Ryoo and Agarwal *Advances in Difference Equations* (2017) 2017:213
DOI 10.1186/s13662-017-1275-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Some identities involving q-poly-tangent numbers and polynomials and distribution of their zeros

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Abstract

In this paper we introduce the q-poly-tangent polynomials and numbers. We also give some properties, explicit formulas, several identities, a connection with poly-tangent numbers and polynomials, and some integral formulas. Finally, we investigate the zeros of the q-poly-tangent polynomials by using a computer.

MSC: 11B68; 11S40; 11S80

Keywords: tangent numbers and polynomials; q-poly-tangent numbers and polynomials; Cauchy numbers; Stirling numbers; Frobenius-Euler polynomials; q-poly-Euler polynomials

1 Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials, poly-Bernoulli numbers and polynomials, poly-Euler numbers and polynomials, and poly-tangent numbers and polynomials (see [1–11]). In this paper, we define q-poly-tangent polynomials and numbers and study some properties of the q-poly-tangent polynomials and numbers. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations (see [11])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{1.1}$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function,

$$\left(e^t - 1\right)^m = m! \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!}. \tag{1.2}$$

We also have

$$m!\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \big(\log(1+t)\big)^m. \tag{1.3}$$

We also need the binomial theorem: for a variable x,

$$\frac{1}{(1-t)^c} = \sum_{n=0}^{\infty}\binom{c+n-1}{n}t^n. \tag{1.4}$$

For $0 \le q < 1$, the q-poly-Bernoulli numbers $B_n^{(k)}$ were introduced by Mansour [6] by using the following generating function:

$$\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}\frac{t^n}{n!} \quad (k \in \mathbb{Z}), \tag{1.5}$$

where

$$\mathrm{Li}_{k,q}(t) = \sum_{n=1}^{\infty}\frac{t^n}{[n]_q^k} \tag{1.6}$$

is the kth q-poly-logarithm function, and $[n]_q = \frac{1-q^n}{1-q}$ is the q-integer (*cf.* [6]).

The q-poly-Euler polynomials $E_{n,q}^{(k)}(x)$ are defined by the generating function

$$\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(k)}(x)\frac{t^n}{n!} \quad (k \in \mathbb{Z}). \tag{1.7}$$

The familiar tangent polynomials $\mathbf{T}_n(x)$ are defined by the generating function [7–9]

$$\left(\frac{2}{e^{2t}+1}\right)e^{xt} = \sum_{n=0}^{\infty}\mathbf{T}_n(x)\frac{t^n}{n!} \quad \big(|2t| < \pi\big). \tag{1.8}$$

When $x = 0$, $\mathbf{T}_n(0) = \mathbf{T}_n$ are called the tangent numbers. The tangent polynomials $\mathbf{T}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{2}{e^{2t}+1}\right)^r e^{xt} = \sum_{n=0}^{\infty}\mathbf{T}_n^{(r)}(x)\frac{t^n}{n!} \quad \big(|2t| < \pi\big). \tag{1.9}$$

It is clear that for $r = 1$ we recover the tangent polynomials $\mathbf{T}_n(x)$.

The Bernoulli polynomials $\mathbf{B}_n^{(r)}(x)$ of order r are defined by the following generating function:

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x)\frac{t^n}{n!} \quad \big(|t| < 2\pi\big). \tag{1.10}$$

The Frobenius-Euler polynomials of order r, denoted by $\mathbf{H}_n^{(r)}(u,x)$, are defined as

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \sum_{n=0}^{\infty}\mathbf{H}_n^{(r)}(u,x)\frac{t^n}{n!}. \tag{1.11}$$

The values at $x = 0$ are called Frobenius-Euler numbers of order r; when $r = 1$, the polynomials or numbers are called ordinary Frobenius-Euler polynomials or numbers.

The poly-tangent polynomials $T_{n,q}^{(k)}(x)$ are defined by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} T_n^{(k)}(x)\frac{t^n}{n!} \quad (k \in \mathbb{Z}), \tag{1.12}$$

where $\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}$ is the kth poly-logarithm function (see [9]).

Many kinds of generalizations of these polynomials and numbers have been presented in the literature (see [1–11]). In the following section, we introduce the q-poly-tangent polynomials and numbers. After that we will investigate some their properties. We also give some relationships both between these polynomials and tangent polynomials and between these polynomials and q-cauchy numbers. Finally, we investigate the zeros of the q-poly-tangent polynomials by using a computer.

2 q-Poly-tangent numbers and polynomials

In this section, we define q-poly-tangent numbers and polynomials and provide some of their relevant properties.

For $0 \le q < 1$, the q-poly-tangent polynomials $T_{n,q}^{(k)}(x)$ are defined by the generating function:

$$\frac{2\,\mathrm{Li}_{k,q}(1-e^{-t})}{e^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x)\frac{t^n}{n!} \quad (k \in \mathbb{Z}). \tag{2.1}$$

When $x = 0$, $T_{n,q}^{(k)}(0) = T_{n,q}^{(k)}(x)$ are called the q-poly-tangent numbers. Observe that $\lim_{q\to 1} T_{n,q}^{(k)}(x) = T_n^{(k)}(x)$. By (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x)\frac{t^n}{n!} &= \left(\frac{2\,\mathrm{Li}_{k,q}(1-e^{-t})}{e^{2t}+1}\right)e^{xt} \\ &= \sum_{n=0}^{\infty} T_{n,q}^{(k)}\frac{t^n}{n!}\sum_{n=0}^{\infty} x^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T_{l,q}^{(k)}x^{n-l}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

By comparing the coefficients on both sides of (2.2), we have the following theorem.

Theorem 2.1 *For $n \in \mathbb{Z}_+$, we have*

$$T_{n,q}^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}T_{l,q}^{(k)}x^{n-l}.$$

The following elementary properties of the q-poly-tangent numbers $T_{n,q}^{(k)}$ and polynomials $T_{n,q}^{(k)}(x)$ are readily derived form (2.1). We, therefore, choose to omit the details involved.

Theorem 2.2 *For $k \in \mathbb{Z}$, we have*

$$(1) \quad T_{n,q}^{(k)}(x+y) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(k)}(x) y^{n-l},$$

$$(2) \quad T_{n,q}^{(k)}(2-x) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} T_{n-l,q}^{(k)}(2) x^l.$$

Theorem 2.3 *For any positive integer n, we have*

$$\begin{aligned}
&(1) \quad T_{n,q}^{(k)}(mx) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(k)}(x)(m-1)^{n-l} x^{n-l} \quad (m \geq 1), \\
&(2) \quad T_{n,q}^{(k)}(x+1) - T_{n,q}^{(k)}(x) = \sum_{l=0}^{n-1} \binom{n}{l} T_{l,q}^{(k)}(x), \\
&(3) \quad \frac{d}{dx} T_{n,q}^{(k)}(x) = n T_{n-1,q}^{(k)}(x), \\
&(4) \quad T_{n,q}^{(k)}(x) = T_{n,q}^{(k)} + n \int_0^x T_{n-1,q}^{(k)}(t)\, dt.
\end{aligned} \tag{2.3}$$

From (1.6), (1.8), and (2.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x) \frac{t^n}{n!} &= \left(2 \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e^{2t}+1} \right) e^{xt} = \sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{[l+1]_q^k} \frac{2e^{xt}}{e^{2t}+1} \\
&= \sum_{l=0}^{\infty} \frac{1}{[l+1]_q^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i \frac{2e^{(x-i)t}}{e^{2t}+1} \\
&= \sum_{l=0}^{\infty} \frac{1}{[l+1]_q^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i \sum_{n=0}^{\infty} \mathbf{T}_n(x-i) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{1}{[l+1]_q^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i \mathbf{T}_n(x-i) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.4}$$

By comparing the coefficients on both sides of (2.4), we have the following theorem.

Theorem 2.4 *For $n \in \mathbb{Z}_+$, we have*

$$T_{n,q}^{(k)}(x) = \sum_{l=0}^{\infty} \frac{1}{[l+1]_q^k} \sum_{j=0}^{l+1} \binom{l+1}{j} (-1)^j \mathbf{T}_n(x-j).$$

By using the definition of tangent polynomials and Theorem 2.4, we have the following corollary.

Corollary 2.5 *For any positive integer n, we have*

$$T_{n,q}^{(k)}(2-x) = (-1)^n \sum_{l=0}^{\infty} \frac{1}{[l+1]_q^k} \sum_{j=0}^{l+1} \binom{l+1}{j} (-1)^j \mathbf{T}_n(x+j).$$

By (2.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x)\frac{t^n}{n!} &= 2\sum_{l=0}^{\infty}(-1)^l e^{2lt}\sum_{l=0}^{\infty}\frac{(1-e^{-t})^{l+1}}{[l+1]_q^k}e^{xt} \\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\frac{2(-1)^{l+j-i}\binom{i+1}{j}}{[i+1]_q^k}e^{(2l-2i-j+x)t} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\frac{2(-1)^{l+j-i}\binom{i+1}{j}(2l-2i-j+x)^n}{[i+1]_q^k}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides, we have the following theorem.

Theorem 2.6 *For $n \in \mathbb{Z}_+$, we have*

$$T_{n,q}^{(k)}(x) = 2\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\frac{(-1)^{l+j-i}\binom{i+1}{j}}{[i+1]_q^k}(2l-2i-j+x)^n.$$

By (1.7), (1.8), and (2.1) and by using the Cauchy product, we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x)\frac{t^n}{n!} &= \frac{1}{2}\left(\frac{2\,\mathrm{Li}_{k,q}(1-e^{-t})}{e^t+1}\right)\frac{2(e^t+1)}{e^{2t}+1}e^{xt} \\ &= \left(\sum_{n=0}^{\infty} E_{n,q}^{(k)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\big(\mathbf{T}_n(1)+\mathbf{T}_n\big)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\big(\mathbf{T}_n(1)+\mathbf{T}_n\big)E_{n-l,q}^{(k)}(x)\right). \qquad (2.5)\end{aligned}$$

By comparing the coefficients on both sides of (2.5), we have the following theorem related the q-poly-Euler polynomials and tangent polynomials.

Theorem 2.7 *For $n \in \mathbb{Z}_+$, we have*

$$T_{n,q}^{(k)}(x) = \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\big(\mathbf{T}_n(1)+\mathbf{T}_n\big)E_{n-l,q}^{(k)}(x).$$

By (1.5), (1.8), and (2.1) and by using the Cauchy product, we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x)\frac{t^n}{n!} &= \left(\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{1-e^{-t}}\right)\frac{2(1-e^{-t})}{e^{2t}+1}e^{xt} \\ &= \left(\sum_{n=0}^{\infty} B_{n,q}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\big(\mathbf{T}_n(x)-\mathbf{T}_n(x-1)\big)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{n=0}^{n}\binom{n}{l}\big(\mathbf{T}_n(x)-\mathbf{T}_n(x-1)\big)B_{n-l,q}^{(k)}\right). \qquad (2.6)\end{aligned}$$

By comparing the coefficients on both sides of (2.6), we have the following theorem related the q-poly-Bernoulli polynomials and tangent polynomials.

Theorem 2.8 *For $n \in \mathbb{Z}_+$, we have*

$$T_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} \big(\mathbf{T}_n(x) - \mathbf{T}_n(x-1)\big) B_{n-l,q}^{(k)}.$$

By (1.2), (1.5), (1.8), and Theorem 2.8, we have the following corollary.

Corollary 2.9 *For $n \in \mathbb{Z}_+$, we have*

$$T_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{(-1)^{m+l} m! S_2(l,m)}{[m+1]_q^k} \big(\mathbf{T}_{n-l}(x) - \mathbf{T}_{n-l}(x-1)\big).$$

3 Some identities involving q-poly-tangent numbers and polynomials

In this section, we give several combinatorics identities involving q-poly-tangent numbers and polynomials in terms of Stirling numbers, falling factorial functions, raising factorial functions, Beta functions, Bernoulli polynomials of higher order, and Frobenius-Euler functions of higher order.

By (2.1) and by using the Cauchy product, we get

$$\begin{aligned} \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x) \frac{t^n}{n!} &= \left(\frac{2\,\mathrm{Li}_{k,q}(1-e^{-t})}{e^{2t}+1} \right) \big(1 - (1-e^{-t})\big)^{-x} \\ &= \frac{2\,\mathrm{Li}_{k,q}(1-e^{-t})}{e^{2t}+1} \sum_{l=0}^{\infty} \binom{x+l-1}{l} (1-e^{-t})^l \\ &= \sum_{l=0}^{\infty} \langle x \rangle_l \frac{(e^t-1)^l}{l!} \left(\frac{2\,\mathrm{Li}_{k,q}(1-e^{-t})}{e^{2t}+1} e^{-lt} \right) \\ &= \sum_{l=0}^{\infty} \langle x \rangle_l \sum_{n=0}^{\infty} S_2(n,l) \frac{t^n}{n!} \sum_{n=0}^{\infty} T_{n,q}^{(k)}(-l) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \sum_{i=l}^{n} \binom{n}{i} S_2(i,l) T_{n-i,q}^{(k)}(-l) \langle x \rangle_l \right) \frac{t^n}{n!}, \end{aligned} \tag{3.1}$$

where $\langle x \rangle_l = x(x+1)\cdots(x+l-1)$ $(l \geq 1)$ with $\langle x \rangle_0 = 1$.

By comparing the coefficients on both sides of (3.1), we have the following theorem.

Theorem 3.1 *For $n \in \mathbb{Z}_+$, we have*

$$T_{n,q}^{(k)}(x) = \sum_{l=0}^{\infty} \sum_{i=l}^{n} \binom{n}{i} \langle x \rangle_l S_2(i,l) T_{n-i,q}^{(k)}(-l).$$

By using the Jackson q-integral (see [1]) and Theorem 2.1, we get

$$\begin{aligned} \int_0^1 T_{n,q}^{(k)}(x)\, d_q x &= \int_0^1 \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(k)} x^{n-l}\, d_q x \\ &= \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(k)} \frac{1}{[n-l+1]_q}. \end{aligned} \tag{3.2}$$

By (3.2) and Theorem 3.1, we have the following theorem.

Theorem 3.2 *For any positive integer n, we have*

$$\sum_{l=0}^{n}\binom{n}{l}T_{l,q}^{(k)}\frac{1}{[n-l+1]_q}=\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i,q}^{(k)}(-l)(-1)^l\hat{c}_{l,q},$$

where $\hat{c}_{l,q}$ *are q-Cauchy numbers of the second kind* (*see* [5]).

By (2.1) and by using the Cauchy product, we get

$$\begin{aligned}\sum_{n=0}^{\infty}T_{n,q}^{(k)}(x)\frac{t^n}{n!}&=\left(\frac{2\operatorname{Li}_{k,q}(1-e^{-t})}{e^{2t}+1}\right)\bigl((e^t-1)+1\bigr)^x\\&=\frac{2\operatorname{Li}_{k,q}(1-e^{-t})}{e^{2t}+1}\sum_{l=0}^{\infty}\binom{x}{l}\bigl(e^t-1\bigr)^l\\&=\sum_{l=0}^{\infty}(x)_l\frac{(e^t-1)^l}{l!}\left(\frac{2\operatorname{Li}_{k,q}(1-e^{-t})}{e^{2t}+1}\right)\\&=\sum_{l=0}^{\infty}(x)_l\sum_{n=0}^{\infty}S_2(n,l)\frac{t^n}{n!}\sum_{n=0}^{\infty}T_{n,q}^{(k)}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_lS_2(i,l)T_{n-i,q}^{(k)}\right)\frac{t^n}{n!}.\end{aligned}\tag{3.3}$$

By comparing the coefficients on both sides of (3.3), we have the following theorem.

Theorem 3.3 *For* $n\in\mathbb{Z}_+$ *and* $0\le q<1$, *we have*

$$T_{n,q}^{(k)}(x)=\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_lS_2(i,l)T_{n-i,q}^{(k)}.$$

By (3.2) and Theorem 3.3, we have the following theorem.

Theorem 3.4 *For any positive integer n, we have*

$$\sum_{l=0}^{n}\binom{n}{l}\frac{T_{n-l,q}^{(k)}}{[l+1]_q}=\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i,q}^{(k)}c_{l,q},$$

where $c_{l,q}$ *are q-Cauchy numbers of the first kind* (*see* [5]).

By Theorem 2.2, we note that

$$\begin{aligned}\int_0^1 y^nT_{n,q}^{(k)}(x+y)\,dy&=\int_0^1 y^n\sum_{l=0}^{n}\binom{n}{l}T_{n-l,q}^{(k)}(x)y^l\,dy\\&=\sum_{l=0}^{n}\binom{n}{l}T_{n-l,q}^{(k)}(x)\int_0^1 y^{n+l}\,dy\\&=\sum_{l=0}^{n}\binom{n}{l}T_{n-l,q}^{(k)}(x)\frac{1}{n+l+1}.\end{aligned}\tag{3.4}$$

From (2.1) and Theorem 2.2, we note that

$$\begin{aligned}\int_0^1 y^n T_{n,q}^{(k)}(x+y)\,dy &= \frac{y^n T_{n+1,q}^{(k)}(x+y)}{n+1}\bigg|_0^1 - \int_0^1 ny^{n-1}\frac{T_{n+1,q}^{(k)}(x+y)}{n+1}\,dy \\ &= \frac{T_{n+1,q}^{(k)}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1}T_{n+1,q}^{(k)}(x+y)\,dy \\ &= \frac{T_{n+1,q}^{(k)}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1}\sum_{l=0}^{n+1}\binom{n+1}{l}T_{l,q}^{(k)}(x)y^{n+1-l}\,dy \\ &= \frac{T_{n+1,q}^{(k)}(x+1)}{n+1} - \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}T_{l,q}^{(k)}(x)\frac{1}{2n-l+1}. \end{aligned} \tag{3.5}$$

Therefore, by (3.4) and (3.5), we obtain the following theorem.

Theorem 3.5 *For $n \in \mathbb{Z}_+$, we have*

$$T_{n+1,q}^{(k)}(x+1) = \sum_{l=0}^{n+1}\binom{n+1}{l}T_{l,q}^{(k)}(x)\frac{n}{2n-l+1} + \sum_{l=0}^{n}\binom{n}{l}T_{n-l,q}^{(k)}(x)\frac{n+1}{n+l+1}.$$

By (1.2), (1.10), (2.1), and by using the Cauchy product, we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x)\frac{t^n}{n!} &= \left(\frac{2\operatorname{Li}_{k,q}(1-e^{-t})}{e^{2t}+1}\right)e^{xt} \\ &= \frac{(e^t-1)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{e^t-1}\right)^r e^{xt}\sum_{n=0}^{\infty}T_{n,q}^{(k)}\frac{t^n}{n!} \\ &= \frac{(e^t-1)^r}{r!}\left(\sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}T_{n,q}^{(k)}\frac{t^n}{n!}\right)\frac{r!}{t^r} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i}\mathbf{B}_i^{(r)}(x)T_{n-l-i,q}^{(k)}\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.6 *For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have*

$$T_{n,q}^{(k)}(x) = \sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i}T_{n-l-i,q}^{(k)}\mathbf{B}_i^{(r)}(x).$$

From (2.1) and Theorem 2.2, we note that

$$\begin{aligned}&\int_0^1 y^n T_{n,q}^{(k)}(x+y)\,dy \\ &\quad= \frac{y^n T_{n+1,q}^{(k)}(x+y)}{n+1}\bigg|_0^1 - \int_0^1 \frac{ny^{n-1}T_{n+1,q}^{(k)}(x+y)}{n+1}\,dy \\ &\quad= \frac{T_{n+1,q}^{(k)}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1\sum_{l=0}^{\infty}\frac{1}{[l+1]_q^k}\sum_{i=0}^{l+1}\binom{l+1}{i}(-1)^i\mathbf{T}_{n+1}(x+y-i)y^{n-1}\,dy\end{aligned}$$

$$\begin{aligned}
&= \frac{T^{(k)}_{n+1,q}(x+1)}{n+1} \\
&\quad - \frac{n}{n+1}\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{j=0}^{n+1}\frac{\binom{l+1}{i}\binom{n+1}{l}}{[l+1]_q^k}(-1)^{n+1+i}\mathbf{T}_{n+1-j}(1-x+i)\int_0^1 y^{n-1}(1-y)^j\,dy \\
&= \frac{T^{(k)}_{n+1,q}(x+1)}{n+1} \\
&\quad - \frac{n}{n+1}\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{j=0}^{n+1}\frac{\binom{l+1}{i}\binom{n+1}{l}}{[l+1]_q^k}(-1)^{n+1+i}\mathbf{T}_{n+1-j}(1-x+i)B(n,j+1),
\end{aligned} \tag{3.6}$$

where $B(n,j)$ is the beta integral (see [1]).

Therefore, by (3.5) and (3.6), we obtain the following theorem.

Theorem 3.7 *For $n \in \mathbb{Z}_+$, we have*

$$\sum_{l=0}^{n+1}\binom{n+1}{l}\frac{T^{(k)}_{l,q}(x)}{2n-l+1} = \sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{j=0}^{n+1}\frac{\binom{l+1}{i}\binom{n+1}{l}}{[l+1]_q^k}(-1)^{n+1+i}\mathbf{T}_{n+1-j}(1-x+i)B(n,j+1).$$

By (1.2), (1.11), (2.1), and by using the Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} T^{(k)}_{n,q}(x)\frac{t^n}{n!} &= \left(\frac{2\,\mathrm{Li}_{k,q}(1-e^{-t})}{e^{2t}+1}\right)e^{xt} \\
&= \frac{(e^t-u)^r}{(1-u)^r}\left(\frac{1-u}{e^t-u}\right)^r e^{xt}\frac{2\,\mathrm{Li}_{k,q}(1-e^{-t})}{e^{2t}+1} \\
&= \sum_{n=0}^{\infty}\mathbf{H}^{(r)}_n(u,x)\frac{t^n}{n!}\sum_{i=0}^{r}\binom{r}{i}e^{it}(-u)^{r-i}\frac{1}{(1-u)^r}\frac{2\,\mathrm{Li}_{k,q}(1-e^{-t})}{e^{2t}+1} \\
&= \frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{n=0}^{\infty}\mathbf{H}^{(r)}_n(u,x)\frac{t^n}{n!}\sum_{n=0}^{\infty}T^{(k)}_{n,q}(i)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{l=0}^{n}\binom{n}{l}\mathbf{H}^{(r)}_l(u,x)T^{(k)}_{n-l}(i)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.8 *For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have*

$$T^{(k)}_{n,q}(x) = \frac{1}{(1-u)^r}\sum_{i=0}^{r}\sum_{l=0}^{n}\binom{r}{i}\binom{n}{l}(-u)^{r-i}\mathbf{H}^{(r)}_l(u,x)T^{(k)}_{n-l,q}(i).$$

For $n \in \mathbb{N}$ with $n \geq 4$, we obtain

$$\begin{aligned}
\int_0^1 y^n T^{(k)}_{n,q}(x+y)\,dy &= y^{n+1}\frac{T^{(k)}_{n,q}(x+y)}{n+1}\bigg|_0^1 - \int_0^1 ny^{n+1}\frac{T^{(k)}_{n-1,q}(x+y)}{n+1}\,dy \\
&= \frac{T^{(k)}_{n,q}(x+1)}{n+1} - \frac{n,q}{n+1}\int_0^1 y^{n+1}T^{(k)}_{n-1,q}(x+y)\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{T_{n,q}^{(k)}(x+1)}{n+1} - \frac{nT_{n-1,q}^{(k)}(x+1)}{(n+1)(n+2)} \\
&\quad + (-1)^2 \frac{n}{n+1}\frac{n-1}{n+2}\int_0^1 y^{n+2} T_{n-2,q}^{(k)}(x+y)\,dy \\
&= \frac{T_{n,q}^{(k)}(x+1)}{n+1} + (-1)\frac{nT_{n-1,q}^{(k)}(x+1)}{(n+1)(n+2)} \\
&\quad + (-1)^2 \frac{n}{n+1}\frac{n-1}{n+2}\frac{T_{n-2,q}^{(k)}(x+1)}{n+3} \\
&\quad + (-1)^3 \frac{n}{n+1}\frac{n-1}{n+2}\frac{n-2}{n+3}\frac{T_{n-3,q}^{(k)}(x+1)}{n+4} \\
&\quad + (-1)^4 \frac{n}{n+1}\frac{n-1}{n+2}\frac{n-2}{n+3}\frac{n-3}{n+4}\int_0^1 y^{n+4} T_{n-4,q}^{(k)}(x+y)\,dy.
\end{aligned}
$$

Continuing this process, we obtain

$$
\begin{aligned}
\int_0^1 y^n T_{n,q}^{(k)}(x+y)\,dy &= \frac{T_{n,q}(x+1)}{n+1} \\
&\quad + \sum_{l=2}^{n} \frac{n(n-1)\cdots(n-l+2)(-1)^{l-1}}{(n+1)(n+2)\cdots(n+l)} T_{n-l+1,q}^{(k)}(x+1) \\
&\quad + (-1)^n \frac{n!}{(n+1)(n+2)\cdots(2n)} \int_0^1 y^{2n} T_{0,q}^{(k)}(x+y)\,dy.
\end{aligned} \tag{3.7}
$$

Hence, by (3.4) and (3.7), we have the following theorem.

Theorem 3.9 *For $n \in \mathbb{N}$ with $n \geq 2$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n} \binom{n}{l} T_{n-l,q}^{(k)}(x) \frac{1}{n+l+1} \\
&= \frac{T_{n,q}^{(k)}(x+1)}{n+1} + \sum_{l=2}^{n} \frac{n(n-1)\cdots(n-l+2)(-1)^{l-1}}{(n+1)(n+2)\cdots(n+l)} T_{n-l+1,q}^{(k)}(x+1).
\end{aligned}
$$

4 Zeros of the q-poly-tangent polynomials

This section aims to demonstrate the benefit of using a numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the poly-tangent polynomials $T_{n,q}^{(k)}(x)$. The q-poly-tangent polynomials $T_{n,q}^{(k)}(x)$ can be determined explicitly. A few of them are

$$
\begin{aligned}
&T_{0,q}^{(k)}(x) = 0, \\
&T_{1,q}^{(k)}(x) = 1, \\
&T_{2,q}^{(k)}(x) = -3 + \frac{2}{[2]_q^k} + 2x, \\
&T_{3,q}^{(k)}(x) = 4 - \frac{12}{[2]_q^k} + \frac{6}{[3]_q^k} - \left(9 - \frac{6}{[2]_q^k}\right)x + 3x^2,
\end{aligned}
$$

$$\begin{aligned}
T_{4,q}^{(k)}(x) &= 3 + \frac{38}{[2]_q^k} - \frac{60}{[3]_q^k} + \frac{24}{[4]_q^k} + \left(16 - \frac{48}{[2]_q^k} + \frac{24}{[3]_q^k}\right)x \\
&\quad - \left(18 - \frac{12}{[2]_q^k}\right)x^2 + 4x^3, \\
T_{5,q}^{(k)}(x) &= -14 - \frac{60}{[2]_q^k} + \frac{330}{[3]_q^k} - \frac{360}{[4]_q^k} + \frac{120}{[5]_q^k} + \left(15 + \frac{190}{[2]_q^k} - \frac{300}{[3]_q^k} + \frac{120}{[4]_q^k}\right)x \\
&\quad + \left(40 - \frac{120}{[2]_q^k} + \frac{60}{[3]_q^k}\right)x^2 - \left(30 + \frac{20}{[2]_q^k}\right)x^3 + 5x^4.
\end{aligned}$$

We investigate the beautiful zeros of the q-poly-tangent polynomials $T_{n,q}^{(k)}(x)$ by using a computer. We plot the zeros of the q-poly-tangent polynomials $T_{n,q}^{(k)}(x)$ for $n = 20$, $q = 1/2, -1/2$, $k = -3, 3$ and $x \in \mathbb{C}$ (Figure 1). In Figure 1 (top-left), we choose $n = 20$, $q = 1/2$, and $k = 3$. In Figure 1 (top-right), we choose $n = 20$, $q = -1/2$, and $k = 3$. In Figure 1 (bottom-left), we choose $n = 20$, $q = 1/2$, and $k = -3$. In Figure 1 (bottom-right), we choose $n = 20$, $q = -1/2$, and $k = -3$.

Stacks of zeros of $T_{n,q}^{(k)}(x)$ for $2 \leq n \leq 40$ from a 3-D structure are presented (Figure 2). In Figure 2, we choose $k = 3$, $q = 1/2$. Our numerical results for approximate solutions of real zeros of $T_{n,q}^{(k)}(x)$ are displayed (Tables 1, 2).

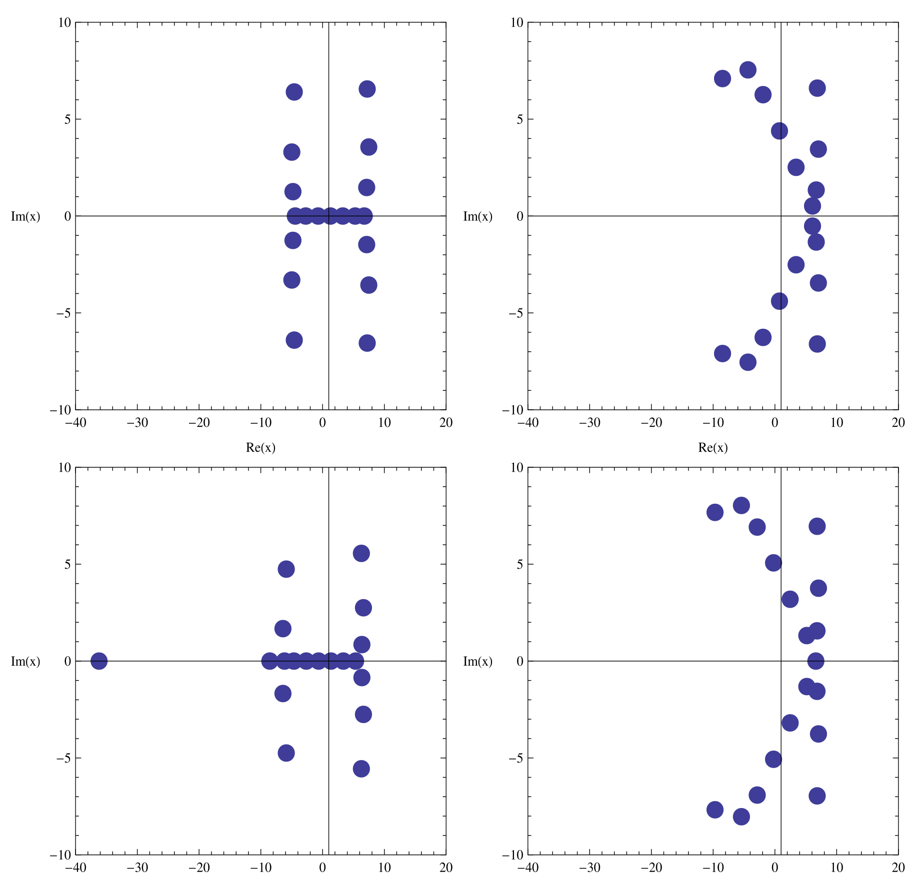

Figure 1 Zeros of $T_{n,q}^{(k)}(x)$.

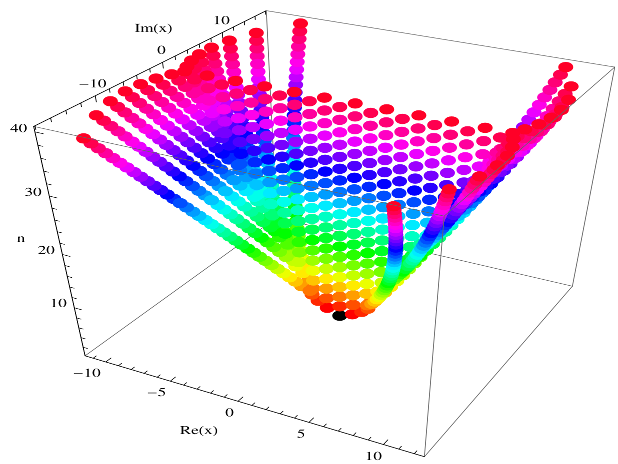

Figure 2 Stacks of zeros of $T^{(k)}_{n,q}(x)$ for $2 \le n \le 40$.

Table 1 Numbers of real and complex zeros of $T^{(k)}_{n,q}(x)$

Degree n	$k = 3, q = 1/2$		$k = -3, q = -1/2$	
	Real zeros	Complex zeros	Real zeros	Complex zeros
2	1	0	1	0
3	2	0	2	0
4	3	0	1	2
5	4	0	0	4
6	1	4	1	4
7	2	4	0	6
8	3	4	1	6
9	4	4	2	6
10	5	4	1	8
11	4	6	0	10
12	3	8	1	10

Table 2 Approximate solutions of $T^{(k)}_{n,q}(x) = 0$, $k = 3$, $q = 1/2$

Degree n	x
2	1.2037
3	0.24060, 2.1668
4	−0.47700, 1.2291, 2.8590
5	−0.96664, 0.23158, 2.2563, 3.2936
6	1.2308
7	0.23043, 2.2296

The plot of real zeros of $T^{(k)}_{n,q}(x)$ for the $2 \le n \le 40$ structure is presented (Figure 3). In Figure 3, we choose $k = 3$.

We observe a remarkable regular structure of the real roots of the q-poly-tangent polynomials $T^{(k)}_{n,q}(x)$. We also hope to verify a remarkable regular structure of the real roots of the q-poly-tangent polynomials $T^{(k)}_{n,q}(x)$ (Table 1).

Next, we calculated an approximate solution satisfying q-poly-tangent polynomials $T^{(k)}_{n,q}(x) = 0$ for $x \in \mathbb{R}$. The results are given in Table 2 and Table 3.

By numerical computations, we will present a series of conjectures.

Figure 3 Real zeros of $T_{n,q}^{(k)}(x)$ for $2 \le n \le 40$.

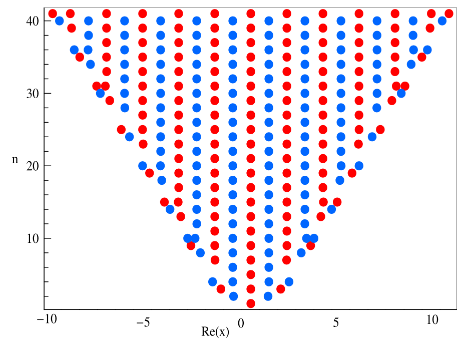

Table 3 Approximate solutions of $T_{n,q}^{(k)}(x) = 0$, $k = -3$, $q = -1/2$

Degree n	x
2	1.3750
3	0.91289, 1.8371
4	2.6383
5	-
6	1.8993
7	-

Conjecture 4.1 *Prove that* $T_{n,q}^{(k)}(x)$, $x \in \mathbb{C}$, *has* $\mathrm{Im}(x) = 0$ *reflection symmetry analytic complex functions. However,* $T_{n,q}^{(k)}(x)$ *has no* $Re(x) = a$ *reflection symmetry for* $a \in \mathbb{R}$.

Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n (see Figures 1, 2, 3).

We are able to decide if $T_{n,q}^{(k)}(x) = 0$ has $n-1$ distinct solutions (see Tables 1, 2, 3).

Conjecture 4.2 *Prove that* $T_{n,q}^{(k)}(x) = 0$ *has* $n-1$ *distinct solutions.*

Since $n-1$ is the degree of the polynomial $T_{n,q}^{(k)}(x)$, the number of real zeros $R_{T_{n,q}^{(k)}(x)}$ lying on the real plane $\mathrm{Im}(x) = 0$ is $R_{T_{n,q}^{(k)}(x)} = n - C_{T_{n,q}^{(k)}(x)}$, where $C_{T_{n,q}^{(k)}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{T_{n,q}^{(k)}(x)}$ and $C_{T_{n,q}^{(k)}(x)}$. The authors have no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the poly-tangent polynomials $T_{n,q}^{(k)}(x)$, which appear in mathematics and physics.

Acknowledgements
This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2017R1A2B4006092).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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Publisher's Note

Received: 8 June 2017 Accepted: 10 July 2017 Published online: 01 August 2017

References

1. Andrews, GE, Askey, R, Roy, R: Special Functions. Encyclopedia of Mathematics and Its Applications, vol. 71. Combridge University Press, Cambridge (1999)
2. Ayoub, R: Euler and zeta function. Am. Math. Mon. **81**, 1067-1086 (1974)
3. Comtet, L: Advances Combinatorics. Reidel, Dordrecht (1974)
4. Kim, D, Kim, T: Some identities involving Genocchi polynomials and numbers. Ars Comb. **121**, 403-412 (2015)
5. Kaneko, M: Poly-Bernoulli numbers. J. Théor. Nr. Bordx. **9**, 199-206 (1997)
6. Mansour, T: Identities for sums of a *q*-analogue of polylogarithm functions. Lett. Math. Phys. **87**, 1-18 (2009)
7. Ryoo, CS: A note on the tangent numbers and polynomials. Adv. Stud. Theor. Phys. **7**(9), 447-454 (2013)
8. Ryoo, CS: A numerical investigation on the zeros of the tangent polynomials. J. Appl. Math. Inform. **32**(3-4), 315-322 (2014)
9. Ryoo, CS: Differential equations associated with tangent numbers. J. Appl. Math. Inform. **34**(5-6), 487-494 (2016)
10. Shin, H, Zeng, J: The *q*-tangent and *q*-secant numbers via continued fractions. Eur. J. Comb. **31**, 1689-1705 (2010)
11. Young, PT: Degenerate Bernoulli polynomials, generalized factorial sums, and their applications. J. Number Theory **128**, 738-758 (2008)