Chen and Zhang *Advances in Difference Equations* (2017) 2017:228
DOI 10.1186/s13662-017-1277-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The hybrid power mean of the generalized Gauss sums

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Abstract

The main purpose of this paper is using the analytic method and the orthogonal properties of the character sums to study the computational problem of one kind hybrid power mean of the generalized Gauss sums mod p, an odd prime with $p \equiv 1 \bmod 3$, and give some exact computational formulas for this kind hybrid power mean.

MSC: 11L05

Keywords: generalized Gauss sums; hybrid power mean; analytic method; computational formula

1 Introduction

Let $q \geq 3$ be a positive integer, χ be any Dirichlet character mod q. For any positive integer $k \geq 1$ and integer m, the generalized Gauss sum $G(m,k,\chi;q)$ is defined as

$$G(m,k,\chi;q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma^k}{q}\right)$$

with $e(y) = e^{2\pi iy}$.

If $k = m = 1$, then $G(m,k,\chi;q)$ becomes the classical Gauss sums $\tau(\chi)$; see [1] for its definition and some basic properties.

About the various arithmetical properties of $G(m,k,\chi;q)$, many authors had studied it, and obtained a series of interesting results; see [2–9]. For example, from Weil's classical result [2] one can obtain the upper bound estimate

$$\left|G(m,k,\chi;p)\right| \leq (k+1)\sqrt{p}.$$

Zhang Wenpeng and Liu Huaning [3] studied the fourth power mean of the generalized third Gauss sums with $3|(p-1)$, and they obtained a complex but exact computational formula. That is, they proved that

$$\sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a^3}{p}\right)\right|^4 = 5p^3 - 18p^2 + 20p + 1 + \frac{U^5}{p} + 5pU - 5U^3 - 4U^2 + 4U,$$

where $U = \sum_{a=1}^{p} e(\frac{a^3}{p})$ is a real constant.

In this paper, we are concerned with the computational problem of the following hybrid power mean:

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{2k}\cdot\left|\sum_{b=1}^{p-1}\left(\frac{b}{p}\right)e\left(\frac{mb^3}{p}\right)\right|^{2h}, \tag{1}$$

where k and h are positive integers, ψ is any three-order character mod p and $(\frac{*}{p})$ denotes the Legendre symbol mod p.

About the hybrid power mean (1), it seems that no one had studied it, at least we have not seen such a related paper before. The characteristic of this hybrid power mean is symmetrical. That is, the two-order character corresponds to the third power, and the three-order character corresponds to the second power. And the interesting thing is that, for any positive integers k and h, one can give an exact computational formula for (1). The main purpose of this paper is to illustrate this point. That is, we shall use the analytic method and the orthogonal properties of the character sums to prove the following main results.

Theorem 1 *Let p be an odd prime with $p\equiv 1 \bmod 3$, k be any positive integer. For any three-order character $\psi \bmod p$, if $p\equiv 1 \bmod 12$, then we have*

$$\begin{aligned}&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{2k}\cdot\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^{2}\\&\quad=3p(p-1)\sum_{i=0}^{[\frac{k}{2}]}\binom{k}{2i}2^{k-2i}p^{k-i}\left(\sum_{a=0}^{p-1}\left(\frac{a^3-1}{p}\right)\right)^{2i}.\end{aligned}$$

If $p\equiv 7 \bmod 12$, then we have

$$\begin{aligned}&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{2k}\cdot\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^{2}\\&\quad=3p(p-1)\sum_{i=0}^{[\frac{k}{2}]}\binom{k}{2i}2^{k-2i}p^{k-i}\left|\sum_{a=1}^{p-1}\psi(a)\left(\frac{a^2-1}{p}\right)\right|^{2i}.\end{aligned}$$

Theorem 2 *Let p be an odd prime with $p\equiv 1 \bmod 3$. For any positive integer k, if $p\equiv 1 \bmod 12$, then we have the identity*

$$\begin{aligned}&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{2k}\cdot\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^{4}\\&\quad=p(p-1)\left(9p+2\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\psi\left(a^3-1\right)\right|^2\right)\\&\qquad\times\sum_{i=0}^{[\frac{k}{2}]}\binom{k}{2i}2^{k-2i}p^{k-i}\left(\sum_{a=0}^{p-1}\left(\frac{a^3-1}{p}\right)\right)^{2i}.\end{aligned}$$

If $p \equiv 7 \bmod 12$, *then we have*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{2k}\cdot\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^4 = p(p-1)\left(9p+2\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\psi\left(a^3-1\right)\right|^2\right)\sum_{i=0}^{[\frac{k}{2}]}\binom{k}{2i}2^{k-2i}p^{k-i}\left|\sum_{a=1}^{p-1}\psi(a)\left(\frac{a^2-1}{p}\right)\right|^{2i}.$$

From Theorem 1 and Theorem 2 we may immediately deduce the following corollaries.

Corollary 1 *For any odd prime p with* $p \equiv 1 \bmod 3$, *we have the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{2}\cdot\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2 = 6p^2(p-1).$$

Corollary 2 *Let p be an odd prime with* $p \equiv 1 \bmod 3$, *then we have*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{2}\cdot\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^4 = 2p^2(p-1)\left(9p+2\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\psi\left(a^3-1\right)\right|^2\right).$$

Corollary 3 *Let p be an odd prime with* $p \equiv 1 \bmod 3$, *then we have*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{4}\cdot\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^4 = p^2(p-1)\left(9p+2\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\psi\left(a^3-1\right)\right|^2\right)\left(4p+\left|C(p)\right|^2\right),$$

where $C(p) = \sum_{a=0}^{p-1}(\frac{a^3-1}{p})$ *or* $\sum_{a=1}^{p-1}\psi(a)(\frac{a^2-1}{p})$ *according to* $p \equiv 1$ *or* $7 \bmod 12$.

Corollary 4 *Let p be an odd prime with* $p \equiv 1 \bmod 3$, *then we have*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^4 = 9p^2(p-1)+2p(p-1)\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\psi\left(a^3-1\right)\right|^2.$$

In fact for any positive integers k and h, by using our method we can give an exact computational formula for (1). But when k and h are larger, the formula becomes more complex, therefore it is not listed here.

It is very regrettable that we cannot get the exact value for $|\sum_{a=1}^{p-1}(\frac{a}{p})\psi(a^3-1)|$, $|\sum_{a=1}^{p-1}\psi(a)(\frac{a^2-1}{p})|$ with $p \equiv 7 \bmod 12$ and $\sum_{a=0}^{p-1}(\frac{a^3-1}{p})$ with $p \equiv 1 \bmod 12$, which makes our theorems and corollaries look less beautiful.

Whether there exists exact values for these sums are three interesting open problems.

2 Some lemmas

In this section, we will give several simple lemmas, which are necessary in the proofs of our theorems. Hereinafter, we shall use many elementary number theory knowledge and the properties of the classical Gauss sums and Dirichlet characters, all of them can be found in reference [1], so we do not repeat them here. First we have the following.

Lemma 1 *Let p be an odd prime with $p \equiv 1 \bmod 3$, ψ be any three-order Dirichlet character* mod p. *Then, for any integer m with $(m,p) = 1$, we have the identity*

$$\left|\sum_{a=1}^{p-1} \psi(a) e\left(\frac{ma^2}{p}\right)\right|^2 = \begin{cases} 2p + (\frac{m}{p})\sqrt{p}\sum_{a=0}^{p-1}(\frac{a^3-1}{p}) & \text{if } p \equiv 1 \bmod 12; \\ 2p + (\frac{m}{p})\tau(\chi_2)\sum_{a=1}^{p-1}\psi(a)(\frac{a^2-1}{p}) & \text{if } p \equiv 7 \bmod 12. \end{cases}$$

Proof For any integer m with $(m,p) = 1$, note that the trigonometric identity

$$\sum_{n=0}^{p-1} e\left(\frac{mn^2}{p}\right) = \left(\frac{m}{p}\right) \cdot \tau(\chi_2), \tag{2}$$

where $\chi_2 = (\frac{*}{p})$ denotes the Legendre symbol mod p.

From (2) and the properties of the reduced residue system mod p we have

$$\begin{aligned} &\left|\sum_{a=1}^{p-1} \psi(a) e\left(\frac{ma^2}{p}\right)\right|^2 \\ &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \psi(a)\overline{\psi}(b) e\left(\frac{m(a^2-b^2)}{p}\right) \\ &= \sum_{a=1}^{p-1} \psi(a) \sum_{b=1}^{p-1} e\left(\frac{mb^2(a^2-1)}{p}\right) \\ &= 2(p-1) + \sum_{\substack{a=1 \\ a\neq\pm1}}^{p-1} \psi(a) \sum_{b=1}^{p-1} e\left(\frac{mb^2(a^2-1)}{p}\right) \\ &= 2p + \sum_{\substack{a=1 \\ a\neq\pm1}}^{p-1} \psi(a) \sum_{b=0}^{p-1} e\left(\frac{mb^2(a^2-1)}{p}\right) - \sum_{a=1}^{p-1}\psi(a) \\ &= 2p + \left(\frac{m}{p}\right)\tau(\chi_2) \sum_{a=1}^{p-1} \psi(a) \left(\frac{a^2-1}{p}\right). \end{aligned} \tag{3}$$

If $p \equiv 1 \bmod 12$, then note that $(\frac{-1}{p}) = 1$, $(\frac{a}{p}) = (\frac{\overline{a}}{p})$, $1 + \psi(a) + \overline{\psi}(a) = 3$ or 0 according to $a \equiv b^3 \bmod p$ or $p \nmid (a - b^3)$ for some $1 \le b \le p-1$, we have

$$\begin{aligned} &\sum_{a=1}^{p-1} \psi(a) \left(\frac{a^2-1}{p}\right) \\ &= \sum_{a=1}^{p-1} \psi(\overline{a}) \left(\frac{\overline{a}^2-1}{p}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{a=1}^{p-1} \overline{\psi}(a)\left(\frac{1-a^2}{p}\right) \\
&= \sum_{a=1}^{p-1} \psi(\overline{a})\left(\frac{\overline{a}^2-1}{p}\right) \\
&= \sum_{a=1}^{p-1} \overline{\psi}(a)\left(\frac{a^2-1}{p}\right) \\
&= \frac{1}{2}\sum_{a=1}^{p-1}\bigl(1+\psi(a)+\psi(\overline{a})\bigr)\left(\frac{a^2-1}{p}\right) - \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a^2-1}{p}\right) \\
&= \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a^6-1}{p}\right) - \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a-1}{p}\right) - \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a-1}{p}\right) \\
&= \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a^3-1}{p}\right) + \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a^3-1}{p}\right) - \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a-1}{p}\right) - \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{1-\overline{a}}{p}\right) \\
&= \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a^3-1}{p}\right) + \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{1-\overline{a}^3}{p}\right) + 1 = \sum_{a=0}^{p-1}\left(\frac{a^3-1}{p}\right). \qquad (4)
\end{aligned}
$$

Note that $\tau(\chi_2) = \sqrt{p}$, if $p \equiv 1 \bmod 4$. From (3) and (4) we may immediately deduce Lemma 1. □

Lemma 2 *Let* p *be an odd prime with* $p \equiv 1 \bmod 3$. *Then we have the identities*

$$
\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2 = 3p(p-1); \qquad \text{(A)}
$$

$$
\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2 = 0. \qquad \text{(B)}
$$

Proof First note the trigonometric identity

$$
\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{if } q \nmid n. \end{cases} \qquad (5)
$$

Since $p \equiv 1 \bmod 3$, the congruence equation $x^3 \equiv 1 \bmod p$ has three solutions. Let these three solutions be $x = 1$, g and g^2, respectively. Note that $(\frac{g}{p}) = (\frac{g}{p})^3 = (\frac{g^3}{p}) = 1$, so from (5) we have

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2 \\
&\quad = \sum_{m=1}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{ab}{p}\right)e\left(\frac{m(a^3-b^3)}{p}\right) \\
&\quad = \sum_{m=1}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{mb^3(a^3-1)}{p}\right)
\end{aligned}
$$

$$= \sum_{b=1}^{p-1}\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\sum_{m=0}^{p-1} e\left(\frac{m(a^3-1)}{p}\right)$$
$$= (p-1)\left(\left(\frac{1}{p}\right)+\left(\frac{g}{p}\right)+\left(\frac{g^2}{p}\right)\right)p + \sum_{\substack{a=1\\ p\nmid(a^3-1)}}^{p-1}\left(\frac{a}{p}\right)\sum_{m=0}^{p-1} e\left(\frac{m(a^3-1)}{p}\right)$$
$$= 3p(p-1).$$

This proves equation (A) of Lemma 2.

On the other hand, from the properties of Gauss sums we have

$$\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2$$
$$= \sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{ab}{p}\right)e\left(\frac{m(a^3-b^3)}{p}\right)$$
$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a}{p}\right)\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)e\left(\frac{mb^3(a^3-1)}{p}\right)$$
$$= \sum_{b=1}^{p-1}\left(\frac{b^3}{p}\right)\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)e\left(\frac{m(a^3-1)}{p}\right)$$
$$= \left(\sum_{b=1}^{p-1}\left(\frac{b}{p}\right)\right)\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)e\left(\frac{m(a^3-1)}{p}\right) = 0.$$

This proves equation (B) of Lemma 2. □

Lemma 3 *Let p be an odd prime with $p \equiv 1 \bmod 3$. Then, for any integer m with $(m,p)=1$, we have the identity*

$$\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2 = 3p + \overline{\psi}(m)\tau(\psi)\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\overline{\psi}(a^3-1) + \psi(m)\tau(\overline{\psi})\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\psi(a^3-1).$$

Proof From the properties of three-order character mod p and Gauss sums we have

$$\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2$$
$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{ab}{p}\right)e\left(\frac{m(a^3-b^3)}{p}\right)$$
$$= \sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\sum_{b=0}^{p-1} e\left(\frac{mb^3(a^3-1)}{p}\right) - \sum_{a=1}^{p-1}\left(\frac{a}{p}\right)$$

$$= 3p + \sum_{\substack{a=1\\ p\nmid(a^3-1)}}^{p-1} \left(\frac{a}{p}\right)\left(1 + \sum_{b=1}^{p-1}\left(1 + \psi(b) + \overline{\psi}(b)\right)e\left(\frac{mb(a^3-1)}{p}\right)\right)$$

$$= 3p + \sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\left(\overline{\psi}\left(m\left(a^3-1\right)\right)\tau(\psi) + \psi\left(m\left(a^3-1\right)\right)\tau(\overline{\psi})\right)$$

$$= 3p + \overline{\psi}(m)\tau(\psi)\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\overline{\psi}\left(a^3-1\right) + \psi(m)\tau(\overline{\psi})\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\psi\left(a^3-1\right).$$

This proves Lemma 3. □

3 Proofs of the theorems

Now we will complete the proofs of our theorems. First we prove Theorem 1. Let

$$A(p) = \sqrt{p}\sum_{a=0}^{p-1}\left(\frac{a^3-1}{p}\right) \quad \text{or} \quad \tau(\chi_2)\sum_{a=1}^{p-1}\psi(a)\left(\frac{a^2-1}{p}\right).$$

Then, for any integer $k \geq 1$, from Lemma 1 and the binomial theorem we have

$$\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{2k} = \left(2p + \left(\frac{m}{p}\right)A(p)\right)^k = \sum_{i=0}^{k}\binom{k}{i}(2p)^{k-i}A^i(p)\left(\frac{m}{p}\right)^i. \tag{6}$$

If i is an odd number, then from (B) of Lemma 2 we know that

$$\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)^i\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2 = \sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2 = 0. \tag{7}$$

Now combining (6), (7) and (A) of Lemma 2 we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{2k}\cdot\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2\\
&= \sum_{i=0}^{k}\binom{k}{i}(2p)^{k-i}A^i(p)\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)^i\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2\\
&= \sum_{i=0}^{[\frac{k}{2}]}\binom{k}{2i}(2p)^{k-2i}A^{2i}(p)\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^2\\
&= 3p(p-1)\sum_{i=0}^{[\frac{k}{2}]}\binom{k}{2i}(2p)^{k-2i}A^{2i}(p).
\end{aligned} \tag{8}$$

Note that if $p \equiv 7 \bmod 12$, then $\tau^2(\chi_2) = -p$ and $A(p)$ is a pure imaginary number, so from (8) and the definition of $A(p)$ we may immediately deduce Theorem 1.

Now we prove Theorem 2. From Lemma 3 we have

$$\begin{aligned}
&\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^4 \\
&= 9p^2 + 6p\left(\overline{\psi}(m)\tau(\psi)\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\overline{\psi}(a^3-1) + \psi(m)\tau(\overline{\psi})\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\psi(a^3-1)\right) \\
&\quad + \psi(m)\tau^2(\psi)\left(\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\overline{\psi}(a^3-1)\right)^2 + \overline{\psi}(m)\tau^2(\overline{\psi})\left(\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\psi(a^3-1)\right)^2 \\
&\quad + 2p\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\overline{\psi}(a^3-1)\right|^2. \qquad (9)
\end{aligned}$$

Note that, for any integer i, we have

$$\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)^i\psi(m) = \sum_{m=1}^{p-1}\left(\frac{m}{p}\right)^i\overline{\psi}(m) = 0.$$

So from (6), (7), (8) and (A) of Lemma 2 we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^{2k}\cdot\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right|^4 \\
&= \left(9p^2 + 2p\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\overline{\psi}(a^3-1)\right|^2\right)\sum_{i=0}^{[\frac{k}{2}]}\binom{k}{2i}(2p)^{k-2i}A^{2i}(p)\sum_{m=1}^{p-1}1 \\
&= p(p-1)\left(9p + 2\left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\psi(a^3-1)\right|^2\right)\sum_{i=0}^{[\frac{k}{2}]}\binom{k}{2i}(2p)^{k-2i}A^{2i}(p). \qquad (10)
\end{aligned}$$

Now Theorem 2 follows from (10) and the definition of $A(p)$. This completes the proofs of all our results.

Acknowledgements
The authors would like to thank the referees for their very helpful and detailed comments, which have significantly improved the presentation of this paper. This work is supported by the N.S.F. (11371291) of P.R. China.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Publisher's Note

Received: 5 June 2017 Accepted: 28 June 2017 Published online: 07 August 2017

References

1. Apostol, TM: Introduction to Analytic Number Theory. Springer, New York (1976)
2. Weil, A: On some exponential sums. Proc. Natl. Acad. Sci. USA **34**, 204-207 (1948)
3. Zhang, W, Liu, H: On the general Gauss sums and their fourth power mean. Osaka J. Math. **42**, 189-199 (2005)

4. Cochrane, T, Pinner, C: Using Stepanov's method for exponential sums involving rational functions. J. Number Theory **116**, 270-292 (2006)
5. Cochrane, T, Pinner, C: A further refinement of Mordell's bound on exponential sums. Acta Arith. **116**, 35-41 (2005)
6. Duke, W, Iwaniec, H: A relation between cubic exponential and Kloosterman sums. Contemp. Math. **143**, 255-258 (1993)
7. Zhang, H, Zhang, W: The fourth power mean of two-term exponential sums and its application. Math. Rep. **19**, 75-81 (2017)
8. Zhang, W: On the number of the solutions of one kind congruence equation mod p. J. Northwest Univ. Nat. Sci. **46**, 313-316 (2016)
9. Shen, S, Zhang, W: On the quartic Gauss sums and their recurrence property. Adv. Differ. Equ. **2017**, 43 (2017).