $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) = I_k^*(u(t_k))$ , we have

$$b_{k} = b_{k-1} + I_{k}^{*}(u(t_{k})),$$
(3.6)

$$b_k = b_p - \sum_{j=k+1}^p I_j^* \left( u(t_j) \right) \quad (k = 1, 2, \dots, p-1).$$
(3.7)

In the same way, using the impulsive condition  $\Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(u(t_k))$ , we have

$$a_k + b_k t_k = a_{k-1} + b_{k-1} t_k + I_k (u(t_k)),$$
(3.8)

which by (3.6) implies that

$$a_{k} = a_{k-1} - I_{k}^{*}(u(t_{k}))t_{k} + I_{k}(u(t_{k})).$$
(3.9)

Thus

$$a_{k} = a_{p} + \sum_{j=k+1}^{p} I_{j}^{*}(u(t_{j}))t_{j} - \sum_{j=k+1}^{p} I_{j}(u(t_{j})) \quad (k = 0, 1, 2, \dots, p-1).$$
(3.10)

Combining (3.4), (3.5), (3.7) with (3.10) yields

$$a_{p} = \frac{(a+b)\gamma_{1}(u) - b\gamma_{2}(u)}{a^{2}} + \frac{b}{a\Gamma(\alpha)} \int_{0}^{1} (1-s)^{\alpha-1} \rho(s) \, ds$$
$$+ \frac{b^{2}}{a^{2}\Gamma(\alpha-1)} \int_{0}^{1} (1-s)^{\alpha-2} \rho(s) \, ds - \left(\frac{b}{a}+1\right) \sum_{j=1}^{p} I_{j}^{*}\left(u(t_{j})\right) \left(t_{j}-\frac{b}{a}\right)$$
$$+ \left(\frac{b}{a}+1\right) \sum_{j=1}^{b} I_{j}\left(u(t_{j})\right), \tag{3.11}$$

$$b_{p} = \frac{\gamma_{2}(u) - \gamma_{1}(u)}{a} - \frac{1}{\Gamma(\alpha)} \int_{0}^{1} (1 - s)^{\alpha - 1} \rho(s) \, ds$$
$$- \frac{b}{a\Gamma(\alpha - 1)} \int_{0}^{1} (1 - s)^{\alpha - 2} \rho(s) \, ds + \sum_{j=1}^{p} I_{j}^{*} \left( u(t_{j}) \right) \left( t_{j} - \frac{b}{a} \right)$$
$$- \sum_{j=1}^{p} I_{j} \left( u(t_{j}) \right). \tag{3.12}$$

Furthermore, by (3.7), (3.10), (3.11) and (3.12), we have

$$\begin{aligned} a_{k} &= a_{p} + \sum_{j=k+1}^{p} I_{j}^{*} \left( u(t_{j}) \right) t_{j} + \sum_{j=k+1}^{p} I_{j} \left( u(t_{j}) \right) \\ &= \frac{(a+b)\gamma_{1}(u) - b\gamma_{2}(u)}{a^{2}} + \frac{b}{a} \frac{1}{\Gamma(\alpha)} \int_{0}^{1} (1-s)^{\alpha-1} \rho(s) \, ds \\ &+ \frac{b^{2}}{a^{2}} \frac{1}{\Gamma(\alpha-1)} \int_{0}^{1} (1-s)^{\alpha-2} \rho(s) \, ds - \left(\frac{b}{a}+1\right) \sum_{j=1}^{p} I_{j}^{*} \left( u(t_{j}) \right) \left(t_{j}-\frac{b}{a}\right) \\ &+ \left(\frac{b}{a}+1\right) \sum_{j=1}^{b} I_{j} \left( u(t_{j}) \right) + \sum_{j=k+1}^{p} I_{j}^{*} \left( u(t_{j}) \right) t_{j} \\ &+ \sum_{j=k+1}^{p} I_{j} \left( u(t_{j}) \right) \quad (k=0,1,2,\ldots,p-1), \end{aligned}$$
(3.13)  
$$b_{k} = b_{p} - \sum_{j=k+1}^{p} I_{j}^{*} \left( u(t_{j}) \right) \\ &= \frac{\gamma_{2}(u) - \gamma_{1}(u)}{a} - \frac{1}{\Gamma(\alpha)} \int_{0}^{1} (1-s)^{\alpha-1} \rho(s) \, ds \\ &- \frac{b}{a} \frac{1}{\Gamma(\alpha-1)} \int_{0}^{1} (1-s)^{\alpha-2} \rho(s) \, ds + \sum_{j=1}^{p} I_{j}^{*} \left( u(t_{j}) \right) \left(t_{j}-\frac{b}{a}\right) \\ &- \sum_{j=1}^{p} I_{j} \left( u(t_{j}) \right) - \sum_{j=k+1}^{p} I_{j}^{*} \left( u(t_{j}) \right) \quad (k=0,1,2,\ldots,p-1). \end{aligned}$$
(3.14)

Hence, for k = 0, 1, 2, ..., p - 1, (3.13) and (3.14) imply

$$a_{k} + b_{k}t = \left(\frac{b}{a} - t\right) \left[\frac{1}{\Gamma(\alpha)} \int_{0}^{1} (1 - s)^{\alpha - 1} \rho(s) \, ds + \frac{b}{a\Gamma(\alpha - 1)} \int_{0}^{1} (1 - s)^{\alpha - 2} \rho(s) \, ds\right] \\ + \left(\frac{b}{a} + 1 - t\right) \sum_{j=1}^{p} I_{j}^{*} \left(u(t_{j})\right) \left(\frac{b}{a} - t_{j}\right) + \left(\frac{b}{a} + 1 - t\right) \sum_{j=1}^{p} I_{j} \left(u(t_{j})\right) \\ - \left(t - t_{j}\right) \sum_{j=k+1}^{p} I_{j}^{*} \left(u(t_{j})\right) - \sum_{j=k+1}^{p} I_{j} \left(u(t_{j})\right) \\ + \frac{1}{a^{2}} \left[\left(a(1 - t) + b\right) \gamma_{1}(u) + (at - b) \gamma_{2}(u)\right].$$
(3.15)

For *k* = *p*, (3.11) and (3.12) imply

$$a_{k} + b_{k}t = \left(\frac{b}{a} - t\right) \left[\frac{1}{\Gamma(\alpha)} \int_{0}^{1} (1 - s)^{\alpha - 1} \rho(s) \, ds + \frac{b}{a\Gamma(\alpha - 1)} \int_{0}^{1} (1 - s)^{\alpha - 2} \rho(s) \, ds\right] \\ + \left(\frac{b}{a} + 1 - t\right) \sum_{j=1}^{p} I_{j}^{*} \left(u(t_{j})\right) \left(\frac{b}{a} - t_{j}\right) + \left(\frac{b}{a} + 1 - t\right) \sum_{j=1}^{p} I_{j} \left(u(t_{j})\right) \\ + \frac{1}{a^{2}} \left[ \left(a(1 - t) + b\right) \gamma_{1}(u) + (at - b) \gamma_{2}(u) \right].$$
(3.16)

Now it is clear that (3.3), (3.15), (3.16) imply that (3.1) holds.

Conversely, assume that u satisfies (3.1). By a direct computation, it follows that the solution given by (3.1) satisfies (3.2). This completes the proof.

To prove our main results, we state the following basic assumptions of this paper.

(H1)  $f : \mathbb{R}_+ \times E \times E \times E \to E$  is continuous and there exists a nondecreasing and upper semicontinuous function  $\varphi \in \Psi$ ; furthermore, there exist nondecreasing continuous functions  $\Phi_1, \Phi_2 : \mathbb{R} \to \mathbb{R}$  with  $\Phi_1(0) = \Phi_2(0) = 0$  such that

$$\left\|f\left(t,u(t),(Tu)(t),(Su)(t)\right)-f\left(t,v(t),(Tv)(t),(Sv)(t)\right)\right\|$$
  
$$\leq\varphi\left(\|u-v\|\right)+\Phi_1\left(\|Tu-Tv\|\right)+\Phi_2\left(\|Su-Sv\|\right).$$

(H2) The function defined by |f(t, 0, 0, 0)| is bounded on *J*, i.e.,

 $M_1 = \sup\{|f(t, 0, 0, 0)| : t \in J\} < \infty.$ 

- (H3)  $\gamma_i$  (i = 1, 2):  $E \to E$  are continuous and compact mappings and there exists a nondecreasing and upper semicontinuous function  $\varphi_i \in \Psi$ ; furthermore, there exist constants  $N_i$  such that  $\|\gamma_i(u) \gamma_i(v)\| \le \varphi_i(\|u v\|)$  and  $\|\gamma_i(u)\| \le N_i$  (i = 1, 2) for any  $u, v \in B_R$ .
- (H4) The functions  $I_k, I_k^* : E \to E$  are continuous, and there exists a nondecreasing and upper semicontinuous function  $\varphi_3, \varphi_4 \in \Psi$ ; furthermore, there exist constants  $\mu > 0$  and  $\rho > 0$  such that

$$\begin{aligned} \|I_k(u) - I_k(v)\| &\leq \varphi_3 \big( \|u - v\| \big) \quad \text{and} \quad \|I_k(u)\| \leq \mu, \\ \text{for all } u, v \in B_R, k = 1, 2, \dots, p, \\ \|I_k^*(u) - I_k^*(v)\| &\leq \varphi_4 \big( \|u - v\| \big) \quad \text{and} \quad \|I_k^*(u)\| \leq \rho, \\ \text{for all } u, v \in B_R, k = 1, 2, \dots, p. \end{aligned}$$

(H5) There exists a positive solution  $r_0$  of the inequality

$$\begin{cases} \frac{(b+a)}{a\Gamma(\alpha+1)} + \frac{b}{a^2\Gamma(\alpha)} \\ \end{cases} \left(\varphi\left(\|u\|\right) + \Phi_1(D_1) + \Phi_2(D_2) + M_1\right) \\ + \frac{p}{a^2}\left(a(b+a)\mu + \left(a^2 + ab + b^2\right)\rho\right) + \frac{a+b}{a^2}(N_1 + N_2) \le r, \end{cases}$$

where D is a positive constant defined by the equality

$$D_1 = \sup\left\{ \left| \int_0^t k(t,s) p_1(s,u(s)) \, ds \right| : t,s \in J, u \in B_R \right\},$$
$$D_2 = \sup\left\{ \left| \int_0^a h(t,s) p_2(s,u(s)) \, ds \right| : t,s \in J, u \in B_R \right\}.$$

Moreover,

.

$$\lim_{t \to \infty} \int_0^t \|k(t,s)[p_1(s,u(s)) - p_1(s,v(s))]\| \, ds = 0$$

and

$$\lim_{a\to\infty}\int_0^a \|h(t,s)[p_2(s,u(s))-p_2(s,v(s))]\|\,ds=0.$$

**Theorem 3.1** Let *E* be a Banach space, suppose that conditions (H1)-(H5) are satisfied. Then problem (1.1) has at least one solution in the space PC(J, E).

*Proof* First we consider the operator  $Q : PC(J, E) \to PC(J, E)$  defined by

$$\begin{aligned} (Qu)(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, u(s), (Tu)(s), (Su)(s)\right) ds \\ &+ \left(\frac{b}{a} - t\right) \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, u(s), (Tu)(s), (Su)(s)\right) ds \\ &+ \frac{b}{a\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f\left(s, u(s), (Tu)(s), (Su)(s)\right) ds \right] \\ &+ \left(\frac{b}{a} + 1 - t\right) \sum_{j=1}^p I_j^* (u(t_j)) \left(\frac{b}{a} - t_j\right) + \left(\frac{b}{a} + 1 - t\right) \sum_{j=1}^p I_j (u(t_j)) \\ &- (t-t_j) \sum_{j=k+1}^p I_j^* (u(t_j)) - \sum_{j=k+1}^p I_j (u(t_j)) \\ &+ \frac{1}{a^2} \left[ \left(a(1-t) + b\right) \gamma_1(u) + (at-b) \gamma_2(u) \right], \quad t \in (t_k, t_{k+1}], k = 1, 2, \dots, p-1. \end{aligned}$$

It is easy to see that the fixed points of Q are the solutions of nonlocal problem (1.1). Set  $B_{r_0} = \{u \in B : ||u(t)|| \le r_0, t \in J\}$ , then  $B_{r_0}$  is a closed ball in PC(J, E) with center  $\theta$  and radius  $r_0$ . For  $\forall u \in B_{r_0}$ , by means of (H1), (H2) and the triangle inequality, we get

$$\|f(s, u(s), (Tu)(s), (Su)(s))\| \le \|f(s, u(s), (Tu)(s), (Su)(s)) - f(s, 0, 0, 0)\| + \|f(s, 0, 0, 0)\| \le \varphi(\|u\|) + \Phi_1(\|Tu\|) + \Phi_2(\|Su\|) + M_1 \le \varphi(\|u\|) + \Phi_1(D_1) + \Phi_2(D_2) + M_1.$$
(3.17)

First, we notice that the continuity of Q(u)(t) for any  $u \in PC(J, E)$  is obvious, and by (3.17), we have

$$\|(Qu)(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,u(s),(Tu)(s),(Su)(s))\| ds + \left|\frac{b}{a} - t\right| \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \|f(s,u(s),(Tu)(s),(Su)(s))\| ds$$

$$\begin{aligned} &+ \frac{b}{a\Gamma(\alpha-1)} \int_{0}^{1} (1-s)^{\alpha-2} \left\| f\left(s,u(s),(Tu)(s),(Su)(s)\right) \right\| ds \right] \\ &+ \left| \frac{b}{a} + 1 - t \right| \sum_{j=1}^{p} \left\| I_{j}^{*}\left(u(t_{j})\right) \right\| \cdot \left| \frac{b}{a} - t_{j} \right| + \left| \frac{b}{a} + 1 - t \right| \sum_{j=1}^{p} \left\| I_{j}\left(u(t_{j})\right) \right\| \\ &- \left| t - t_{j} \right| \sum_{j=k+1}^{p} \left\| I_{j}^{*}\left(u(t_{j})\right) \right\| - \sum_{j=k+1}^{p} \left\| I_{j}(u(t_{j})) \right\| \\ &+ \frac{1}{a^{2}} \Big[ \left| a(1-t) + b \right| \cdot \left\| \gamma_{1}(u) \right\| + \left| at - b \right| \cdot \left\| \gamma_{2}(u) \right\| \Big] \\ &\leq \left( \varphi\left( \left\| u \right\| \right) + \Phi_{1}(D_{1}) + \Phi_{2}(D_{2}) + M_{1} \right) \left\{ \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &+ \frac{b}{a} \left[ \int_{0}^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{b}{a} \int_{0}^{1} \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \right] \right\} \\ &+ \frac{b}{a} \left( \frac{b}{a} + 1 \right) \sum_{j=1}^{p} \left\| I_{j}^{*}\left(u(t_{j})\right) \right\| + \left( \frac{b}{a} + 1 \right) \sum_{j=1}^{p} \left\| I_{j}(u(t_{j})) \right\| \\ &+ \sum_{j=k+1}^{p} \left\| I_{j}^{*}\left(u(t_{j})\right) \right\| + \sum_{j=k+1}^{p} \left\| I_{j}(u(t_{j})) \right\| + \frac{a+b}{a^{2}} (N_{1}+N_{2}) \\ &\leq \left\{ \frac{(b+a)}{a\Gamma(\alpha+1)} + \frac{b}{a^{2}\Gamma(\alpha)} \right\} (\varphi\left( \| u \| \right) + \Phi_{1}(D_{1}) + \Phi_{2}(D_{2}) + M_{1}) \\ &+ \frac{p}{a^{2}} (a(b+a)\mu + (a^{2} + ab + b^{2})\rho ) + \frac{a+b}{a^{2}} (N_{1}+N_{2}), \end{aligned}$$

and  $D_1$ ,  $D_2$  are given by (H5). Thus

$$\|(Qu)(t)\| \le \left\{\frac{(b+a)}{a\Gamma(\alpha+1)} + \frac{b}{a^2\Gamma(\alpha)}\right\} (\varphi(\|u\|) + \Phi_1(D_1) + \Phi_2(D_2) + M_1) + \frac{p}{a^2} (a(b+a)\mu + (a^2 + ab + b^2)\rho) + \frac{a+b}{a^2} (N_1 + N_2) \le r.$$

Now *Q* is well defined, we have  $Q(B_{r_0}) \subset B_{r_0}$ , where  $r_0$  is a constant appearing in assumption (H5).

We shall show that *Q* is continuous from  $B_{r_0}$  into  $B_{r_0}$ . To show this, take  $u, v \in B_{r_0}$  and  $\varepsilon > 0$  arbitrarily such that  $||u - v|| < \varepsilon$ , for  $t \in J$ , we have

$$\begin{split} \|Qu(t) - Qv(t)\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(t,u(t),(Tu)(t),(Su)(t)) - f(t,v(t),(Tv)(t),(Sv)(t))\| \, ds \\ &+ \left|\frac{b}{a} - t\right| \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(t,u(t),(Tu)(t),(Su)(t)) - f(t,v(t),(Tv)(t),(Sv)(t))\| \, ds \\ &+ \frac{b}{a} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(t,u(t),(Tu)(t),(Su)(t)) - f(t,v(t),(Tv)(t),(Sv)(t))\| \, ds \right] \\ &+ \left|\frac{b}{a} + 1 - t_k\right| \sum_{j=1}^p \|I_j^*(u(t_j)) - I_j^*(v(t_j))\| \cdot \left|\frac{b}{a} - t_j\right| \end{split}$$

$$+ \left| \frac{b}{a} + 1 - t_{k} \right| \sum_{j=1}^{p} \left\| I_{j}(u(t_{j})) - I_{j}(v(t_{j})) \right\| - |t - t_{j}| + \sum_{j=k+1}^{p} \left\| I_{j}^{*}(u(t_{j})) - I_{j}^{*}(v(t_{j})) \right\|$$

$$+ \sum_{j=k+1}^{p} \left\| I_{j}(u(t_{j})) - I_{j}(v(t_{j})) \right\| + \frac{1}{a^{2}} \left[ (a(1 - t) + b) \right\| \gamma_{1}(u) - \gamma_{1}(v) \right\|$$

$$+ (at - b) \left\| \gamma_{2}(u) - \gamma_{2}(v) \right\| \right]$$

$$\leq \left\{ \frac{(b + a)}{a\Gamma(\alpha + 1)} + \frac{b}{a^{2}\Gamma(\alpha)} \right\} \left\{ \varphi(\|u - v\|) \right\}$$

$$+ \Phi_{1} \left( \left| \int_{0}^{t} k(t,s) \left[ p_{1}(s, u(s)) - p_{1}(s, v(s)) \right] ds \right| \right) \right\} + \frac{p}{a^{2}} (a(b + a)\varphi_{3}(\|u - v\|)$$

$$+ \left( a^{2} + ab + b^{2} \right) \varphi_{4}(\|u - v\|) \right) + \frac{b + a}{a^{2}} (\varphi_{1}(\|u - v\|) + \varphi_{2}(\|u - v\|))$$

$$+ \left( \frac{(b + a)}{a\Gamma(\alpha + 1)} + \frac{b}{a^{2}\Gamma(\alpha)} \right\} \left\{ \varphi(\|u - v\|) + \Phi_{1}(\delta(t, u, v)) + \Phi_{2}(\gamma(t, u, v)) \right\}$$

$$+ \frac{p}{a^{2}} (a(b + a)\varphi_{3}(\|u - v\|) + (a^{2} + ab + b^{2})\varphi_{4}(\|u - v\|))$$

$$+ \frac{b + a}{a^{2}} (\varphi_{1}(\|u - v\|) + \varphi_{2}(\|u - v\|)),$$

$$(3.18)$$

where

$$\delta(t, u, v) = \left| \int_0^t k(t, s) [p_1(s, u(s)) - p_1(s, v(s))] ds \right|,$$
  
$$\gamma(t, u, v) = \left| \int_0^a h(t, s) [p_2(s, u(s)) - p_2(s, v(s))] ds \right|.$$

Furthermore, considering conditions (H1) and (H5), there exists a > 0 such that for  $t \ge a$  we have

$$\Phi_1(\delta(t, u, v)) < \varepsilon, \qquad \Phi_2(\gamma(t, u, v)) < \varepsilon.$$
(3.19)

Then, from (3.18) and (3.19) for  $t \ge a$ , we have

$$\begin{aligned} \left\| (Qu)(t) - (Qv)(t) \right\| \\ &\leq \left\{ \frac{(b+a)}{a\Gamma(\alpha+1)} + \frac{b}{a^2\Gamma(\alpha)} + \frac{p}{a^2}(a(b+a) + (a^2+ab+b^2) + \frac{b+a}{a^2} \right\} \varphi_{\max}(\varepsilon), \end{aligned}$$

where  $\varphi_{\max}(\varepsilon) := \max\{\varphi(\varepsilon), \varphi_i(\varepsilon), i = 1, 2, 3, 4\}.$ 

Now we assume that  $t \in J$ , then by using the continuity of  $p_1$ ,  $p_2$  on  $J \times J \times [-r, r]$  and condition (H5), we can obtain

$$\Phi_1(\delta(t, u, v)) \to 0, \qquad \Phi_2(\gamma(t, u, v)) \to 0,$$

as  $\varepsilon \to 0$ . Thus, we proved that  $Q: B_{r_0} \to B_{r_0}$  is a continuous operator.

Now we show that for any nonempty set  $X \subset B_{r_0}$ ,  $\mu(QX) \leq \phi(\mu(X))$ . First, we demonstrate that the operator  $Q: B_{r_0} \to B_{r_0}$  is equicontinuous. In view of (3.17), we define  $f_{\max} = \sup_{(t,u)\in J\times B_{r_0}} ||f(s, u, Tu, Su)||$ . For any  $u \in B_{r_0}$  and  $t_{k-1} \leq \tau_1 < \tau_2 \leq t_k \in J$  with  $|\tau_1 - \tau_2| \leq \varepsilon$ , we get that

$$\begin{split} \left\| (Qu)(\tau_{2}) - (Qu)(\tau_{1}) \right\| \\ &= \left\| \int_{\tau_{1}}^{\tau_{2}} \frac{(\tau_{2} - s)^{\alpha - 1}}{\Gamma(\alpha)} f\left(s, u(s), (Tu)(s), (Su)(s)\right) ds \right. \\ &+ \int_{0}^{\tau_{1}} \left( \frac{(\tau_{2} - s)^{\alpha - 1} - (\tau_{1} - s)^{\alpha - 1}}{\Gamma(\alpha)} \right) f\left(s, u(s), (Tu)(s), (Su)(s)\right) ds \\ &+ (\tau_{1} - \tau_{2}) \left[ \int_{0}^{1} \frac{(1 - s)^{\alpha - 1}}{\Gamma(\alpha)} f\left(s, u(s), (Tu)(s), (Su)(s)\right) ds \right. \\ &+ \frac{b}{a} \int_{0}^{1} \frac{(1 - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} f\left(s, u(s), (Tu)(s), (Su)(s)\right) ds \right] \\ &+ (\tau_{2} - \tau_{1}) \sum_{j=1}^{p} I_{j}^{*}(u(t_{j})) \left( \frac{b}{a} - t_{j} \right) + (\tau_{2} - \tau_{1}) \sum_{j=1}^{p} I_{j}(u(t_{j})) + (\tau_{1} - \tau_{2}) \sum_{j=k+1}^{p} I_{j}^{*}(u(t_{j})) \right. \\ &+ \frac{(\tau_{1} - \tau_{2})}{a} (\gamma_{2}(u) - \gamma_{1}(u)) \right\| \\ &\leq f_{\max} \left\{ \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{b}{a\Gamma(\alpha)} \right) |\tau_{1} - \tau_{2}| + \frac{2}{\Gamma(\alpha + 1)} (\tau_{2} - \tau_{1})^{\alpha} \right\} \\ &+ p \left( \left( \frac{b}{a} + 1 \right) \rho + \mu \right) |\tau_{2} - \tau_{1}| + \frac{|\tau_{1} - \tau_{2}|}{a} (\mu + \rho). \end{split}$$

In conclusion ,  $\omega(Qu,\varepsilon) \to 0$  as  $\varepsilon \to 0$ , which implies that  $Q(B_{r_0})$  is equicontinuous.

Linking these statements with the above estimate and formula (2.1), we deduce the following inequality:

$$\omega_0(QX) \le \left\{ \frac{(b+a)}{a\Gamma(\alpha+1)} + \frac{b}{a^2\Gamma(\alpha)} \right\} \varphi(\omega_0(X)).$$

Also, for fixed  $t \in J$  and  $u, v \in B_{r_0}$ , from (3.18) we obtain

$$\begin{split} \|(Qu)(t) - (Qv)(t)\| \\ &\leq \left\{ \frac{p}{a^2} (a(b+a) + (a^2 + ab + b^2)) + \frac{b+a}{a^2} \right\} \varphi_{\max} (\|u-v\|) \\ &+ \left\{ \frac{(b+a)}{a\Gamma(\alpha+1)} + \frac{b}{a^2\Gamma(\alpha)} \right\} \left\{ \varphi_{\max} (\|u-v\|) + \Phi_1 (\delta(t,u,v)) + \Phi_2 (\gamma(t,u,v)) \right\}, \end{split}$$

where  $\varphi_{\max}(\|u - v\|) := \max\{\varphi(\|u - v\|), \varphi_i(\|u - v\|), i = 1, 2, 3, 4\}$ . By using condition (2.1) and  $t \to \infty$ , we deduce that

$$\lim_{t \to \infty} \sup \operatorname{diam}(QX)(t) \le \left[ \frac{p}{a^2} \left( a(b+a) + \left( a^2 + ab + b^2 \right) \right) + \frac{b+a}{a^2} + \left\{ \frac{(b+a)}{a\Gamma(\alpha+1)} + \frac{b}{a^2\Gamma(\alpha)} \right\} \right] \lim_{t \to \infty} \sup \operatorname{diam} \varphi_{\max}(X(t)).$$

Consequently, by considering  $\mu$  defined by (2.1), we have

$$\mu(QX) \leq \left[\frac{p}{a^2}\left(a(b+a) + \left(a^2 + ab + b^2\right)\right) + \frac{b+a}{a^2} + \frac{(b+a)}{a\Gamma(\alpha+1)} + \frac{b}{a^2\Gamma(\alpha)}\right]\varphi(\mu(X)),$$

where  $\phi(s) = \left[\frac{p}{a^2}(a(b+a) + (a^2 + ab + b^2)) + \frac{b+a}{a^2} + \frac{(b+a)}{a\Gamma(\alpha+1)} + \frac{b}{a^2\Gamma(\alpha)}\right]\varphi(s) \in \Psi$ . Combining the above estimate with all properties of the operator *Q*, by Lemma 2.4, we complete the proof.

**Theorem 3.2** Let  $\xi : E \times E \to \mathbb{R}^+$  be a given function. Assume that the following conditions *hold*:

(A) There exists  $\psi \in \Psi$  such that

$$\begin{split} \left\| f\left(s, u(s), (Tu)(s), (Su)(s)\right) - f\left(s, v(s), (Tv)(s), (Sv)(s)\right) \right\| \\ &\leq \frac{a^2 \Gamma(\alpha) \Gamma(\alpha + 1)}{4[a \Gamma(\alpha)(a + b) + b \Gamma(\alpha + 1)]} \psi \left( \|u - v\| \right), \\ \left\| \gamma_i(u) - \gamma_i(v) \right\| &\leq \frac{a^2}{8(a + b)} \psi \left( \|u - v\| \right), \\ \left\| I_k\left(u(t_k^-)\right) - I_k\left(v(t_k^-)\right) \right\| &\leq \frac{a^2}{4p(a(b + a))} \psi \left( \|u - v\| \right), \\ \left\| I_k^*\left(u(t_k^-)\right) - I_k^*\left(v(t_k^-)\right) \right\| &\leq \frac{a^2}{4p(a^2 + ab + b^2)} \psi \left( \|u - v\| \right). \end{split}$$

for all  $t \in J$  and for all  $u, v \in E$  with  $\xi(u, v) \ge 0$ , i = 1, 2, k = 1, 2, ..., p.

(B) There exists  $u_0 \in PC(J, E)$  such that  $\xi(u_0(t), Qu_0(t)) \ge 0$  for all  $t \in J$ , where a mapping  $Q : PC(J, E) \rightarrow PC(J, E)$  is defined by

$$\begin{split} (Qu)(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, u(s), (Tu)(s), (Su)(s)\right) ds \\ &+ \left(\frac{b}{a} - t\right) \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, u(s), (Tu)(s), (Su)(s)\right) ds \\ &+ \frac{b}{a\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f\left(s, u(s), (Tu)(s), (Su)(s)\right) ds \right] \\ &+ \left(\frac{b}{a} + 1 - t\right) \sum_{j=1}^p I_j^* \left(u(t_j)\right) \left(\frac{b}{a} - t_j\right) + \left(\frac{b}{a} + 1 - t\right) \sum_{j=1}^p I_j(u(t_j)) \\ &- (t-t_j) \sum_{j=k+1}^p I_j^* \left(u(t_j)\right) - \sum_{j=k+1}^p I_j(u(t_j)) \\ &+ \frac{1}{a^2} \left[ \left(a(1-t) + b\right) \gamma_1(u) + (at-b) \gamma_2(u) \right], \quad t \in (t_k, t_{k+1}], k = 1, 2, \dots, p-1. \end{split}$$

(C) For each  $t \in J$ , and  $u, v \in PC(J, E)$ ,  $\xi(u(t), v(t)) \ge 0$  implies that  $\xi(Qu(t), Qv(t)) \ge 0$ .

(D) For each  $t \in J$ , if  $\{u_n\}$  is a sequence in PC(J, E) such that  $u_n \to u$  in PC(J, E) and  $\xi(u_n(t), u_{n+1}(t)) \ge 0$  for all  $n \in \mathbb{N}$ , then

$$\xi(u_n(t),u(t))\geq 0$$

for all  $n \in \mathbb{N}$ . Then problem (1.1) has at least one solution.

*Proof* First of all, let E = PC(J, E). By Lemma 3.1, it is easy to see that  $u \in E$  is a solution of (1.1) given by (3.1), then problem (1.1) is equivalent to finding  $u^* \in E$  which is a fixed point of Q.

Now, let  $u, v \in E$  such that  $\xi(u(t), v(t)) \ge 0$  for all  $t \in J$ . By condition (A), we have

$$\begin{split} \|Qu(t) - Qv(t)\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \|f(t,u(t),(Tu)(t),(Su)(t)) - f(t,v(t),(Tv)(t),(Sv)(t))\| \, ds \\ &+ \left|\frac{b}{a} - t\right| \left[\int_{0}^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(t,u(t),(Tu)(t),(Su)(t)) - f(t,v(t),(Tv)(t),(Sv)(t))\| \, ds \right] \\ &+ \frac{b}{a} \int_{0}^{1} \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(t,u(t),(Tu)(t),(Su)(t)) - f(t,v(t),(Tv)(t),(Sv)(t))\| \, ds \right] \\ &+ \left|\frac{b}{a} + 1 - t_{k}\right| \sum_{j=1}^{p} \|I_{j}^{*}(u(t_{j})) - I_{j}^{*}(v(t_{j}))\| \cdot \left|\frac{b}{a} - t_{j}\right| \\ &+ \left|\frac{b}{a} + 1 - t_{k}\right| \sum_{j=1}^{p} \|I_{j}(u(t_{j})) - I_{j}(v(t_{j}))\| - |t - t_{j}| + \sum_{j=k+1}^{p} \|I_{j}^{*}(u(t_{j})) - I_{j}^{*}(v(t_{j}))\| \\ &+ \sum_{j=k+1}^{p} \|I_{j}(u(t_{j})) - I_{j}(v(t_{j}))\| + \frac{1}{a^{2}} [(a(1-t)+b)\|\gamma_{1}(u) - \gamma_{1}(v)\| \\ &+ (at-b)\|\gamma_{2}(u) - \gamma_{2}(v)\|] \\ &\leq \psi (\|u - v\|). \end{split}$$

This implies that for each  $u, v \in E$  with  $\xi(u(t), v(t)) \ge 0$  for all  $t \in J$ , we obtain that

$$\left\|Qu(t) - Qv(t)\right\| \le \psi\left(\|u - v\|\right) \tag{3.20}$$

for all  $u, v \in E$ . Now, we define the function  $\gamma : E \times E \rightarrow [0, \infty)$  by

$$\gamma(u, v) = \begin{cases} 1 & \text{if } \xi(u(t), v(t)) \ge 0 \text{ for all } t \in J, \\ 0 & \text{otherwise,} \end{cases}$$

and also we define the *w*-distance *p* on *E* by p(u, v) = ||u - v||. From (3.20), we have

$$\gamma(u,v)p(Qu,Qv) \leq \psi(p(u,v))$$

for all  $u, v \in E$ . This implies that Q is a  $(\gamma, \psi, p)$ -contractive mapping. From condition (B), there exists  $u_0 \in E$  such that  $\gamma(u_0, Qu_0) \ge 1$ . Next, by using condition (C), the following assertions hold for all  $u, v \in E$ :

$$\begin{aligned} \gamma(u,v) \ge 1 \quad \Rightarrow \quad \xi(u(t),v(t)) \ge 0 \\ \Rightarrow \quad \xi(Qu(t),Qv(t)) \ge 0 \\ \Rightarrow \quad \gamma(Qu,Qv) \ge 1 \end{aligned}$$

and hence Q is a  $\gamma$ -admissible mapping. Finally, from condition (D) we get that condition (iii) of Lemma 2.5 holds. Therefore, by Lemma 2.5, we find  $x^* \in E$  such that  $x^* = Qx^*$ , and so  $x^*$  is a solution of problem (1.1), which completes the proof.

# 4 An example

In this section we give an example to illustrate the usefulness of our results.

**Example 4.1** We consider the following impulsive fractional differential equation:

$$\begin{cases} {}^{c}D_{0^{+}}^{\frac{3}{2}}u(t) = \frac{t^{2}}{2+2t^{4}}\ln(1+|u(t)|) + \int_{0}^{t}\frac{se^{-t}\cos u(t)}{1+|\sin u(t)|} \, ds \\ + \int_{0}^{t}\frac{s|\cos u(t)|+e^{s}(1+\sin^{2}u(t))}{e^{t}(1+\sin u(t))} \, ds, \end{cases}$$

$$\Delta u(\frac{1}{4}) = \frac{|u(\frac{1}{4})|}{15+|u(\frac{1}{4})|}, \qquad \Delta u'(\frac{1}{4}) = \frac{|u(\frac{1}{4})|}{10+|u(\frac{1}{4})|},$$

$$3u(0) + u'(0) = \sum_{i=1}^{m} \eta_{i}u(\frac{1}{2}), \qquad 3u(0) + u'(0) = \sum_{i=1}^{m} \widetilde{\eta_{i}}\widetilde{u}(\frac{1}{2}),$$

$$(4.1)$$

where  $0 < \eta_1 < \eta_2 < \cdots < 1$ ,  $0 < \widetilde{\eta_1} < \widetilde{\eta_2} < \cdots < 1$ , and  $\eta_i$ ,  $\widetilde{\eta_i}$  are given positive constants with  $\sum_{i=1}^{m} \eta_i < \frac{2}{15}$  and  $\sum_{j=1}^{m} \widetilde{\eta_j} < \frac{3}{15}$ . Take J := [0,1] and  $\alpha = \frac{3}{2}$ , p = 1, a = 3, b = 1.

Let

$$\begin{aligned} Tu &= \int_0^t \frac{se^{-t}\cos u(t)}{1+|\sin u(t)|} \, ds, \qquad Su = \int_0^t \frac{s|\cos u(t)| + e^s(1+\sin^2 u(t))}{e^t(1+\sin u(t))} \, ds, \\ f(t,u,Tu,Su) &= \frac{t^2}{2+2t^4} \ln(1+|u(t)|) + Fu + Gu, \\ I_k(u) &= \frac{|u(\frac{1}{4})|}{15+|u(\frac{1}{4})|}, \qquad I_k^*(u) = \frac{|u(\frac{1}{4})|}{10+|u(\frac{1}{4})|}, \\ \gamma_1(u) &= \sum_{i=1}^m \eta_i u\left(\frac{1}{2}\right), \qquad \gamma_2(u) = \sum_{i=1}^m \widetilde{\eta_i} \widetilde{u}\left(\frac{1}{2}\right), \end{aligned}$$

then the impulsive fractional differential equation (4.1) can be transformed into the abstract form of problem (1.1). We show that all the conditions of Theorem 3.1 are satisfied for problem (4.1). Next, let  $u, v \in PC(J, E)$ , we calculate

$$\begin{aligned} \left\| f(t, u, Tu, Su) - f(t, v, Tv, Sv) \right\| \\ &= \left\| \frac{t^2}{2 + 2t^4} \left( \ln(1 + |u(t)|) - \ln(1 + |v(t)|) \right) \\ &+ (Tu - Tv) + (Su - Sv) \right\| \\ &\leq \frac{1}{4} \left\| \ln\left( 1 + \frac{|u| - |v|}{1 + |u|} \right) \right\| + \Phi_1 (\|Tu - Tv\|) + \Phi_2 (\|Su - Sv\|) \\ &\leq \frac{1}{2} \left[ \frac{1}{2} \ln(1 + |u - v|) \right] + \Phi_1 (\|Tu - Tv\|) + \Phi_2 (\|Su - Sv\|) \\ &\leq \frac{1}{2} \ln\left( \frac{2 + |u - v|}{2} \right) + \Phi_1 (\|Tu - Tv\|) + \Phi_2 (\|Su - Sv\|) \end{aligned}$$

$$\leq \frac{1}{2} \ln \left( 1 + \frac{|u - v|}{2} \right) + \Phi_1 \left( ||Tu - Tv|| \right) + \Phi_2 \left( ||Su - Sv|| \right)$$
  
$$\leq \frac{1}{2} \ln \left( 1 + |u - v| \right) + \Phi_1 \left( ||Tu - Tv|| \right) + \Phi_2 \left( ||Su - Sv|| \right)$$
  
$$= \varphi \left( ||u - v| \right) + \Phi_1 \left( ||Tu - Tv|| \right) + \Phi_2 \left( ||Su - Sv|| \right).$$

Obviously, the function  $\varphi(t) = \frac{1}{2} \ln(1 + t)$  is nondecreasing on *J* and  $\varphi(t) < t$  for all t > 0,  $\Phi_1(t) = \Phi_2(t) = t$ , hence condition (H1) holds.

On the other hand, the fact that  $M_1 = \sup\{|f(t, 0, 0, 0)|, t \in \mathbb{R}_+\} = 1$  shows condition (H2) is valid. Moreover, we have

$$\begin{split} \left\| I_{k}(u) - I_{k}(v) \right\| &= \left\| \frac{|u(\frac{1}{4})|}{15 + |u(\frac{1}{4})|} - \frac{|v(\frac{1}{4})|}{15 + |v(\frac{1}{4})|} \right\| \leq \frac{1}{15} \|u - v\|, \\ \left\| I_{k}^{*}(u) - I_{k}^{*}(v) \right\| &= \left\| \frac{|u(\frac{1}{4})|}{10 + |u(\frac{1}{4})|} - \frac{|v(\frac{1}{4})|}{10 + |v(\frac{1}{4})|} \right\| \leq \frac{1}{10} \|u - v\|, \\ \left\| \gamma_{1}(u) - \gamma_{1}(v) \right\| &= \left\| \sum_{i=1}^{m} \eta_{i}u\left(\frac{1}{2}\right) - \sum_{i=1}^{m} \eta_{i}v\left(\frac{1}{2}\right) \right\| \leq \frac{2}{15} \|u - v\|, \\ \left\| \gamma_{2}(u) - \gamma_{2}(v) \right\| &= \left\| \sum_{i=1}^{m} \widetilde{\eta}_{i}\widetilde{u}\left(\frac{1}{2}\right) - \sum_{i=1}^{m} \widetilde{\eta}_{i}\widetilde{v}\left(\frac{1}{2}\right) \right\| \leq \frac{3}{15} \|u - v\|, \\ \left\| I_{k}(u) \right\| &= \left\| \frac{|u(\frac{1}{4})|}{15 + |u(\frac{1}{4})|} \right\| \leq \frac{1}{15} \|u\|, \qquad \left\| I_{k}^{*}(u) \right\| = \left\| \frac{|u(\frac{1}{4})|}{10 + |u(\frac{1}{4})|} \right\| \leq \frac{1}{10} \|u\|, \\ \left\| \gamma_{1}(u) \right\| &= \left\| \sum_{i=1}^{m} \eta_{i}u \right\| \leq \frac{2}{15} \|u\|, \qquad \left\| \gamma_{2}(u) \right\| = \left\| \sum_{i=1}^{m} \widetilde{\eta}_{i}\widetilde{u}\left(x, \frac{1}{2}\right) \right\| \leq \frac{3}{15} \|u\|, \\ \left| k(t,s)[p_{1}(s,u) - p_{1}(s,v)] \right| \leq \frac{s}{e^{t}}, \\ \left| h(t,s)[p_{2}(s,u) - p_{2}(s,v)] \right| \leq \frac{s}{2e^{t}}, \\ \lim_{t \to \infty} \int_{0}^{t} \left| k(t,s)[p_{1}(s,u(s)) - p_{2}(s,v(s))] \right| ds \leq \lim_{t \to \infty} \frac{s}{e^{t}} = \lim_{t \to \infty} \frac{t^{2}}{2e^{t}} = 0 \end{split}$$

and

$$\lim_{t\to\infty}\int_0^t \left|h(t,s)\left[p_2(s,u(s))-p_2(s,v(s))\right]\right|ds \le \lim_{t\to\infty}\frac{s}{2e^t}=\lim_{t\to\infty}\frac{t^2}{4e^t}=0$$

for any  $u, v \in PC(J, E)$ . Also, we have

$$\int_{0}^{t} k(t,s)p_{1}(s,u(s)) ds \leq \int_{0}^{t} \left| k(t,s)p_{1}(s,u(s)) \right| ds \leq \int_{0}^{t} \frac{s}{e^{t}} ds = \frac{t^{2}}{2e^{t}},$$
(4.2)

$$\left| \int_{0}^{t} h(t,s) p_{2}(s,u(s)) \, ds \right| \leq \int_{0}^{t} \left| h(t,s) p_{2}(s,u(s)) \right| \, ds \leq \int_{0}^{t} \frac{s+e^{s}}{2e^{t}} \, ds = \frac{t^{2}+4e^{t}-4}{4e^{t}}.$$
(4.3)

Using (4.2), (4.3), we compute

$$D_{1} = \sup \left\{ \left| \int_{0}^{t} k(t,s)p_{1}(s,u(s)) ds \right| : t,s \in J, u \in C(J,E) \right\}$$

$$\leq \sup \left\{ \frac{t^{2}}{2e^{t}} : t > 0 \right\}$$

$$= 2e^{-2},$$

$$D_{2} = \sup \left\{ \left| \int_{0}^{a} h(t,s)p_{2}(s,u(s)) ds \right| : t,s \in J, u \in C(J,E) \right\}$$

$$\leq \sup \left\{ \frac{t^{2} + 4e^{t} - 4}{4e^{t}} : t > 0 \right\}$$

$$= \frac{4 + 4e^{2} - 4}{4e^{2}}$$

$$= 1.$$

Finally, let us consider the first inequality in assumption (H5). On the basis of the above calculations, we see that each number  $r \ge 4$  satisfies the inequality in condition (H5), i.e.,

$$\left\{\frac{(b+a)}{a\Gamma(\alpha+1)} + \frac{b}{a^2\Gamma(\alpha)}\right\} \left(\varphi(||u||) + \Phi_1(D_1) + \Phi_2(D_2) + M_1\right) \\ + \frac{p}{a^2} \left(a(b+a)\mu + \left(a^2 + ab + b^2\right)\rho\right) + \frac{a+b}{a^2}(N_1 + N_2) \le r.$$

Thus, as the number  $r_0$ , we can take  $r_0 = 4$ . Consequently, all the conditions of Theorem 3.1 are satisfied. Thus, problem (4.1) has at least one solution belonging to the ball  $B_{r_0}$  in the space PC(J, E).

# **5** Conclusions

The aim of this paper is to discuss the existence of solutions for a class of mixed boundary value problems of impulsive integrodifferential equations of fractional order  $\alpha \in (1, 2]$ . Our results improve and generalize some known results.

#### Acknowledgements

The authors wish to thank the referees for their valuable comments. This work is supported by the National Natural Science Foundation of China (11061031).

#### **Competing interests**

The authors declare that they have no competing interests.

#### Authors' contributions

All authors read and approved the final manuscript.

#### Publisher's Note

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#### Received: 23 March 2017 Accepted: 13 June 2017 Published online: 01 August 2017

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