Advances in Difference Equations
a SpringerOpen Journal

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Coupled fixed point theorems with applications to fractional evolution equations

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Abstract

In this paper, we first prove some coupled fixed point theorems in partially ordered Φ-orbitally complete normed linear spaces. And then apply the obtained fixed point theorems to a class of semilinear evolution systems of fractional order for proving the existence of coupled mild solutions under some weaker monotone conditions. An example is given to illustrate the application of the abstract results.

MSC: 34A08; 47H10

Keywords: coupled fixed point theorem; nonlinear fractional evolution system; equi-continuous C_0-semigroup; coupled mild solution; existence

1 Introduction

Let E be a nonempty set, $\Psi : E \to E$ a mapping. If for $x \in E$, one has $\Psi x = x$, then $x \in E$ is called a fixed point of Ψ in E. Fixed point theory plays an important role in nonlinear functional analysis. Different types of fixed point theorems have been used to prove the existence of solutions for differential and integral equations; see [1–17]. The Banach contraction principle is one of the most powerful fixed point theorems in nonlinear analysis for proving the existence and uniqueness of fixed points in metric spaces. It is interesting to improve and extend the conditions of the Banach contraction principle. Recently, by weakening the requirement on the contraction in partially ordered metric spaces, a series of fixed point theorems are established for monotone mappings by Agarwal *et al.* [1], Harjani and Sadarangani [7], Nieto and Rodriguez-López [12, 13], O'Regan and Petrusel [14] and Ran and Reurings [15].

We recall some definitions of the monotone mapping. A mapping $\Psi : E \to E$ is monotone means it is monotone nondecreasing or monotone nonincreasing. Assume that $(E, \leq)$ is a partially ordered set and $\Psi : E \to E$. For $x, y \in E$, if $x \leq y$ implies $\Psi(x) \leq \Psi(y)$, Ψ is called a monotone nondecreasing mapping in E. Similarly, we can define a monotone nonincreasing mapping in E. If for $x_1, x_2 \in E$, $x_1 \leq x_2$ implies $\Psi(x_1, y) \leq \Psi(x_2, y)$ for all $y \in E$, while for $y_1, y_2 \in E$, $y_1 \leq y_2$ implies $\Psi(x, y_1) \geq \Psi(x, y_2)$ for all $x \in E$, then we say that Ψ is a mixed monotone mapping in E.

The above mentioned fixed point theorems, see [1, 7, 12–15], are all for the monotone mapping. In [2], Bhaskar and Lakshmikantham extended the fixed point theorems obtained in [1, 7, 12–15] to the mixed monotone mapping in partially ordered metric spaces. At first, they introduced a definition of the coupled fixed point. And then some coupled

fixed point theorems were proved in partially ordered metric spaces. Recently, these coupled fixed point theorems were refined and improved by Lakshmikantham and Ćirić [9], Luong and Thuan [11] and Samet [16].

In the above mentioned results, the assumptions of mixed monotone property and contraction property of the mapping Ψ are essential. The purpose of this paper is to delete or weaken these conditions. In this paper, by utilizing a different technique, we prove some coupled fixed point theorems in a partially ordered Φ-orbitally complete normed linear space. In our results, we neither assume that the mapping Ψ is mixed monotone, nor assume that it is a contraction. We divide the mapping Ψ into two parts, and assume that every part satisfies some conditions, by using an existing Krasnoselskii-type fixed point theorem, a coupled fixed point theorem for the mapping Ψ is proved. As applications, we apply the obtained coupled fixed point theorem to a certain abstract fractional evolution systems for proving the existence of coupled mild solutions.

The rest of this paper is organized as follows. In Section 2, some definitions are recalled and an existing Krasnoselskii-type fixed point theorem is introduced. In Section 3, coupled fixed point theorems are proved. In Section 4, we apply the obtained coupled fixed point theorem to a certain abstract fractional evolution systems. A specific example is given in Section 5 to illustrate the abstract results.

2 Preliminaries

Let E be a partially ordered normed linear space with partial order $\leq$ and the norm $\|\cdot\|_E$. If two elements $x, y \in E$ satisfy either $x \leq y$ or $x \geq y$, we say that they are comparable. If E is complete with respect to the norm $\|\cdot\|_E$, we called it a partially ordered complete normed linear space.

Definitions 2.1-2.5 can be found in [4, 5].

Definition 2.1 Let $\Psi : E \to E$ be a mapping. For any $x \in E$, we define an orbit $\Phi(x; \Psi)$ by

$$\Phi(x; \Psi) = \left\{ x, \Psi x, \Psi^2 x, \ldots, \Psi^n x, \ldots \right\}.$$

If for any sequence $\{x_n\} \subset \Phi(x; \Psi)$, $x_n \to x^*$ implies $\Psi x_n \to \Psi x^*$ for each $x \in E$, Ψ is said to be Φ-orbitally continuous in E. A normed linear space $(E, \|\cdot\|_E)$ is called Φ-orbitally complete if every Cauchy sequence $\{x_n\} \subset \Phi(x; \Psi)$ converges to a point x^* in E.

Definition 2.2 A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a $\mathfrak{D}$-function if it is a monotone nondecreasing and upper semi-continuous function satisfying $\phi(0) = 0$.

Definition 2.3 A mapping $\Psi : E \to E$ is called partially nonlinear $\mathfrak{D}$-Lipschitz if for all comparable elements $x, y \in E$, there is a $\mathfrak{D}$-function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\Psi x - \Psi y\|_E \leq \phi\big(\|x - y\|_E\big).$$

Furthermore, if $\phi(r) < r$ for $r > 0$, Ψ is called a partially nonlinear $\mathfrak{D}$-contraction in E.

Definition 2.4 A mapping $\Psi : E \to E$ is said to be partially compact if for all totally ordered sets or chains $C \subset E$, $\Psi(C)$ is a relatively compact subset of E.

Definition 2.5 The norm $\| \cdot \|_E$ and the order relation $\leq$ on a partially ordered normed linear space $(E, \leq, \| \cdot \|_E)$ are said to be compatible if for any monotone sequence $\{x_n\}$ in E, subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*.

Remark 2.6 From Definition 2.1, if the normed linear space $(E, \| \cdot \|_E)$ is complete, then it is Φ-orbitally complete. But the converse expression may not be true.

Remark 2.7 The $\mathfrak{D}$-functions and the partially nonlinear $\mathfrak{D}$-Lipschitz conditions are much useful in research of solutions for nonlinear differential equations via fixed point theorems; see [6].

The following Krasnoselskii-type fixed point theorem was proved by Dhage in [4].

Lemma 2.8 *Let E be a partially ordered complete normed linear space with the partial order $\leq$ and the norm $\| \cdot \|_E$ such that $\leq$ and $\| \cdot \|_E$ are compatible. Suppose that $A_1, A_2 : E \to E$ are two monotone nondecreasing mappings satisfying*:

(a) *A_1 is continuous and a partially nonlinear $\mathfrak{D}$-contraction,*
(b) *A_2 is continuous and partially compact,*
(c) *there is an element $v_0 \in E$ satisfying $v_0 \leq A_1 v_0 + A_2 y$ for all $y \in E$, and*
(d) *every pair of elements has an upper and a lower bound in E.*

Then $x = A_1 x + A_2 x$ has a solution in E.

3 Fixed point theorems

Let $(E, \| \cdot \|_E)$ be a Φ-orbitally complete normed linear space. Define a positive cone K in E by

$$K = \{x \in E : x \geq 0\}.$$

Then E becomes now a partially ordered Φ-orbitally complete normed linear space with the partial order $\leq$ induced by K. It is well known that the partially order $\leq$ and the norm $\| \cdot \|_E$ are compatible if cone K is normal.

By Lemma 2.8, we first prove the following fixed point theorem.

Theorem 3.1 *Let E be a partially ordered Φ-orbitally complete normed linear space with the norm $\| \cdot \|_E$ and the partial order $\leq$, whose positive cone K is normal, and let D be a nonempty closed subset of E. Assume that $A_1, A_2 : D \to D$ are two monotone nondecreasing mappings satisfying*:

(a)′ *A_1 is Φ-orbitally continuous and*

$$\|A_1 x - A_1 y\|_E < \|x - y\|_E$$

for all comparable elements $x, y \in D$ with $x \not\equiv y$,
(b)′ *A_2 is Φ-orbitally continuous and partially compact,*

(c) *there exists an element $v_0 \in D$ such that $v_0 \le A_1 v_0 + A_2 y$ for all $y \in D$, and*

(d) *every pair of elements in D has an upper and a lower bound.*

Then the equation $x = A_1 x + A_2 x$ has a solution in D.

Proof By Remark 2.7 of [4] and the proof of Lemma 2.8, if the continuity of operators A_1 and A_2 is replaced by the Φ-orbitally continuity in conditions (a) and (b) of Lemma 2.8, the conclusion of Lemma 2.8 is still true. On the other hand, since D is a nonempty closed subset of E, it follows that D is Φ-orbitally complete and has partial order $\le$. For any comparable elements $x, y \in D$, $x \not\equiv y$, by the condition (a)$'$, it follows that there is $\tau \in (0,1)$ satisfying

$$\|A_1 x - A_1 y\|_E \le \tau \|x - y\|_E < \|x - y\|_E.$$

Let $\phi(r) = \tau r$. Then ϕ is a $\mathfrak{D}$-function and $\phi(r) < r$ for any $r > 0$. This implies that the condition (a) of Lemma 2.8 is satisfied. By Lemma 2.8, we obtain the desired conclusion.

$\square$

If the inequality given in assumption (c) of Lemma 2.8 is reverse, more precisely, the condition (c) of Lemma 2.8 is replaced by

(c)$'$ there is $v_0 \in D$ satisfying $v_0 \ge A_1 v_0 + A_2 y$ for all $y \in D$.

By Theorem 3.7 of [4], the conclusion of Lemma 2.8 is still true. Hence we can obtain the following fixed point theorem. Since the proof is similar to Theorem 3.1, we omit the details here.

Theorem 3.2 *Let E be a partially ordered Φ-orbitally complete normed linear space with the norm $\| \cdot \|_E$ and the partial order $\le$, whose positive cone K is normal, and let D be a nonempty closed subset of E. Assume that $A_1, A_2 : D \to D$ are two monotone nondecreasing mappings satisfying the conditions (a)$'$, (b)$'$, (c)$'$ and (d). Then $x = A_1 x + A_2 x$ has a solution in D.*

Let $\widehat{E} := E \times E$. Define a sum and a scalar multiplication in $\widehat{E}$ by

$$w + v = (w_1, w_2) + (v_1, v_2) = (w_1 + v_1, w_2 + v_2),$$

$$\lambda w = \lambda(w_1, w_2) = (\lambda w_1, \lambda w_2)$$

for all $w = (w_1, w_2), v = (v_1, v_2) \in \widehat{E}$ and $\lambda \in \mathbb{R}$. And define a positive cone and a norm in $\widehat{E}$ by

$$K_{\widehat{E}} = \left\{ w = (w_1, w_2) \in \widehat{E} : w_1, w_2 \in K \right\},$$

$$\|v\|_{\widehat{E}} = \left\| (v_1, v_2) \right\|_{\widehat{E}} = \|v_1\|_E + \|v_2\|_E, \quad v = (v_1, v_2) \in \widehat{E}.$$

Then $(\widehat{E}, \| \cdot \|_{\widehat{E}})$ is a partially ordered normed linear space with the order relation $\le$ induced by $K_{\widehat{E}}$. Let D be a nonempty closed subset of E. Then $D \times D$ is a nonempty closed subset of $\widehat{E}$. It is clear that cone $K_{\widehat{E}}$ is normal if K is normal.

Let $Q : \widehat{E} \to E$. A pair of elements $(x, y) \in \widehat{E}$ is called a coupled fixed point of $Q : \widehat{E} \to E$ if and only if it satisfies

$$x = Q(x, y), \qquad y = Q(y, x).$$

By Theorem 3.1, the following coupled fixed point theorem is obtained.

Theorem 3.3 *Let E be a partially ordered Φ-orbitally complete normed linear space with the partial order $\leq$ and the norm $\| \cdot \|_E$, whose positive cone K is normal, and let D be a nonempty closed subset of E. Assume that $A_1, A_2 : D \to D$ are two monotone nondecreasing mappings satisfying*

(i)   *A_1 is Φ-orbitally continuous and a partially nonlinear $\mathfrak{D}$-contraction,*
(ii) *A_2 is Φ-orbitally continuous and partially compact,*
(iii) *there is an element $v_0 \in D$ satisfying $v_0 \leq A_1 v_0 + A_2 y$ for all $y \in D$, and*
(iv) *every pair of elements in D has an upper and a lower bound.*

Then $Q(x, y) = A_1 x + A_2 y$ has a coupled fixed point in $\widehat{E}$.

Proof Since $(E, \leq, \| \cdot \|_E)$ is a partially ordered Φ-orbitally complete normed linear space, and positive cone K is normal, it follows that $(\widehat{E}, \leq, \| \cdot \|_{\widehat{E}})$ is a partially ordered Φ-orbitally complete normed linear space and positive cone $K_{\widehat{E}}$ is normal. Since D is a nonempty closed subset of E, it follows that $D \times D$ is a nonempty closed subset of $\widehat{E}$.

Let $\widehat{D} = D \times D$. Define two operators $\widehat{A}_1, \widehat{A}_2 : \widehat{D} \to \widehat{D}$ by

$$\widehat{A}_1 u = (A_1 x, A_1 y), \qquad \widehat{A}_2 u = (A_2 y, A_2 x), \quad \forall u = (x, y) \in \widehat{D}.$$

If the operator equation $u = \widehat{A}_1 u + \widehat{A}_2 u$ has a solution $u = (x, y) \in \widehat{D}$, namely,

$$(x, y) = u = \widehat{A}_1 u + \widehat{A}_2 u = \widehat{A}_1(x, y) + \widehat{A}_2(x, y),$$

then we obtain

$$\begin{aligned} (x, y) &= \widehat{A}_1 u + \widehat{A}_2 u \\ &= (A_1 x, A_1 y) + (A_2 y, A_2 x) \\ &= (A_1 x + A_2 y, A_1 y + A_2 x) \\ &= \big(Q(x, y), Q(y, x)\big). \end{aligned}$$

This implies that the operator $Q(x, y)$ has a coupled fixed point in $\widehat{E}$. We will apply Theorem 3.1 to prove that the operator equation $u = \widehat{A}_1 u + \widehat{A}_2 u$ has a solution in $\widehat{E}$. The proof will be given in several steps.

Step I. $\widehat{A}_1$ is Φ-orbitally continuous and

$$\|\widehat{A}_1 w - \widehat{A}_1 v\|_{\widehat{E}} < \|w - v\|_{\widehat{E}},$$

for all comparable elements $w, v \in \widehat{D}$ with $w \not\equiv v$.

Since A_1 is Φ-orbitally continuous, by the definition of $\widehat{A}_1$, it is easy to see that $\widehat{A}_1$ is Φ-orbitally continuous.

For all comparable elements $w = (w_1, w_2), v = (v_1, v_2) \in \widehat{D}$ with $w \not\equiv v$, we have

$$
\begin{aligned}
\|\widehat{A}_1 w - \widehat{A}_1 v\|_{\widehat{E}} &= \left\|(A_1 w_1, A_1 w_2) - (A_1 v_1, A_1 v_2)\right\|_{\widehat{E}} \\
&= \left\|(A_1 w_1 - A_1 v_1), (A_1 w_2 - A_1 v_2)\right\|_{\widehat{E}} \\
&= \|A_1 w_1 - A_1 v_1\|_E + \|A_1 w_2 - A_1 v_2\|_E \\
&\leq \varphi\big(\|w_1 - v_1\|_E\big) + \varphi\big(\|w_2 - v_2\|_E\big) \\
&< \|w_1 - v_1\|_E + \|w_2 - v_2\|_E \\
&= \left\|(w_1 - v_1, w_2 - v_2)\right\|_{\widehat{E}} \\
&= \left\|(w_1, w_2) - (v_1, v_2)\right\|_{\widehat{E}} \\
&= \|w - v\|_{\widehat{E}}.
\end{aligned}
$$

Hence, we obtain

$$
\|\widehat{A}_1 w - \widehat{A}_1 v\|_{\widehat{E}} < \|w - v\|_{\widehat{E}}
$$

for all comparable elements $w, v \in \widehat{D}$ with $w \not\equiv v$.

Step II. $\widehat{A}_2$ is Φ-orbitally continuous and partially compact.

Since A_2 is Φ-orbitally continuous, by the definition of $\widehat{A}_2$, it is easy to see that $\widehat{A}_2$ is Φ-orbitally continuous.

Let $C \subset D$ be a bounded chain. Since A_2 is partially compact in D, it follows that $A_2(C)$ is equi-continuous and uniformly bounded in D. Set $\widehat{C} = C \times C$. Then $\widehat{C}$ is a bounded chain in $\widehat{D}$. Next, we claim that $\widehat{A}_2(\widehat{C})$ is equi-continuous and uniformly bounded in $\widehat{D}$

Since $A_2(C) \subset D$ is uniformly bounded, there is a constant $\overline{M} > 0$ satisfying $\|A_2 z\|_E \leq \overline{M}$ for any $z \in C$. For any $Z \in \widehat{A}_2(\widehat{C})$, there are $x, y \in C$ satisfying $u = (x, y) \in \widehat{C}$ such that $Z = \widehat{A}_2 u$. Thus,

$$
\begin{aligned}
\|Z\|_{\widehat{E}} = \|\widehat{A}_2 u\|_{\widehat{E}} &= \left\|(A_2 y, A_2 x)\right\|_{\widehat{E}} \\
&= \|A_2 y\|_E + \|A_2 x\|_E \\
&\leq 2\overline{M}.
\end{aligned}
$$

This implies that $\widehat{A}_2(\widehat{C})$ is uniformly bounded in $\widehat{D}$.

Since $A_2(C)$ is equi-continuous in D, for any $z \in C$ and $t_2 > t_1$, we have

$$
\left\|(A_2 z)(t_2) - (A_2 z)(t_1)\right\|_E \to 0
$$

as $t_2 - t_1 \to 0$. Hence for any $Z = \widehat{A}_2 u = \widehat{A}_2(x, y) \in \widehat{A}_2(\widehat{C})$, we have

$$
\begin{aligned}
\left\|Z(t_2) - Z(t_1)\right\|_{\widehat{E}} &= \left\|(\widehat{A}_2 u)(t_2) - (\widehat{A}_2 u)(t_1)\right\|_{\widehat{E}} \\
&= \left\|\big((A_2 y)(t_2), (A_2 x)(t_2)\big) - \big((A_2 y)(t_1), (A_2 x)(t_1)\big)\right\|_{\widehat{E}} \\
&= \left\|(A_2 y)(t_2) - (A_2 y)(t_1), (A_2 x)(t_2) - (A_2 x)(t_1)\right\|_{\widehat{E}}
\end{aligned}
$$

$$= \left\| (A_2 y)(t_2) - (A_2 y)(t_1) \right\|_E + \left\| (A_2 x)(t_2) - (A_2 x)(t_1) \right\|_E$$
$$\to 0$$

as $t_2 - t_1 \to 0$. This implies that $\widehat{A}_2(\widehat{C})$ is equi-continuous in $\widehat{D}$.

Therefore, by the Arzela-Ascoli theorem, $\widehat{A}_2(\widehat{C}) \subset \widehat{D}$ is relatively compact. Consequently, $\widehat{A}_2 : \widehat{D} \to \widehat{D}$ is partially compact.

Step III. There is an element $u_0 \in \widehat{D}$ satisfying

$$u_0 \le \widehat{A}_1 u_0 + \widehat{A}_2 u$$

for all $u \in \widehat{D}$.

Let $u_0 = (v_0, v_0)$. For any $u = (x, y) \in \widehat{D}$, by the condition (iii), we have

$$\widehat{A}_1 u_0 + \widehat{A}_2 u = (A_1 v_0, A_1 v_0) + (A_2 y, A_2 x)$$
$$= (A_1 v_0 + A_2 y, A_1 v_0 + A_2 x)$$
$$\ge (v_0, v_0)$$
$$= u_0.$$

Hence we obtain the desired conclusion.

Step IV. Every pair of elements in $\widehat{D}$ has an upper and a lower bound.

For every pair of elements $w = (w_1, w_2), v = (v_1, v_2) \in \widehat{D}$, by condition (iv), there exist $z_1, z_2, z_3, z_4 \in D$ such that

$$z_1 \le w_1, \qquad z_1 \le v_1, \qquad z_3 \ge w_1, \qquad z_3 \ge v_1,$$
$$z_2 \le w_2, \qquad z_2 \le v_2, \qquad z_4 \ge w_2, \qquad z_4 \ge v_2.$$

Thus, we have

$$(z_1, z_2) \le (w_1, w_2) \le (z_3, z_4),$$
$$(z_1, z_2) \le (v_1, v_2) \le (z_3, z_4).$$

Consequently, every pair of elements $w, v \in \widehat{D}$ has an upper and a lower bound.

Therefore, by Theorem 3.1, the operator equation $u = \widehat{A}_1 u + \widehat{A}_2 u$ has a solution in $\widehat{E}$.
□

By Theorems 3.2 and 3.3, the following coupled fixed point theorem is obtained. Because its proof is similar to Theorem 3.3, we omit the details.

Theorem 3.4 *Let E be a partially ordered Φ-orbitally complete normed linear space with the norm $\| \cdot \|_E$ and the order relation $\le$, positive cone K be normal, and let D be a nonempty closed subset of E. Assume that $A_1, A_2 : D \to D$ are two monotone nondecreasing mappings satisfying*

(i) *A_1 is Φ-orbitally continuous and a partially nonlinear $\mathfrak{D}$-contraction,*
(ii) *A_2 is Φ-orbitally continuous and partially compact,*

(iii)′ *there is an element $v_0 \in D$ satisfying $v_0 \geq A_1 v_0 + A_2 y$ for all $y \in D$, and*

(iv) *every pair of elements in D has an upper and a lower bound.*

Then $Q(x, y) = A_1 x + A_2 y$ has a coupled fixed point in $\widehat{E}$.

Remark 3.5 The hypothesis (iv) of Theorems 3.3 and 3.4 holds if the partially ordered set E is a lattice. We known that the set $C(J, X)$ is a lattice, where $C(J, X)$ is the set of all continuous X-valued functions on $J \in \mathbb{R}$, X is a partially ordered set. For any $x, y \in C(J, X)$, $\max\{x, y\}$ and $\min\{x, y\}$ are the upper and lower bounds, respectively.

Remark 3.6 The assumptions of mixed monotone property and contractive property of the mapping Q are essential in [2, 9, 11, 16]. But in Theorems 3.3 and 3.4, we neither assume that the mapping Q is mixed monotone, nor assume that the mapping Q is a contraction. We only suppose that the mapping Q is a nondecreasing mapping and a part of Q (namely, the operator A_1) is a partially nonlinear $\mathcal{D}$-contraction. Plus with other assumptions we obtain the coupled fixed point theorems. Hence Theorems 3.3 and 3.4 extend the main results of [2, 9, 11, 16].

4 Existence results for fractional evolution systems

Let $(X, \|\cdot\|)$ be a Φ-orbitally complete normed linear space. Define its positive cone as $\overline{K} = \{x \in X : x \geq 0\}$. Then X becomes a partially ordered Φ-orbitally complete normed linear space with the norm $\|\cdot\|$ and the partial order $\leq$ induced by the cone $\overline{K}$. In this section, we always assume that $\overline{K}$ is normal. Investigate the existence of coupled mild solutions to the initial value problem of the fractional hybrid evolution system

$$\begin{cases} {}^{C}D_t^{\sigma} x(t) + Ax(t) = f(t, x(t)) + h(t, y(t)), \\ {}^{C}D_t^{\sigma} y(t) + Ay(t) = f(t, y(t)) + h(t, x(t)), \quad t \in J, \\ x(0) = \tau_0, y(0) = \tau_0 \in X, \end{cases} \tag{4.1}$$

where $J = [0, b]$, $b > 0$ is a constant, ${}^{C}D_t^{\sigma}$ denotes the $\sigma \in (0, 1)$ order Caputo fractional derivative, $-A$ generates a C_0-semigroup $S(t)$ $(t \geq 0)$ of uniformly bounded linear operator in X, f and h are given functions.

For the C_0-semigroup $S(t)$ $(t \geq 0)$, if $S(t)x \geq 0$ for all $x \geq 0$, it is called a positive C_0-semigroup. Throughout this section, we always assume that $-A$ generates a positive C_0-semigroup $S(t)$ $(t \geq 0)$ of uniformly bounded linear operator in X. Namely, there is a constant $M > 0$ such that $\|S(t)\| \leq M$ for all $t \geq 0$.

Definitions 4.1 and 4.2 can be found in [8, 10, 17].

Definition 4.1 The fractional integral of order $\sigma > 0$ with the lower limits zero for a function $f \in L^1(J, E)$ is defined by

$$I^{\sigma} f(t) = \frac{1}{\Gamma(\sigma)} \int_0^t \frac{f(s)}{(t - s)^{1 - \sigma}} \, ds, \quad t > 0,$$

where Γ is the gamma function.

Definition 4.2 The Riemann-Liouville derivative of order $n - 1 < \sigma < n$, $n \in \mathbb{N}$ with the lower limits zero for a function $f \in L^1(J, E)$ can be defined as

$$^L D_t^\sigma f(t) = \frac{1}{\Gamma(n - \sigma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t - s)^{\sigma + 1 - n}} \, ds, \quad t > 0.$$

The Caputo fractional derivative of order $0 < \sigma < 1$ with the lower limits zero for a function $f \in L^1(J, E)$ can be defined as

$$^C D_t^\sigma f(t) = {}^L D_t^\sigma \big(f(t) - f(0) \big), \quad t > 0.$$

Define two operator families $\{\mathcal{U}_\sigma(t)\}_{t \geq 0}$ and $\{\mathcal{V}_\sigma(t)\}_{t \geq 0}$ as

$$\mathcal{U}_\sigma(t) x = \int_0^\infty \zeta_\sigma(\tau) S(t^\sigma \tau) x \, d\tau,$$

$$\mathcal{V}_\sigma(t) x = \sigma \int_0^\infty \tau \zeta_\sigma(\tau) S(t^\sigma \tau) x \, d\tau, \quad 0 < \sigma < 1,$$

where

$$\zeta_\sigma(\tau) = \frac{1}{\sigma} \tau^{-1 - \frac{1}{\sigma}} \varrho_\sigma \left(\tau^{-\frac{1}{\sigma}} \right),$$

$$\varrho_\sigma(\tau) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \tau^{-\sigma n - 1} \frac{\Gamma(n\sigma + 1)}{n!} \sin(n\pi\sigma), \quad \tau \in (0, \infty).$$

Lemma 4.3

(i) *For any $x \in X$ and fixed $t \geq 0$, one has*

$$\big\| \mathcal{U}_\sigma(t) x \big\| \leq M \|x\|, \qquad \big\| \mathcal{V}_\sigma(t) x \big\| \leq \frac{M}{\Gamma(\sigma)} \|x\|.$$

(ii) *If $S(t)$ $(t \geq 0)$ is an equi-continuous semigroup, $\mathcal{V}_\sigma(t)$ is equi-continuous in X for $t > 0$.*

(iii) *If $S(t)$ $(t \geq 0)$ is a positive C_0-semigroup, $\mathcal{U}_\sigma(t)$ and $\mathcal{V}_\sigma(t)$ are positive operators for all $t \geq 0$.*

Proof (i) and (ii) can be found in reference [10, 17]. (iii) is easily seen from the definitions of $\mathcal{U}_\sigma(t)$ and $\mathcal{V}_\sigma(t)$. So, we omit the details here. □

Let $C(J, X)$ be a set of all continuous X-valued functions on the interval J and let

$$\overline{K}_C = \big\{ x \in C(J, X) : x(t) \in \overline{K}, t \in J \big\}.$$

Then $C(J, X)$ is a partially ordered Φ-orbitally complete normed linear space with the norm $\|x\|_C := \sup\{\|x(t)\| : t \geq 0\}$ and the partial order $\leq$ induced by $\overline{K}_C$. It is clear that $\overline{K}_C$ is normal because $\overline{K}$ is normal.

Let $E = C(J, X)$ and $K = \overline{K}_C$. Define two operators $A_1, A_2 : E \to E$ by

$$(A_1 x)(t) = \mathcal{U}_\sigma(t) \tau_0 + \int_0^t (t - s)^{\sigma - 1} \mathcal{V}_\sigma(t - s) f\big(s, x(s)\big) \, ds, \quad t \in J, \tag{4.2}$$

$$(A_2 x)(t) = \int_0^t (t-s)^{\sigma-1} \mathcal{V}_\sigma (t-s) h(s, x(s)) \, ds, \quad t \in J. \tag{4.3}$$

Definition 4.4 An element $(x, y) \in E \times E$ is called a coupled mild solution of the system (4.1) if and only if it satisfies the following system of operator equations:

$$\begin{cases} x(t) = (A_1 x)(t) + (A_2 y)(t), & t \in J, \\ y(t) = (A_1 y)(t) + (A_2 x)(t), & t \in J. \end{cases} \tag{4.4}$$

We shall use Theorem 3.3 to prove that (4.4) has a coupled fixed point in $E \times E$. For this purpose, we consider the following hypotheses:

(H1) The positive C_0-semigroup $S(t)$ $(t \geq 0)$ is equi-continuous.

(H2) The function $f : J \times X \to X$ is continuous in x for all $t \in J$ and there exist a constant $\rho \in \mathbb{R}$ with $0 < \rho < \frac{\Gamma(\sigma+1)}{Mb^\sigma}$ and a $\mathfrak{D}$-function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(r) < r$ for any $r > 0$ satisfying

$$0 \leq f(t, u) - f(t, v) \leq \rho \phi(\|u - v\|), \quad \forall u, v \in X, u \geq v$$

for all $t \in J$.

(H3) The function $h(t, x) : J \times X \to X$ is continuous, nondecreasing and bounded in x.

(H4) There is an element $\bar{x} \in E$ satisfying

$$\begin{cases} {}^C D_t^\sigma \bar{x}(t) + A\bar{x}(t) \leq f(t, \bar{x}(t)) + h(t, y(t)), & t \in J, \\ \bar{x}(0) \leq \tau_0 \in X \end{cases}$$

for all $y \in E$.

Since, by (H3), $h(t, x)$ is bounded in x for all $t \in J$, there is a constant $\widetilde{M} > 0$ such that $\|h(t, x)\| \leq \widetilde{M}$ for all $t \in J$ and $x \in X$. For

$$r^* \geq \max\left\{ \frac{M\widetilde{M}b^\sigma}{\Gamma(\sigma+1)}, \left(1 - \frac{M\rho b^\sigma}{\Gamma(\sigma+1)}\right)^{-1} \left(M\|\tau_0\| + \frac{MF^* b^\sigma}{\Gamma(\sigma+1)}\right) \right\} + 1, \tag{4.5}$$

where $F^* = \sup_{t \in J} \|f(t, 0)\|$, we define an open ball $\mathfrak{B}(x_0, r)$ in E by

$$\mathfrak{B}(x_0, r) = \{x \in E : \|x_0 - x\|_C < r\},$$

where $r = \|x_0\|_C + r^*$. Let $D = \overline{\mathfrak{B}}(x_0, r)$. Then D is a closed and bounded subset in E. By virtue of the assumptions (H1)-(H3), we have the following lemmas.

Lemma 4.5 *Assume that the hypothesis* (H2) *holds. Then the operator* $A_1 : D \to D$ *is* Φ-*orbitally continuous, nondecreasing and a partially nonlinear* $\mathfrak{D}$-*contraction in* E.

Proof By (H2), (4.2) and (4.5), for any $x \in E$ with $\|x\|_C \leq r^*$, we have

$$\|(A_1 x)(t)\| \leq M\|\tau_0\| + \frac{M}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \left[\|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\| \right] ds$$

$$\leq M\|\tau_0\| + \frac{M}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \left[\rho\phi(\|x\|_C) + F^* \right] ds$$

$$\leq M\|\tau_0\| + \frac{M\rho b^\sigma}{\Gamma(\sigma + 1)}r^* + \frac{MF^* b^\sigma}{\Gamma(\sigma + 1)}$$

$$\leq r^*.$$

This implies that $\|A_1 x\|_C \leq r^*$ for any $x \in E$ with $\|x\|_C \leq r^*$. Moreover, we have

$$\|x_0 - A_1 x\|_C \leq \|x_0\|_C + \|A_1 x\|_C \leq \|x_0\|_C + r^* = r.$$

This implies that A_1 maps D into itself.

Since $S(t)$ ($t \geq 0$) is a positive C_0-semigroup, by (H2) and Lemma 4.3, it follows that $A_1 : D \to D$ is nondecreasing. Take a sequence $\{x_n\} \subset \Phi(x; A_1)$ for any $x \in D$ with $x_n \to x^*$ as $n \to \infty$. Since f is continuous in x for all $t \in J$, by assumption (H2) and dominated convergence theorem, we have

$$\lim_{n \to \infty} (A_1 x_n)(t) = \mathcal{U}_\sigma(t)\tau_0 + \int_0^t (t - s)^{\sigma-1} \mathcal{V}_\sigma(t - s) \lim_{n \to \infty} f(s, x_n(s))\, ds$$

$$= \mathcal{U}_\sigma(t)\tau_0 + \int_0^t (t - s)^{\sigma-1} \mathcal{V}_\sigma(t - s) f(s, x^*(s))\, ds$$

$$= (A_1 x^*)(t), \quad t \in J.$$

This implies that $A_1 : D \to D$ is Φ-orbitally continuous.

For any comparable elements $x, y \in D$, without loss of generality, we assume that $x \geq y$. By (H2), for any $t \in J$, we have

$$\|(A_1 x)(t) - (A_1 y)(t)\| \leq \int_0^t (t - s)^{\sigma-1} \|\mathcal{V}_\sigma(t - s)[f(s, x(s)) - f(s, y(s))]\|\, ds$$

$$\leq \frac{M\rho}{\Gamma(\sigma)} \int_0^t (t - s)^{\sigma-1} \phi(\|x(s) - y(s)\|)\, ds$$

$$\leq \frac{Mb^\sigma \rho}{\Gamma(\sigma + 1)} \phi(\|x - y\|_C).$$

This implies that

$$\|A_1 x - A_1 y\|_C \leq \phi(\|x - y\|_C)$$

because of $0 < \rho < \frac{\Gamma(\sigma+1)}{Mb^\sigma}$. Hence $A_1 : D \to D$ is a partially nonlinear $\mathfrak{D}$-contraction in E. $\qquad \square$

Lemma 4.6 *Let the hypotheses* (H1) *and* (H3) *hold. Then the operator* $A_2 : D \to D$ *is* Φ-*orbitally continuous, nondecreasing and partially compact in* E.

Proof By the assumption (H3), (4.3) and (4.5), for any $x \in E$ with $\|x\|_C \leq r^*$, we have

$$\|(A_2 x)(t)\| \leq \frac{M}{\Gamma(\sigma)} \int_0^t (t - s)^{\sigma-1} \|h(s, x(s))\|\, ds \leq \frac{M\widetilde{M}b^\sigma}{\Gamma(\sigma + 1)} \leq r^*.$$

This follows that $\|A_2 x\|_C \leq r^*$ for any $x \in E$ with $\|x\|_C \leq r^*$. Further, we have

$$\|x_0 - A_2 x\|_C \leq \|x_0\|_C + \|A_2 x\|_C \leq \|x_0\|_C + r^* = r.$$

This implies that A_2 maps D into itself.

Since $S(t)$ $(t \geq 0)$ is a positive C_0-semigroup, by (H3) and Lemma 4.3, a similar proof as in Lemma 4.5 shows that $A_2 : D \to D$ is Φ-orbitally continuous and nondecreasing.

For any $t_1, t_2 \in J$ with $t_1 < t_2$, denote

$$G_1 = \left\| \int_0^{t_1} \left[(t_2 - s)^{\sigma - 1} - (t_1 - s)^{\sigma - 1} \right] \mathcal{V}_\sigma(t_2 - s) h(s, x(s)) \, ds \right\|,$$

$$G_2 = \left\| \int_0^{t_1} (t_1 - s)^{\sigma - 1} \left[\mathcal{V}_\sigma(t_2 - s) - \mathcal{V}_\sigma(t_1 - s) \right] h(s, x(s)) \, ds \right\|,$$

$$G_3 = \left\| \int_{t_1}^{t_2} (t_2 - s)^{\sigma - 1} \mathcal{V}_\sigma(t_2 - s) h(s, x(s)) \, ds \right\|.$$

Then, by Lemma 4.3, we have

$$G_1 \leq \frac{M \widetilde{M}}{\Gamma(\sigma)} \int_0^{t_1} \left| (t_2 - s)^{\sigma - 1} - (t_1 - s)^{\sigma - 1} \right| \, ds,$$

$$G_3 \leq \frac{M \widetilde{M} b^\sigma}{\Gamma(\sigma + 1)} (t_2 - t_1)^\sigma.$$

This implies that $G_1 \to 0$ and $G_3 \to 0$ as $t_2 - t_1 \to 0$. If $t_1 \equiv 0$ and $0 < t_2 \leq b$, it is clear that $G_2 \equiv 0$. For $t_1 > 0$ and $\delta \in (0, t_1)$ small enough, we have

$$G_2 \leq \left\| \int_0^{t_1 - \delta} (t_1 - s)^{\sigma - 1} \left[\mathcal{V}_\sigma(t_2 - s) - \mathcal{V}_\sigma(t_1 - s) \right] h(s, x(s)) \, ds \right\|$$

$$+ \left\| \int_{t_1 - \delta}^{t_1} (t_1 - s)^{\sigma - 1} \left[\mathcal{V}_\sigma(t_2 - s) - \mathcal{V}_\sigma(t_1 - s) \right] h(s, x(s)) \, ds \right\|$$

$$\leq \frac{\widetilde{M}(t_1^\sigma - \delta^\sigma)}{\sigma} \sup_{s \in [0, t_1 - \delta]} \left\| \mathcal{V}_\sigma(t_2 - s) - \mathcal{V}_\sigma(t_1 - s) \right\|$$

$$+ \frac{2 M \widetilde{M} \delta^\sigma}{\Gamma(\sigma + 1)}.$$

Hence $G_2 \to 0$ as $t_2 - t_1 \to 0$ and $\delta \to 0$ because of (H1).

Let $C \subset D$ be an arbitrary chain. For any $Z \in A_2(C)$, there is $x \in C$ such that $Z(t) = (A_2 x)(t)$ for all $t \in J$. So, for any $t_1, t_2 \in J$ with $t_1 < t_2$, by the definition of the operator A_2, the inequality

$$\left\| Z(t_2) - Z(t_1) \right\| = \left\| (A_2 x)(t_2) - (A_2 x)(t_1) \right\| \leq G_1 + G_2 + G_3$$

implies that

$$\left\| Z(t_2) - Z(t_1) \right\| \to 0$$

as $t_2 - t_1 \to 0$. This further implies that $A_2(C)$ is equi-continuous on J.

On the other hand, by (H3), we have

$$\left\| Z(t) \right\| = \left\| (A_2 x)(t) \right\| \leq \int_0^t (t - s)^{\sigma - 1} \left\| \mathcal{V}_\sigma(t - s) h(s, x(s)) \right\| \, ds \leq \frac{M \widetilde{M} b^\sigma}{\Gamma(\sigma + 1)}, \quad \forall t \in J.$$

It follows that $\|Z\|_C \leq \frac{M\widetilde{M}b^\sigma}{\Gamma(\sigma+1)}$. Hence $A_2(C)$ is uniformly bounded in E. By the Arzela-Ascoli theorem, $A_2 : D \to D$ is partially compact. □

Theorem 4.7 *Let the hypotheses* (H1)-(H4) *hold. Then the fractional evolution system* (4.1) *has a coupled mild solution on J.*

Proof Define two operators A_1 and A_2 as in (4.2) and (4.3). By Lemmas 4.5 and 4.6, we deduce that $A_1 : D \to D$ is Φ-orbitally continuous, nondecreasing and a partially nonlinear $\mathfrak{D}$-contraction as well as $A_2 : D \to D$ is Φ-orbitally continuous, nondecreasing and partially compact.

To apply Theorem 3.3, it remains to prove that there is an element $\overline{x} \in D$ satisfying $\overline{x} \leq A_1\overline{x} + A_2 y$ for all $y \in D$. By (H4), there is an elements $\overline{x} \in D$ satisfying

$$
\begin{cases}
{}^C D_t^\sigma \overline{x}(t) + A\overline{x}(t) \leq f(t,\overline{x}(t)) + h(t,y(t)), & t \in J, \\
\overline{x}(0) \leq \tau_0 \in X,
\end{cases}
$$

for all $y \in D$. Let $F(t) = D^\sigma \overline{x}(t) + A\overline{x}(t)$, $t \in J$. Then we have

$$
\begin{aligned}
\overline{x}(t) &= \mathcal{U}_\sigma(t)\overline{x}(0) + \int_0^t (t-s)^{\sigma-1}\mathcal{V}_\sigma(t-s)F(s)\,ds \\
&\leq \mathcal{U}_\sigma(t)\tau_0 + \int_0^t (t-s)^{\sigma-1}\mathcal{V}_\sigma(t-s)\big[f\big(t,\overline{x}(t)\big) + h\big(t,y(t)\big)\big]\,ds \\
&= (A_1\overline{x})(t) + (A_2 y)(t)
\end{aligned}
$$

for all $y \in D$ and $t \in J$. Hence all the conditions of Theorem 3.3 are satisfied. By Theorem 3.3, the system (4.4) has a coupled fixed point in $\widehat{E}$. Therefore, the fractional evolution system (4.1) has a coupled mild solution in $\widehat{E}$. □

By Theorem 4.7, we can obtain the following corollaries easily.

Corollary 4.8 *Let the hypotheses* (H1), (H3) *and* (H4) *hold. In addition, the following condition is satisfied*:

(H2)′ *The function $f : J \times X \to X$ is continuous in x for all $t \in J$ and there is a constant $\beta \in (0, \frac{\Gamma(\sigma+1)}{Mb^\sigma})$ such that*

$$
0 \leq f(t,u) - f(t,v) \leq \beta(u-v), \quad \forall u \geq v, t \in J.
$$

Then the fractional evolution system (4.1) *has a coupled mild solution on J.*

Corollary 4.9 *Let the hypotheses* (H1), (H3) *and* (H4) *hold. In addition, the following condition is satisfied*:

(H2)″ *The function $f : J \times X \to X$ is continuous in x for all $t \in J$ and there is a constant $\gamma \in (0, \frac{\Gamma(\sigma+1)}{Mb^\sigma})$ such that*

$$
0 \leq f(t,u) - f(t,v) \leq \frac{\gamma\|u-v\|}{1+\|u-v\|}, \quad \forall u \geq v, t \in J.
$$

Then the fractional evolution system (4.1) *has a coupled mild solution on J.*

5 Applications

In this section, we apply the obtained abstract results to the following fractional hybrid dynamic system:

$$\begin{cases} {}^{C}D_t^{\frac{1}{2}}x(z,t) + \frac{\partial x(z,t)}{\partial z} = f(z,t,x(z,t)) + h(z,t,y(z,t)), & (z,t) \in I \times I, \\ {}^{C}D_t^{\frac{1}{2}}y(z,t) + \frac{\partial y(z,t)}{\partial z} = f(z,t,y(z,t)) + h(z,t,x(z,t)), & (z,t) \in I \times I, \\ x(0,t) = x(1,t) = 0, \qquad y(0,t) = y(1,t) = 0, \quad t \in I, \\ x(z,0) = y(z,0) = \tau_0(z), \quad z \in (0,1), \end{cases} \tag{5.1}$$

where $I = [0,1]$, $h : I \times I \times \mathbb{R} \to \mathbb{R}$ is defined by

$$h(z,t,u) = \begin{cases} 2, & u \leq 0, \\ 2 + \frac{5u}{1+2u}, & 0 < u < 2, \\ 4, & u \geq 2. \end{cases}$$

It is clear that $h : I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing and

$$\left| h(z,t,u) \right| \leq 4.$$

This implies that the condition (H3) holds.

Theorem 5.1 *Suppose that the following conditions are satisfied*:

(P1) *The function $f : I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a constant $\rho \in (0, \frac{\sqrt{\pi}}{2})$ and a $\mathcal{D}$-function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(r) < r$ for $r > 0$ such that*

$$0 \leq f\big(z,t,u(z,t)\big) - f\big(z,t,v(z,t)\big) \leq \rho\phi\big(\left|u(z,t) - v(z,t)\right|\big)$$

for all $(z,t) \in I \times I$ and $u,v \in C(I \times I, \mathbb{R})$ with $u \geq v$.

(P2) *There exists a function $\hat{x} \in C(I \times I, \mathbb{R})$ such that*

$$\begin{cases} {}^{C}D_t^{\frac{1}{2}}\hat{x}(z,t) + \frac{\partial \hat{x}(z,t)}{\partial z} \leq f(z,t,\hat{x}(z,t)) + h(z,t,y(z,t)), & (z,t) \in I \times I, \\ \hat{x}(z,0) \leq \tau_0(z), \quad z \in (0,1), \end{cases}$$

for any $y \in C(I \times I, \mathbb{R})$.

Then the fractional hybrid dynamic system (5.1) *has a coupled mild solution.*

Proof Let $X = C(I, \mathbb{R})$. Then X is a Φ-orbitally complete normed linear space with the norm $\|x(t)\|_C = \max_{t \in I} |x(t)|$. Define a positive cone in X by $\overline{K} = \{x \in X : x \geq 0\}$. Then $\overline{K}$ is a closed convex cone in X, which is normal. Define an operator $A : D(A) \subset X \to X$ by

$$Au = u', \quad u \in D(A) := \big\{u \in X : u' \in X, u(0) = u(1) = 0\big\}.$$

It is well known that A generates a C_0-semigroup $S(t)$ $(t \geq 0)$ given by

$$S(t)u(z) = u(t+z), \quad t \geq 0, u \in X.$$

Then $S(t)$ $(t \geq 0)$ is an equi-continuous C_0-semigroup, but it is not compact, and $\sup_{t \in I} \|S(t)\| \leq 1$. This implies that the condition (H1) holds.

Let

$$x(t)(z) = x(z, t),$$

$$y(t)(z) = y(z, t),$$

$$f(t, x(t))(z) = f(z, t, x(z, t)),$$

$$h(t, x(t))(z) = h(z, t, x(z, t)).$$

Then the fractional hybrid dynamic system (5.1) can be rewritten into the abstract fractional evolution system (4.1).

By the assumptions (P1) and (P2), the conditions (H2) and (H4) hold. Hence by Theorem 4.7, the abstract fractional evolution system (4.1) has a coupled mild solution, which is also the coupled mild solution of the fractional hybrid dynamic system (5.1). □

Similarly, using Corollaries 4.8 and 4.9, we can obtain the following theorems.

Theorem 5.2 *Assume that the condition* (P2) *and the following condition are satisfied*:

(P3) *The function $f : I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a constant $\beta \in (0, \frac{\sqrt{\pi}}{2})$ such that*

$$0 \leq f(z, t, u(z, t)) - f(z, t, v(z, t)) \leq \beta(u(z, t) - v(z, t))$$

for all $(z, t) \in I \times I$ and $u, v \in C(I \times I, \mathbb{R})$ with $u \geq v$.

Then the fractional hybrid dynamic system (5.1) *has a coupled mild solution.*

Theorem 5.3 *Let the condition* (P2) *and the following condition hold*:

(P4) *The function $f : I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a constant $\gamma \in (0, \frac{\sqrt{\pi}}{2})$ such that*

$$0 \leq f(z, t, u(z, t)) - f(z, t, v(z, t)) \leq \frac{\gamma \|u - v\|_C}{1 + \|u - v\|_C}$$

for all $(z, t) \in I \times I$ and $u, v \in C(I \times I, \mathbb{R})$ with $u \geq v$.

Then the fractional hybrid dynamic system (5.1) *has a coupled mild solution.*

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