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Existence and uniqueness results for a coupled fractional order systems with the multi-strip and multi-point mixed boundary conditions

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Abstract

This paper is concerned with the existence and uniqueness of solutions for a coupled system of fractional differential equations supplemented with the multi-strip and multi-point mixed boundary conditions. The existence of solutions is derived by applying Leray-Schauder's alternative, while the uniqueness of the solution is established via Banach's contraction principle. We also show the existence and uniqueness results of a positive solution by applying the Krasnoselskii fixed point theorem.

MSC: 26A33; 34B15; 34B18

Keywords: fractional differential systems; Leray-Schauder's alternative; fixed point theorems; multi-strip and multi-point mixed boundary conditions

1 Introduction

Fractional differential equations arise from the studies of complex problems in many engineering and scientific as the mathematical modeling of systems and processes in the fields of physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data. Fractional differential equations are also an excellent tool for the description of hereditary properties of various materials and processes.

Therefore, fractional order boundary value problems, fractional evolution equations, impulsive fractional differential equations, fractional partial differential equations are found to be of significant interest for many researchers [1–4]. The fractional order boundary value problems are especially popular. Since recently, there are several kinds of boundary conditions in the study of fractional order boundary value problem such as two-point, multi-point, periodic/anti-periodic, nonlocal and integral boundary conditions. We can see the details of several recent works on the subject in [5–9].

Another interesting field of recent research is the coupled system of fractional differential equations. It have been shown to be more accurate and realistic that have many applications in real-world problems such as synchronization of chaotic systems [10–12], anomalous diffusion [13], disease models [14, 15], and ecological models [16].

In [17], the authors considered a nonlinear coupled system of Liouville-Caputo type fractional differential equations:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), y(t), {}^cD^{\sigma_1} y(t)), & t \in [0,1], \\ {}^cD^p y(t) = g(t, x(t), {}^cD^{\sigma_2} x(t), y(t)), & t \in [0,1], \end{cases}$$

$$\begin{cases} x(0) = \Psi_1(y), & x'(0) = e_1 y'(w_1), & x(1) = a_1 \int_0^\xi y(s)\,ds + b_1 \sum_{i=1}^{m-2} \alpha_i y(\eta_i), \\ y(0) = \Psi_2(x), & y'(0) = e_2 x'(w_2), & y(1) = a_2 \int_0^\xi x(s)\,ds + b_2 \sum_{i=1}^{m-2} \beta_i x(\eta_i), \end{cases}$$

where $2 < p, q \leq 3$, $1 < \sigma_1, \sigma_2 < 2$, $0 < w_1 < w_2 < \xi < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$. The existence and uniqueness results were obtained by applying Banach's fixed point theorem and Leray-Schauder's alternative. We recommend the reader a series of papers studying on the coupled systems of fractional differential equations [18–27].

Motivated by the above mentioned work, we consider the existence and uniqueness results for the following fractional order differential systems:

$$\begin{cases} D_{0+}^{\alpha_1} u(t) + f_1(t, u(t), v(t)) = 0, & t \in (0,1), \\ D_{0+}^{\alpha_2} v(t) + f_2(t, u(t), v(t)) = 0, & t \in (0,1), \end{cases} \tag{1.1}$$

with the coupled integral and discrete mixed boundary conditions:

$$\begin{cases} u(0) = u'(0) = 0, & u(1) = \sum_{i=1}^n \int_{\xi_i}^{\eta_i} v(s)\,dA_i(s) + \sum_{i=1}^{m-2} b_i v(\sigma_i), \\ v(0) = v'(0) = 0, & v(1) = \sum_{i=1}^n \int_{\xi_i}^{\eta_i} u(s)\,dA_i(s) + \sum_{i=1}^{m-2} b_i u(\sigma_i), \end{cases} \tag{1.2}$$

where $2 < \alpha_k \leq 3$, D^{α_k} is the standard Riemann-Liouville fractional derivative of order α_k, $k = 1,2$; $0 < \xi_i < \eta_i < 1$, $A_i(s)$ is a nondecreasing function of bounded variation in $[0,1]$, $i = 1,2,\ldots,n$; $0 < \sigma_i < 1$, $b_i \geq 0$, $i = 1,2,\ldots,m-2$.

We emphasize that the multi-strip and multi-point mixed boundary conditions in (1.2) state that the value of the unknown function at the right end point $t = 1$ of the given interval is equal to the summation of the Riemann-Stieljes integral values of the unknown function on the sub-interval $[\xi_i, \eta_i]$ ($i = 1,2,\ldots,n$) plus the linear combination of discrete values of the unknown function on σ_i ($i = 1,2,\ldots,m-2$).

By applying Leray-Schauder's alternative, Banach's contraction principle and the fixed point theorems of cone expansion and compression of norm type, the sufficient conditions for the existence and uniqueness results to a general class of multi-strip and multi-point mixed boundary value problem for a coupled system of fractional differential equations are obtained.

2 Preliminaries

We present here the definitions, some lemmas from the theory of fractional calculus and some auxiliary results that will be used to prove our main theorems.

Definition 2.1 ([1]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$\left(I_{0+}^\alpha y\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds, \quad t > 0,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}\,dt$, for $\alpha>0$.

Definition 2.2 ([1]) The Riemann-Liouville fractional derivative of order $\alpha\geq 0$ for a function $y:(0,\infty)\to\mathbb{R}$ is given by

$$\left(D_{0+}^{\alpha}y\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t y(s)(t-s)^{n-\alpha-1}\,ds,\quad t>0,$$

where $n=[\alpha]+1$, provided that the right-hand side is pointwise defined on $[0,\infty)$. The notation $[\alpha]$ stands for the largest integer not greater than α. We also denote the Riemann-Liouville fractional derivative of y by $D_{0+}^{\alpha}y(t)$. If $\alpha=m\in\mathbb{N}$ then $D_{0+}^{m}y(t)=y^{(m)}(t)$ for $t>0$, and if $\alpha=0$ then $D_{0+}^{0}y(t)=y(t)$ for $t>0$.

Lemma 2.1 *Let $\alpha>0$ and $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}$; that is, n is the smallest integer greater than or equal to α. Then the solutions of the fractional differential equation $D_{0+}^{\alpha}u(t)=0$, $0<t<1$, are*

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},\quad 0<t<1,$$

where $c_1,c_2,\ldots,c_n$ are arbitrary real constants.

Lemma 2.2 *Let $\alpha>0$, n be the smallest integer greater than or equal to α $(n-1<\alpha\leq n)$ and $y\in L^1(0,1)$. The solutions of the fractional equation $D_{0+}^{\alpha}u(t)+y(t)=0$, $0<t<1$, are*

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+c_1t^{\alpha-1}+\cdots+c_nt^{\alpha-n},\quad 0<t<1,$$

where $c_1,c_2,\ldots,c_n$ are arbitrary real constants.

Now we derive the corresponding Green's function for the boundary value problem (1.1)-(1.2) and obtain some properties of the Green's function.

Lemma 2.3 *For $x,y\in L^1(0,1)$, boundary value problem*

$$\begin{cases}D_{0+}^{\alpha_1}u(t)+x(t)=0, & D_{0+}^{\alpha_2}v(t)+y(t)=0,\quad t\in(0,1),\\ u(0)=u'(0)=0, & u(1)=\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}v(s)\,dA_i(s)+\sum_{i=1}^{m-2}b_iv(\sigma_i),\\ v(0)=v'(0)=0, & v(1)=\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}u(s)\,dA_i(s)+\sum_{i=1}^{m-2}b_iu(\sigma_i)\end{cases}\tag{2.1}$$

has an integral representation

$$\begin{cases}u(t)=\int_0^1K_1(t,s)x(s)\,ds+\int_0^1H_1(t,s)y(s)\,ds,\\ v(t)=\int_0^1K_2(t,s)y(s)\,ds+\int_0^1H_2(t,s)x(s)\,ds,\end{cases}\tag{2.2}$$

where

$$\begin{aligned} K_1(t,s) &= g_1(t,s) + \frac{l_2 t^{\alpha_1-1}}{1-l_1 l_2}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} g_1(t,s)\,dA_i(t) + \sum_{i=1}^{m-2} b_i g_1(\sigma_i,s)\right], \\ H_1(t,s) &= \frac{t^{\alpha_1-1}}{1-l_1 l_2}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} g_2(t,s)\,dA_i(t) + \sum_{i=1}^{m-2} b_i g_2(\sigma_i,s)\right], \end{aligned} \tag{2.3}$$

$$\begin{aligned} K_2(t,s) &= g_2(t,s) + \frac{l_1 t^{\alpha_2-1}}{1-l_1 l_2}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} g_2(t,s)\,dA_i(t) + \sum_{i=1}^{m-2} b_i g_2(\sigma_i,s)\right], \\ H_2(t,s) &= \frac{t^{\alpha_2-1}}{1-l_1 l_2}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} g_1(t,s)\,dA_i(t) + \sum_{i=1}^{m-2} b_i g_1(\sigma_i,s)\right], \end{aligned} \tag{2.4}$$

$$g_k(t,s) = \frac{1}{\Gamma(\alpha_k)}\begin{cases} t^{\alpha_k-1}(1-s)^{\alpha_k-1} - (t-s)^{\alpha_k-1}, & 0 \le s \le t \le 1, \\ t^{\alpha_k-1}(1-s)^{\alpha_k-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.5}$$

$$l_k = \sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} t^{\alpha_k-1}\,dA_i(t) + \sum_{i=1}^{m-2} b_i \sigma_i^{\alpha_k-1}, \quad k = 1,2. \tag{2.6}$$

Proof From Lemma 2.2, we can reduce (2.1) to the following equivalent integral equations:

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} x(s)\,ds + c_{11}t^{\alpha_1-1} + c_{12}t^{\alpha_1-2} + c_{13}t^{\alpha_1-3}, \\ v(t) &= -\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} y(s)\,ds + c_{21}t^{\alpha_2-1} + c_{22}t^{\alpha_2-2} + c_{23}t^{\alpha_2-3}. \end{aligned} \tag{2.7}$$

From $u(0) = u'(0) = v(0) = v'(0) = 0$, we have $c_{12} = c_{13} = c_{22} = c_{23} = 0$. Thus (2.7) reduces to

$$\begin{aligned} u(t) &= c_{11}t^{\alpha_1-1} - \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} x(s)\,ds, \\ v(t) &= c_{21}t^{\alpha_2-1} - \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} y(s)\,ds. \end{aligned} \tag{2.8}$$

Using the right side boundary conditions of (1.2) and combining (2.8), we have

$$\begin{aligned} u(t) &= t^{\alpha_1-1}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} v(s)\,dA_i(s) + \sum_{i=1}^{m-2} b_i v(\sigma_i)\right] + \int_0^1 g_1(t,s)x(s)\,ds, \\ v(t) &= t^{\alpha_2-1}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} u(s)\,dA_i(s) + \sum_{i=1}^{m-2} b_i u(\sigma_i)\right] + \int_0^1 g_2(t,s)y(s)\,ds. \end{aligned} \tag{2.9}$$

Then we can get

$$\begin{aligned} &\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} v(t)\,dA_i(t) + \sum_{i=1}^{m-2} b_i v(\sigma_i) \\ &\quad = \sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\int_0^1 g_2(t,s)y(s)\,ds\,dA_i(t) + \sum_{i=1}^{m-2} b_i \int_0^1 g_2(\sigma_i,s)y(s)\,ds \\ &\qquad + \left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} t^{\alpha_2-1}\,dA_i(t) + \sum_{i=1}^{m-2} b_i \sigma_i^{\alpha_2-1}\right]\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} u(t)\,dA_i(t) + \sum_{i=1}^{m-2} b_i u(\sigma_i)\right], \end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} u(t)\,dA_i(t)+\sum_{i=1}^{m-2} b_i u(\sigma_i)\\
&\quad=\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\int_0^1 g_1(t,s)x(s)\,ds\,dA_i(t)+\sum_{i=1}^{m-2} b_i\int_0^1 g_1(\sigma_i,s)x(s)\,ds\\
&\qquad+\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} t^{\alpha_1-1}\,dA_i(t)+\sum_{i=1}^{m-2} b_i\sigma_i^{\alpha_1-1}\right]\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} v(t)\,dA_i(t)+\sum_{i=1}^{m-2} b_i v(\sigma_i)\right]. \qquad (2.11)
\end{aligned}$$

Combining (2.6), (2.10) and (2.11), we get

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} v(t)\,dA_i(t)+\sum_{i=1}^{m-2} b_i v(\sigma_i)\\
&\quad=\frac{1}{1-l_1l_2}\left[\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\int_0^1 g_2(t,s)y(s)\,ds\,dA_i(t)+\sum_{i=1}^{m-2} b_i\int_0^1 g_2(\sigma_i,s)y(s)\,ds\right)\right.\\
&\qquad\left.+l_2\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\int_0^1 g_1(t,s)x(s)\,ds\,dA_i(t)+\sum_{i=1}^{m-2} b_i\int_0^1 g_1(\sigma_i,s)x(s)\,ds\right)\right]. \qquad (2.12)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} u(t)\,dA_i(t)+\sum_{i=1}^{m-2} b_i u(\sigma_i)\\
&\quad=\frac{1}{1-l_1l_2}\left[\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\int_0^1 g_1(t,s)x(s)\,ds\,dA_i(t)+\sum_{i=1}^{m-2} b_i\int_0^1 g_1(\sigma_i,s)x(s)\,ds\right)\right.\\
&\qquad\left.+l_1\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\int_0^1 g_2(t,s)y(s)\,ds\,dA_i(t)+\sum_{i=1}^{m-2} b_i\int_0^1 g_2(\sigma_i,s)y(s)\,ds\right)\right], \qquad (2.13)
\end{aligned}$$

where l_k ($k=1,2$) is defined by (2.6). From (2.9), (2.12) and (2.13), we have

$$\begin{aligned}
u(t)&=\frac{t^{\alpha_1-1}}{1-l_1l_2}\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\int_0^1 g_2(t,s)y(s)\,ds\,dA_i(t)+\sum_{i=1}^{m-2} b_i\int_0^1 g_2(\sigma_i,s)y(s)\,ds\right)\\
&\quad+\frac{l_2t^{\alpha_1-1}}{1-l_1l_2}\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\int_0^1 g_1(t,s)x(s)\,ds\,dA_i(t)+\sum_{i=1}^{m-2} b_i\int_0^1 g_1(\sigma_i,s)x(s)\,ds\right)\\
&\quad+\int_0^1 g_1(t,s)x(s)\,ds\\
&=\int_0^1 g_1(t,s)x(s)\,ds+\frac{l_2t^{\alpha_1-1}}{1-l_1l_2}\int_0^1 x(s)\,ds\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} g_1(t,s)\,dA_i(t)+\sum_{i=1}^{m-2} b_i g_1(\sigma_i,s)\right)\\
&\quad+\frac{t^{\alpha_1-1}}{1-l_1l_2}\int_0^1 y(s)\,ds\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} g_2(t,s)\,dA_i(t)+\sum_{i=1}^{m-2} b_i g_2(\sigma_i,s)\right)\\
&=\int_0^1 K_1(t,s)x(s)\,ds+\int_0^1 H_1(t,s)y(s)\,ds,
\end{aligned}$$

$$
\begin{aligned}
v(t) &= \frac{t^{\alpha_2-1}}{1-l_1l_2}\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\int_0^1 g_1(t,s)x(s)\,ds\,dA_i(t)+\sum_{i=1}^{m-2}b_i\int_0^1 g_1(\sigma_i,s)x(s)\,ds\right)\\
&\quad+\frac{l_1t^{\alpha_2-1}}{1-l_1l_2}\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\int_0^1 g_2(t,s)y(s)\,ds\,dA_i(t)+\sum_{i=1}^{m-2}b_i\int_0^1 g_2(\sigma_i,s)y(s)\,ds\right)\\
&\quad+\int_0^1 g_2(t,s)y(s)\,ds\\
&=\int_0^1 g_2(t,s)y(s)\,ds+\frac{l_1t^{\alpha_2-1}}{1-l_1l_2}\int_0^1 y(s)\,ds\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} g_2(t,s)\,dA_i(t)+\sum_{i=1}^{m-2}b_ig_2(\sigma_i,s)\right)\\
&\quad+\frac{t^{\alpha_2-1}}{1-l_1l_2}\int_0^1 x(s)\,ds\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} g_1(t,s)\,dA_i(t)+\sum_{i=1}^{m-2}b_ig_1(\sigma_i,s)\right)\\
&=\int_0^1 K_2(t,s)y(s)\,ds+\int_0^1 H_2(t,s)x(s)\,ds.
\end{aligned}
$$

This completes the proof of the lemma. □

Lemma 2.4 *Let* $\psi_k(t)=t^{\alpha_k-1}(1-t)$, $\varphi_k(t)=(1-t)^{\alpha_k-1}t$, *for* $t\in[0,1]$, $k=1,2$. *The function* $g_k(t,s)$ *defined by* (2.5) *has the following properties*:

(1) $g_k(t,s)$ *is continuous for* $(t,s)\in[0,1]\times[0,1]$ *and* $g_k(t,s)>0$ *for* $t,s\in(0,1)$;

(2) $\psi_k(t)\varphi_k(s)\le\Gamma(\alpha_k)g_k(t,s)\le(\alpha_k-1)\varphi_k(s)$ *for* $t,s\in[0,1]$;

(3) $\psi_k(t)\varphi_k(s)\le\Gamma(\alpha_k)g_k(t,s)\le(\alpha_k-1)\psi_k(t)$ *for* $t,s\in[0,1]$.

Proof (1) The continuity of g_k is easily checked.

(2) For $0\le s\le t\le 1$, we have

$$
\begin{aligned}
\Gamma(\alpha_k)g_k(t,s) &= t^{\alpha_k-1}(1-s)^{\alpha_k-1}-(t-s)^{\alpha_k-1}\\
&=(\alpha_k-1)\int_{t-s}^{t(1-s)}x^{\alpha_k-2}\,dx\\
&\le(\alpha_k-1)t^{\alpha_k-2}(1-s)^{\alpha_k-2}\big[(t-ts)-(t-s)\big]\\
&=(\alpha_k-1)t^{\alpha_k-2}(1-s)^{\alpha_k-2}(1-t)s\\
&\le(\alpha_k-1)(1-s)^{\alpha_k-2}(1-s)s\\
&=(\alpha_k-1)\varphi_k(s),\\
\Gamma(\alpha_k)g_k(t,s) &= t^{\alpha_k-1}(1-s)^{\alpha_k-1}-(t-s)^{\alpha_k-1}\\
&=t^{\alpha_k-2}(1-s)^{\alpha_k-2}(t-ts)-(t-s)^{\alpha_k-2}(t-s)\\
&\ge t^{\alpha_k-2}(1-s)^{\alpha_k-2}\big[(t-ts)-(t-s)\big]\\
&=t^{\alpha_k-2}(1-s)^{\alpha_k-2}(s-ts)\\
&\ge(1-s)^{\alpha_k-2}(s-ts)t^{\alpha_k-2}t(1-s)\\
&=\psi_k(t)\varphi_k(s).
\end{aligned}
$$

For $0 \le t \le s \le 1$, we have

$$\begin{aligned}
\Gamma(\alpha_k)g_k(t,s) &= t^{\alpha_k-1}(1-s)^{\alpha_k-1} = t^{\alpha_k-2}(1-s)^{\alpha_k-1}t \\
&\le (\alpha_k-1)(1-s)^{\alpha_k-1}s \\
&= (\alpha_k-1)\varphi_k(s), \\
\Gamma(\alpha_k)g_k(t,s) &= t^{\alpha_k-1}(1-s)^{\alpha_k-1} \ge t^{\alpha_k-1}(1-t)(1-s)^{\alpha_k-1}s \\
&= \psi_k(t)\varphi_k(s).
\end{aligned}$$

(3) We just need to prove the right end of the inequalities. For $0 \le s \le t \le 1$, we get

$$\begin{aligned}
\Gamma(\alpha_k)g_k(t,s) &\le (\alpha_k-1)t^{\alpha_k-2}(1-s)^{\alpha_k-2}(1-t)s \\
&\le (\alpha_k-1)t^{\alpha_k-2}(1-t)t \\
&= (\alpha_k-1)\psi_k(t).
\end{aligned}$$

For $0 \le t \le s \le 1$, we get

$$\begin{aligned}
\Gamma(\alpha_k)g_k(t,s) &= t^{\alpha_k-1}(1-s)^{\alpha_k-1} \le t^{\alpha_k-1}(1-s) \\
&\le (\alpha_k-1)t^{\alpha_k-1}(1-s) \\
&\le (\alpha_k-1)t^{\alpha_k-1}(1-t) \\
&= (\alpha_k-1)\psi_k(t).
\end{aligned}$$

This completes the proof of the lemma. □

Denote

$$\begin{aligned}
\varrho_1 &= \frac{l_2}{\Gamma(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\psi_1(t)\,dA_i(t) + \sum_{i=1}^{m-2}b_i\psi_1(\sigma_i)\right], \\
\varrho_2 &= \frac{1}{\Gamma(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\psi_2(t)\,dA_i(t) + \sum_{i=1}^{m-2}b_i\psi_2(\sigma_i)\right], \\
\varrho_3 &= \frac{l_1}{\Gamma(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\psi_2(t)\,dA_i(t) + \sum_{i=1}^{m-2}b_i\psi_2(\sigma_i)\right], \\
\varrho_4 &= \frac{1}{\Gamma(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}\psi_1(t)\,dA_i(t) + \sum_{i=1}^{m-2}b_i\psi_1(\sigma_i)\right], \\
\rho_1 &= \frac{\alpha_1-1}{\Gamma(\alpha_1)}\left[1 + \frac{l_2}{1-l_1l_2}\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}dA_i(t) + \sum_{i=1}^{m-2}b_i\right)\right], \\
\rho_2 &= \frac{\alpha_2-1}{\Gamma(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}dA_i(t) + \sum_{i=1}^{m-2}b_i\right], \\
\rho_3 &= \frac{\alpha_2-1}{\Gamma(\alpha_2)}\left[1 + \frac{l_1}{1-l_1l_2}\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i}dA_i(t) + \sum_{i=1}^{m-2}b_i\right)\right],
\end{aligned}$$

$$\rho_4 = \frac{\alpha_1 - 1}{\Gamma(\alpha_1)(1 - l_1 l_2)} \left[\sum_{i=1}^{n} \int_{\xi_i}^{\eta_i} dA_i(t) + \sum_{i=1}^{m-2} b_i \right],$$

$$\varrho = \min\{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}, \qquad \rho = \max\{\rho_1, \rho_2, \rho_3, \rho_4\}.$$

Lemma 2.5 *For* $(t,s) \in [0,1] \times [0,1]$, *the functions* $K_k(t,s)$ *and* $H_k(t,s)$ $(k = 1,2)$ *defined by* (2.3) *and* (2.4) *satisfy the following results*:

(1) $K_k(t,s)$ *and* $H_k(t,s)$ *are continuous*, $K_k(t,s) \ge 0$ *and* $H_k(t,s) \ge 0$;

(2) $\varrho t^{\alpha_k - 1}\varphi_k(s) \le K_k(t,s) \le \rho\varphi_k(s)$;

(3) $\varrho t^{\alpha_1 - 1}\varphi_2(s) \le H_1(t,s) \le \rho\varphi_2(s)$, $\varrho t^{\alpha_2 - 1}\varphi_1(s) \le H_2(t,s) \le \rho\varphi_1(s)$;

(4) $H_k(t,s) \le \rho t^{\alpha_k - 1}$, $K_k(t,s) \le \rho t^{\alpha_k - 1}$.

Proof The continuity of K_k and H_k $(k = 1,2)$ is easily checked. According to the property (2) of Lemma 2.4 and (2.3), we have

$$\begin{aligned}
K_1(t,s) &= g_1(t,s) + \frac{l_2 t^{\alpha_1 - 1}}{1 - l_1 l_2} \left[\sum_{i=1}^{n} \int_{\xi_i}^{\eta_i} g_1(t,s)\, dA_i(t) + \sum_{i=1}^{m-2} b_i g_1(\sigma_i, s) \right] \\
&\ge \frac{l_2 t^{\alpha_1 - 1}}{1 - l_1 l_2} \left[\sum_{i=1}^{n} \int_{\xi_i}^{\eta_i} \frac{\psi_1(t)\varphi_1(s)}{\Gamma(\alpha_1)}\, dA_i(t) + \sum_{i=1}^{m-2} b_i \frac{\psi_1(\sigma_i)\varphi_1(s)}{\Gamma(\alpha_1)} \right] \\
&= \varrho_1 t^{\alpha_1 - 1}\varphi_1(s), \\
K_1(t,s) &\le \frac{(\alpha_1 - 1)\varphi_1(s)}{\Gamma(\alpha_1)} + \frac{l_2 t^{\alpha_1 - 1}}{1 - l_1 l_2} \left[\sum_{i=1}^{n} \int_{\xi_i}^{\eta_i} \frac{(\alpha_1 - 1)\varphi_1(s)}{\Gamma(\alpha_1)}\, dA_i(t) + \sum_{i=1}^{m-2} b_i \frac{(\alpha_1 - 1)\varphi_1(s)}{\Gamma(\alpha_1)} \right] \\
&\le \frac{\alpha_1 - 1}{\Gamma(\alpha_1)} \left[1 + \frac{l_2}{1 - l_1 l_2} \left(\sum_{i=1}^{m-2} \int_{\xi_i}^{\eta_i} dA_i(t) + \sum_{i=1}^{m-2} b_i \right) \right] \varphi_1(s) \\
&= \rho_1 \varphi_1(s), \\
H_1(t,s) &= \frac{t^{\alpha_1 - 1}}{1 - l_1 l_2} \left[\sum_{i=1}^{n} \int_{\xi_i}^{\eta_i} g_2(t,s)\, dA_i(t) + \sum_{i=1}^{m-2} b_i g_2(\sigma_i, s) \right] \\
&\ge \frac{t^{\alpha_1 - 1}}{1 - l_1 l_2} \left[\sum_{i=1}^{n} \int_{\xi_i}^{\eta_i} \frac{\psi_2(t)\varphi_2(s)}{\Gamma(\alpha_2)}\, dA_i(t) + \sum_{i=1}^{m-2} b_i \frac{\psi_2(\sigma_i)\varphi_2(s)}{\Gamma(\alpha_2)} \right] \\
&= \varrho_2 t^{\alpha_1 - 1}\varphi_2(s),
\end{aligned}$$

and

$$\begin{aligned}
H_1(t,s) &\le \frac{t^{\alpha_1 - 1}}{1 - l_1 l_2} \left[\sum_{i=1}^{n} \int_{\xi_i}^{\eta_i} \frac{(\alpha_2 - 1)\varphi_2(s)}{\Gamma(\alpha_2)}\, dA_i(t) + \sum_{i=1}^{m-2} b_i \frac{(\alpha_2 - 1)\varphi_2(s)}{\Gamma(\alpha_2)} \right] \\
&\le \frac{\alpha_2 - 1}{\Gamma(\alpha_2)(1 - l_1 l_2)} \left[\sum_{i=1}^{n} \int_{\xi_i}^{\eta_i} dA_i(t) + \sum_{i=1}^{m-2} b_i \right] \varphi_2(s) \\
&= \rho_2 \varphi_2(s).
\end{aligned}$$

Similarly, from the property (2) of Lemma 2.4 and (2.4), we get

$$\begin{aligned}
&\varrho_3 t^{\alpha_2-1}\varphi_2(s) \le K_2(t,s) \le \rho_3\varphi_2(s), \\
&\varrho_4 t^{\alpha_2-1}\varphi_1(s) \le H_2(t,s) \le \rho_4\varphi_1(s).
\end{aligned}$$

On the other hand, according to the property (3) of Lemma 2.4 and (2.3), we obtain

$$\begin{aligned}
K_1(t,s) &= g_1(t,s) + \frac{l_2 t^{\alpha_1-1}}{1-l_1 l_2}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} g_1(t,s)\,dA_i(t) + \sum_{i=1}^{m-2} b_i g_1(\sigma_i,s)\right] \\
&\le \frac{\alpha_1-1}{\Gamma(\alpha_1)}\left[\psi_1(t) + \frac{l_2 t^{\alpha_1-1}}{1-l_1 l_2}\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} \psi_1(t)\,dA_i(t) + \sum_{i=1}^{m-2} b_i\psi_1(\sigma_i)\right)\right] \\
&\le \frac{\alpha_1-1}{\Gamma(\alpha_1)}\left[t^{\alpha_1-1} + \frac{l_2 t^{\alpha_1-1}}{1-l_1 l_2}\left(\sum_{i=1}^{n}\int_{\xi_i}^{\eta_1} dA_i(t) + \sum_{i=1}^{m-2} b_i\right)\right] \\
&= \rho_1 t^{\alpha_1-1}, \\
H_1(t,s) &= \frac{t^{\alpha_1-1}}{1-l_1 l_2}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} g_2(t,s)\,dA_i(t) + \sum_{i=1}^{m-2} b_i g_2(\sigma_i,s)\right] \\
&\le \frac{(\alpha_2-1)t^{\alpha_1-1}}{\Gamma(\alpha_2)(1-l_1 l_2)}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} t^{\alpha_2-1}(1-t)\,dA_i(t) + \sum_{i=1}^{m-2} b_i\sigma_i^{\alpha_2-1}(1-\sigma_i)\right] \\
&\le \frac{(\alpha_2-1)t^{\alpha_1-1}}{\Gamma(\alpha_2)(1-l_1 l_2)}\left[\sum_{i=1}^{n}\int_{\xi_i}^{\eta_i} dA_i(t) + \sum_{i=1}^{m-2} b_i\right] = \rho_2 t^{\alpha_1-1}.
\end{aligned}$$

Similarly, from the property (3) of Lemma 2.4 and (2.4), we get

$$K_2(t,s) \le \rho_3 t^{\alpha_2-1}, \qquad H_2(t,s) \le \rho_4 t^{\alpha_2-1}.$$

This completes the proof of the lemma. □

Remark 2.1 From the Lemma 2.5, for $t,\tau,s \in [0,1]$, we have

$$\begin{aligned}
&K_1(t,s) \ge \omega t^{\alpha_1-1}H_2(\tau,s), \qquad K_2(t,s) \ge \omega t^{\alpha_2-1}H_1(\tau,s), \\
&H_1(t,s) \ge \omega t^{\alpha_1-1}K_2(\tau,s), \qquad H_2(t,s) \ge \omega t^{\alpha_2-1}K_1(\tau,s), \\
&K_i(t,s) \ge \omega t^{\alpha_i-1}K_i(\tau,s), \qquad H_i(t,s) \ge \omega t^{\alpha_i-1}H_i(\tau,s),
\end{aligned}$$

where $\omega = \frac{\varrho}{\rho}$ and $0 < \omega < 1$.

Lemma 2.6 (Leray-Schauder alternative) *Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\varepsilon(F) = \{x \in E : x = \lambda F(x), 0 < \lambda < 1\}.$$

Then either the set $\varepsilon(F)$ is unbounded, or F has at least one fixed point.

Lemma 2.7 *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that either*

(a) $\|Sw\| \leq \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \geq \|w\|$, $w \in P \cap \partial\Omega_2$, *or*

(b) $\|Sw\| \geq \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \leq \|w\|$, $w \in P \cap \partial\Omega_2$.

Then S has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let us introduce the space $X = \{u(t) \mid u(t) \in C[0,1]\}$ endowed with the norm $\|u\| = \max_{\in[0,1]} |u(t)|$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{v(t) \mid v(t) \in C[0,1]\}$ endowed with the norm $\|v\| = \max_{t\in[0,1]} |v(t)|$. The product space $(X \times Y, \|(u,v)\|)$ is also a Banach space with the norm $\|(u,v)\| = \|u\| + \|v\|$.

In view of Lemma 2.3, we define the operator $T : X \times Y \to X \times Y$ by

$$T(u,v)(t) = \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix},$$

where

$$T_1(u,v)(t) = \int_0^1 K_1(t,s) f_1\bigl(s,u(s),v(s)\bigr)\,ds + \int_0^1 H_1(t,s) f_2\bigl(s,u(s),v(s)\bigr)\,ds, \tag{2.14}$$

$$T_2(u,v)(t) = \int_0^1 K_2(t,s) f_2\bigl(s,u(s),v(s)\bigr)\,ds + \int_0^1 H_2(t,s) f_1\bigl(s,u(s),v(s)\bigr)\,ds. \tag{2.15}$$

Lemma 2.8 *The operator $T : X \times Y \to X \times Y$ is completely continuous.*

Proof By continuity of the functions f_1 and f_2, the operator T is continuous.

Let $\Omega \subset X \times Y$ be bounded. Then there exist positive constants A_1 and A_2 such that

$$\bigl|f_1\bigl(t,u(t),v(t)\bigr)\bigr| \leq A_1, \qquad \bigl|f_2\bigl(t,u(t),v(t)\bigr)\bigr| \leq A_2, \quad \forall (u,v) \in \Omega.$$

Then, for any $(u,v) \in \Omega$, we have

$$\begin{aligned} \bigl|T_1(u,v)(t)\bigr| &\leq \left|A_1 \int_0^1 K_1(t,s)\,ds + A_2 \int_0^1 H_1(t,s)\,ds\right| \\ &\leq A_1 \rho t^{\alpha_1 - 1} + A_2 \rho t^{\alpha_1 - 1} \\ &= \rho(A_1 + A_2) t^{\alpha_1 - 1}, \end{aligned}$$

which implies that

$$\bigl\|T_1(u,v)\bigr\| \leq \rho(A_1 + A_2).$$

Similarly, we get

$$\bigl\|T_2(u,v)\bigr\| \leq \rho(A_1 + A_2).$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. Let $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$. Then we have

$$\begin{aligned}&\bigl|T_1\bigl(u(t_2), v(t_2)\bigr) - T_1\bigl(u(t_1), v(t_1)\bigr)\bigr| \\ &\quad \leq A_1\left|\int_0^1 \bigl(K_1(t_2,s) - K_1(t_1,s)\bigr)\,ds\right| + A_2\left|\int_0^1 \bigl(H_1(t_2,s) - H_1(t_1,s)\bigr)\,ds\right|.\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}&\bigl|T_2\bigl(u(t_2), v(t_2)\bigr) - T_2\bigl(u(t_1), v(t_1)\bigr)\bigr| \\ &\quad \leq A_1\left|\int_0^1 \bigl(H_2(t_2,s) - H_2(t_1,s)\bigr)\,ds\right| + A_2\left|\int_0^1 \bigl(K_2(t_2,s) - K_2(t_1,s)\bigr)\,ds\right|.\end{aligned}$$

Therefore, the operator $T(u,v)$ is equicontinuous, and thus the operator $T(u,v)$ is completely continuous. □

3 Existence and uniqueness results

Let us introduce the following hypotheses which are used hereafter.

(H_1) Assume there exist real constants $\mu_i, \lambda_i \geq 0$ $(i = 1,2)$ and $\mu_0, \lambda_0 > 0$ such that $\forall u_i \in \mathbb{R}$, $i = 1,2$, we have

$$\begin{aligned}&\bigl|f_1(t,u_1,u_2)\bigr| \leq \mu_0 + \mu_1|u_1| + \mu_2|u_2|, \\ &\bigl|f_2(t,u_1,u_2)\bigr| \leq \lambda_0 + \lambda_1|u_1| + \lambda_2|u_2|.\end{aligned}$$

(H_2) Assume that $f_1, f_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants m_i, n_i, $i = 1,2$ such that, for all $t \in [0,1]$ and $u_i, v_i \in \mathbb{R}$, $i = 1,2$,

$$\begin{aligned}&\bigl|f_1(t,u_1,u_2) - f_1(t,v_1,v_2)\bigr| \leq m_1|u_1 - v_1| + m_2|u_2 - v_2|, \\ &\bigl|f_2(t,u_1,u_2) - f_2(t,v_1,v_2)\bigr| \leq n_1|u_1 - v_1| + n_2|u_2 - v_2|.\end{aligned}$$

For the sake of convenience, we set

$$M = \min\bigl\{1 - 2\rho(\mu_1 + \lambda_1), 1 - 2\rho(\mu_2 + \lambda_2)\bigr\}. \tag{3.1}$$

The first result is based on Leray-Schauder's alternative.

Theorem 3.1 *Assume that* (H_1) *holds. In addition it is assumed that*

$$2\rho(\mu_1 + \lambda_1) < 1, \qquad 2\rho(\mu_2 + \lambda_2) < 1.$$

Then the boundary value problem (1.1)-(1.2) *has at least one solution.*

Proof It will be verified that the set $\varepsilon = \{(u,v) \in X \times Y \mid (u,v) = \lambda T(u,v), 0 \leq \lambda \leq 1\}$ is bounded. Let $(u,v) \in \varepsilon$, then $(u,v) = \lambda T(u,v)$. For any $t \in [0,1]$, we have

$$u(t) = \lambda T_1(u,v)(t), \qquad v(t) = \lambda T_2(u,v)(t).$$

From (H_1), we have

$$\begin{aligned}|u(t)| &= |\lambda T_1(u,v)(t)| \leq |T_1(u,v)(t)| \\ &\leq \rho t^{\alpha_1-1}\big(\mu_0 + \mu_1\|u\| + \mu_2\|v\| + \lambda_0 + \lambda_1\|u\| + \lambda_2\|v\|\big)\end{aligned}$$

and

$$\begin{aligned}|v(t)| &= |\lambda T_2(u,v)(t)| \leq |T_2(u,v)(t)| \\ &\leq \rho t^{\alpha_2-1}\big(\mu_0 + \mu_1\|u\| + \mu_2\|v\| + \lambda_0 + \lambda_1\|u\| + \lambda_2\|v\|\big).\end{aligned}$$

Hence we have

$$\begin{aligned}&\|u\| \leq \rho\big(\mu_0 + \mu_1\|u\| + \mu_2\|v\| + \lambda_0 + \lambda_1\|u\| + \lambda_2\|v\|\big), \\ &\|v\| \leq \rho\big(\mu_0 + \mu_1\|u\| + \mu_2\|v\| + \lambda_0 + \lambda_1\|u\| + \lambda_2\|v\|\big),\end{aligned}$$

which imply that

$$\|u\| + \|v\| \leq 2\rho\big(\mu_0 + \lambda_0 + (\mu_1 + \lambda_1)\|u\| + (\mu_2 + \lambda_2)\|v\|\big).$$

Consequently,

$$\|(u,v)\| \leq \frac{2\rho(\mu_0 + \lambda_0)}{M},$$

for any $t \in [0,1]$, where M is defined by (3.1), which proves that ε is bounded. Thus, by Lemma 2.6, the operator T has at least one fixed point. Hence the boundary value problem (1.1)-(1.2) has at least one solution. The proof is complete. □

In the second result, we prove existence and uniqueness of solutions of the boundary value problem (1.1)-(1.2) via Banach's contraction principle.

Theorem 3.2 *Assume that* (H_2) *holds. In addition, assume that*

$$2\rho(m_1 + m_2 + n_1 + n_2) < 1.$$

Then the boundary value problem (1.1)-(1.2) *has a unique solution.*

Proof Define $\sup_{t\in[0,1]} f_1(t,0,0) = N_1 < \infty$ and $\sup_{t\in[0,1]} f_2(t,0,0) = N_2 < \infty$ such that

$$r \geq \frac{2\rho(N_1 + N_2)}{1 - 2\rho(m_1 + m_2 + n_1 + n_2)}.$$

We show that $TB_r \subset B_r$, where $B_r = \{(u,v) \in X \times Y : \|(u,v)\| \leq r\}$. For $(u,v) \in B_r$, we have

$$\begin{aligned}|T_1(u,v)(t)| = \max_{0\leq t\leq 1}\bigg|&\int_0^1 K_1(t,s)\big|f_1\big(s,u(s),v(s)\big) - f_1(s,0,0)\big| + \big|f_1(s,0,0)\big|\,ds \\ &+ \int_0^1 H_1(t,s)\big|f_2\big(s,u(s),v(s)\big) - f_2(s,0,0)\big| + \big|f_2(s,0,0)\big|\,ds\bigg|\end{aligned}$$

$$\le \rho t^{\alpha_1-1}\big(m_1\|u\| + m_2\|v\| + N_1 + n_1\|u\| + n_2\|v\| + N_2\big)$$
$$\le \rho\big[(m_1+m_2)r + (n_1+n_2)r + N_1 + N_2\big].$$

Hence

$$\big\|T_1(u,v)\big\| \le \rho\big[(m_1+m_2+n_1+n_2)r + N_1 + N_2\big].$$

In the same way, we obtain

$$\big\|T_2(u,v)\big\| \le \rho\big[(m_1+m_2+n_1+n_2)r + N_1 + N_2\big].$$

Consequently, $\|T(u,v)\| \le r$.

Now for $(u_2,v_2),(u_1,v_1) \in X \times Y$, and for any $t \in [0,1]$, we get

$$\begin{aligned}
&\big|T_1(u_2,v_2)(t) - T_1(u_1,v_1)(t)\big| \\
&\quad\le \int_0^1 K_1(t,s)\big|f_1(s,u_2(s)),v_2(s) - f_1\big(s,u_1(s)v_1(s)\big)\big|\,ds \\
&\qquad + \int_0^1 H_1(t,s)\big|f_2\big(s,u_2(s),v_2(s)\big) - f_2\big(s,u_1(s),v_1(s)\big)\big|\,ds \\
&\quad\le \rho t^{\alpha_1-1}\big(m_1|u_2-u_1| + m_2|v_2-v_1| + n_1|u_2-u_1| + n_2|v_2-v_1|\big) \\
&\quad\le \rho\big[(m_1+n_1)\|u_2-u_1\| + (m_2+n_2)\|v_2-v_1\|\big] \\
&\quad\le \rho(m_1+m_2+n_1+n_2)\big(\|u_2-u_1\| + \|v_2-v_1\|\big),
\end{aligned}$$

and consequently we obtain

$$\big\|T_1(u_2,v_2) - T_1(u_1,v_1)\big\| \le \rho(m_1+m_2+n_1+n_2)\big(\|u_2-u_1\| + \|v_2-v_1\|\big). \tag{3.2}$$

Similarly,

$$\big\|T_2(u_2,v_2) - T_2(u_1,v_1)\big\| \le \rho(m_1+m_2+n_1+n_2)\big(\|u_2-u_1\| + \|v_2-v_1\|\big). \tag{3.3}$$

It follows from (3.2) and(3.3) that

$$\big\|T(u_2,v_2) - T(u_1,v_1)\big\| \le 2\rho(m_1+m_2+n_1+n_2)\big(\|u_2-u_1\| + \|v_2-v_1\|\big).$$

Since $2\rho(m_1+m_2+n_1+n_2) < 1$, T is a contraction operator. So, by Banach's fixed point theorem, the operator T has a unique fixed point, which is the unique solution of problem (1.1)-(1.2). This completes the proof. □

Remark 3.1 In the case that the left parts of boundary conditions (1.2) are also the multi-strip and multi-point mixed boundary conditions, the existence and uniqueness conclusions could be obtained by using the definition of the Caputo type fractional q-derivative. This might be our forthcoming work.

Example 3.1 Consider the following fractional boundary value problem:

$$\begin{cases} D_{0+}^{2.5}u(t) = \frac{1}{8(t+2)^2}u(t) + \frac{1}{32}\sin \nu(t) + 1, \\ D_{0+}^{2.5}\nu(t) = \frac{1}{64\pi}\sin(2\pi u(t)) + \frac{1}{32}\nu(t) + \frac{1}{2}, \\ u(0) = u'(0) = 0, \qquad u(1) = \sum_{i=1}^{2}\int_{\xi_i}^{\eta_i} \nu(s)\,dA_i(s) + \sum_{i=1}^{2} b_i\nu(\sigma_i), \\ \nu(0) = \nu'(0) = 0, \qquad \nu(1) = \sum_{i=1}^{2}\int_{\xi_i}^{\eta_i} u(s)\,dA_i(s) + \sum_{i=1}^{2} b_i u(\sigma_i), \end{cases} \tag{3.4}$$

where

$$\begin{aligned} &A_1(t) = 0.5t, \qquad \xi_1 = \frac{1}{8}, \qquad \eta_1 = \frac{3}{8}, \qquad b_1 = \frac{1}{3}, \qquad \sigma_1 = \frac{1}{3}, \\ &A_2(t) = 0.8t, \qquad \xi_2 = \frac{5}{8}, \qquad \eta_2 = \frac{7}{8}, \qquad b_2 = \frac{2}{3}, \qquad \sigma_2 = \frac{2}{3}. \end{aligned}$$

Then considering the condition (H_1)

$$\begin{aligned} &\left|f_1(t,x_1,x_2)\right| \le 1 + \frac{1}{32}|x_1| + \frac{1}{32}|x_2|, \\ &\left|f_2(t,x_1,x_2)\right| \le \frac{1}{2} + \frac{1}{32}|x_1| + \frac{1}{32}|x_2|. \end{aligned}$$

We get $\mu_1 = \frac{1}{32}$, $\mu_2 = \frac{1}{32}$, $\lambda_1 = \frac{1}{32}$, $\lambda_2 = \frac{1}{32}$. As for the condition (H_2)

$$\begin{aligned} &\left|f_1(t,u_1,u_2) - f_1(t,\nu_1,\nu_2)\right| \le \frac{1}{32}|u_1 - u_2| + \frac{1}{32}|\nu_1 - \nu_2|, \\ &\left|f_2(t,u_1,u_2) - f_2(t,\nu_1,\nu_2)\right| \le \frac{1}{32}|u_1 - u_2| + \frac{1}{32}|\nu_1 - \nu_2|. \end{aligned}$$

We get $m_1 = \frac{1}{32}$, $m_2 = \frac{1}{32}$, $n_1 = \frac{1}{32}$, $n_2 = \frac{1}{32}$. By simple computation, we have

$$\begin{aligned} &\rho_1 = \rho_3 = \frac{1.5}{\Gamma(2.5)}\left[1 + \frac{1}{1 - 1_1 l_2}\left(\sum_{i=1}^{2}\int_{\xi_i}^{\eta_i} dA_i(t) + \sum_{i=1}^{2} b_i\right)\right] \approx 3.3559, \\ &\rho_2 = \rho_4 = \frac{1.5}{\Gamma(2.5)(1 - l_1 l_2)}\left[\sum_{i=1}^{2}\int_{\xi_i}^{\eta_i} dA_i(t) + \sum_{i=1}^{2} b_i\right] \approx 2.2276, \end{aligned}$$

with

$$l_1 = l_2 = \sum_{i=1}^{2}\int_{\xi_i}^{\eta_i} t^{1.5}\,dA_i(t) + \sum_{i=1}^{2} b_i\sigma_i^{1.5} \approx 0.5734.$$

Then $\rho = \max\{\rho_1, \rho_2, \rho_3, \rho_4\} \approx 3.3559$. Therefore,

$$2\rho(\mu_1 + \lambda_1) \approx 0.4194 < 1, \qquad \rho(\mu_2 + \lambda_2) \approx 0.4194 < 1.$$

By Theorem 3.1, the coupled boundary value problem (3.4) has at least one solution. We also have

$$2\rho(m_1 + m_2 + n_1 + n_2) \approx 0.8390 < 1.$$

By Theorem 3.2, the coupled boundary value problem (3.4) has a unique solution.

4 Existence and uniqueness results of a positive solution

In this section, X is the same Banach space as the space defined in Section 3. Denote

$$P = \{(u,v) \in X : u(t) \geq \omega t^{\alpha_1 - 1}\|(u,v)\|, v(t) \geq \omega t^{\alpha_2 - 1}\|(u,v)\|, t \in [0,1]\},$$

where ω is defined in Remark 2.1. It can easily be seen that P is a cone in X. For any real constants r and R with $0 < r < R$, define

$$P_r = \{(u,v) \in P : \|(u,v)\| < r\}, \qquad P_{[r,R]} = \{(u,v) \in P : r \leq \|(u,v)\| \leq R\}.$$

In what follows, we list the following assumptions for convenience.

(H_3) $f_1 : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, $f_1(t,u,v)$ is nondecreasing in u and nonincreasing in v, and there exist two constants $\theta_1, \vartheta_1 \in [0,1)$ such that

$$\begin{aligned} &\kappa^{\theta_1} f_1(t,u,v) \leq f_1(t,\kappa u,v), \\ &f_1(t,u,\kappa v) \leq \kappa^{-\vartheta_1} f_1(t,u,v), \quad \forall u,v > 0, \kappa \in (0,1). \end{aligned} \tag{4.1}$$

(H_4) $f_2 : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, $f_2(t,u,v)$ is nonincreasing in u and nondecreasing in v, and there exist two constants $\theta_2, \vartheta_2 \in [0,1)$ such that

$$\begin{aligned} &\kappa^{\theta_2} f_2(t,u,v) \leq f_2(t,u,\kappa v), \\ &f_2(t,\kappa u,v) \leq \kappa^{-\vartheta_2} f_2(t,u,v), \quad \forall u,v > 0, \kappa \in (0,1). \end{aligned} \tag{4.2}$$

(H_5) The following inequalities hold:

$$0 < \int_0^1 \varphi_1(s) f_1(s,1,s^{\alpha_2 - 1})\,ds < +\infty, \qquad 0 < \int_0^1 \varphi_2(s) f_2(s,s^{\alpha_1 - 1},1)\,ds < +\infty,$$

where φ_1 and φ_2 are defined in Lemma 2.4.

Remark 4.1 From assumptions (H_3) and (H_4), we have

$$f_1(s,s^{\alpha_2 - 1},1) \leq f_1(s,1,s^{\alpha_2 - 1}), \qquad f_2(s,1,s^{\alpha_1 - 1}) \leq f_2(s,s^{\alpha_1 - 1},1).$$

This together with (H_5) yields

$$\begin{aligned} &0 < \int_0^1 \varphi_1(s) f_1(s,s^{\alpha_2 - 1},1)\,ds \leq \int_0^1 \varphi_1(s) f_1(s,1,s^{\alpha_2 - 1})\,ds < +\infty, \\ &0 < \int_0^1 \varphi_2(s) f_2(s,1,s^{\alpha_1 - 1})\,ds \leq \int_0^1 \varphi_2(s) f_2(s,s^{\alpha_1 - 1},1)\,ds < +\infty. \end{aligned}$$

Remark 4.2 The inequalities (4.1) and (4.2) imply that

$$f_1(t,\kappa u,v) \leq \kappa^{\theta_1} f_1(t,u,v), \qquad f_1(t,u,v) \leq \kappa^{\vartheta_1} f_1(t,u,\kappa v), \quad \forall u,v > 0, \kappa \in (1,\infty); \tag{4.3}$$

$$f_2(t,u,\kappa v) \leq \kappa^{\theta_2} f_2(t,u,v), \qquad f_2(t,u,v) \leq \kappa^{\vartheta_2} f_2(t,\kappa u,v), \quad \forall u,v > 0, \kappa \in (1,\infty). \tag{4.4}$$

From the above assumptions (H_3), (H_4) and (H_5), for any $(u,v) \in P \setminus \{(0,0)\}$, define the operator $T : P \setminus \{(0,0)\} \to P$ introduced by (2.14) and (2.15). Obviously, (u,v) is a positive solutions of the boundary value problem (1.1)-(1.2) if and only if (u,v) is a fixed point of T in $P \setminus \{(0,0)\}$.

Lemma 4.1 *Assume that* (H_3)-(H_5) *hold. For any* $0 < r_1 < r_2 < +\infty$, $T : P_{[r_1,r_2]} \to P$ *is a completely continuous operator.*

Proof For any $(u,v) \in P \setminus \{(0,0)\}$, we can see that

$$\begin{aligned} &\omega t^{\alpha_1 - 1}\big\|(u,v)\big\| \le u(t) \le \big\|(u,v)\big\|, \\ &\omega t^{\alpha_2 - 1}\big\|(u,v)\big\| \le v(t) \le \big\|(u,v)\big\|, \quad t \in [0,1]. \end{aligned} \tag{4.5}$$

Let $\kappa > 1$ such that $\|(u,v)\|/\kappa < 1$. From (H_3), (H_4), (4.3), (4.4) and (4.5), we have

$$\begin{aligned} f_1\big(t,u(t),v(t)\big) &\le f_1\big(t,\kappa,\omega t^{\alpha_2-1}\big\|(u,v)\big\|\big) \le \kappa^{\theta_1} f_1\left(t,1,\frac{\omega\|(u,v)\|}{\kappa}t^{\alpha_2-1}\right) \\ &\le \kappa^{\theta_1+\vartheta_1}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_1} f_1\big(t,1,t^{\alpha_2-1}\big), \\ f_2\big(t,u(t),v(t)\big) &\le f_2\big(t,\omega t^{\alpha_1-1}\big\|(u,v)\big\|,\kappa\big) \le \kappa^{\theta_2} f_2\left(t,\frac{\omega\|(u,v)\|}{\kappa}t^{\alpha_1-1},1\right) \\ &\le \kappa^{\theta_2+\vartheta_2}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_2} f_2\big(t,t^{\alpha_1-1},1\big). \end{aligned} \tag{4.6}$$

Hence, for any $t \in [0,1]$, by Lemma 2.5 and (4.6), we get

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,s) f_1\big(s,u(s),v(s)\big)\,ds + \int_0^1 H_1(t,s) f_2\big(s,u(s),v(s)\big)\,ds \\ &\le \rho\bigg(\kappa^{\theta_1+\vartheta_1}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_1}\int_0^1 \varphi_1(s) f_1\big(s,1,s^{\alpha_2-1}\big)\,ds \\ &\quad + \kappa^{\theta_2+\vartheta_2}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_2}\int_0^1 \varphi_2(s) f_2\big(s,s^{\alpha_1-1},1\big)\,ds\bigg) \\ &< +\infty, \\ T_2(u,v)(t) &= \int_0^1 K_2(t,s) f_2\big(s,u(s),v(s)\big)\,ds + \int_0^1 H_2(t,s) f_1\big(s,u(s),v(s)\big)\,ds \\ &\le \rho\bigg(\kappa^{\theta_1+\vartheta_1}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_1}\int_0^1 \varphi_1(s) f_1\big(s,1,s^{\alpha_2-1}\big)\,ds \\ &\quad + \kappa^{\theta_2+\vartheta_2}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_2}\int_0^1 \varphi_2(s) f_2\big(s,s^{\alpha_1-1},1\big)\,ds\bigg) \\ &< +\infty. \end{aligned}$$

Together with the continuity of $K_k(t,s)$ and $H_k(t,s)$ ($k = 1,2$), it is easy to see that $T_k \in C[0,1]$. Therefore, $T : P \setminus \{(0,0)\} \to P$ is well defined.

For any $(u,v) \in P_{[r_1,r_2]}$ and $t,\tau \in [0,1]$, by Remark 2.1, we obtain

$$\begin{aligned}
T_1(u,v)(t) &= \int_0^1 K_1(t,s)f_1\big(s,u(s),v(s)\big)\,ds + \int_0^1 H_1(t,s)f_2\big(s,u(s),v(s)\big)\,ds \\
&\geq \int_0^1 \omega t^{\alpha_1-1}K_1(\tau,s)f_1\big(s,u(s),v(s)\big)\,ds + \int_0^1 \omega t^{\alpha_1-1}H_1(\tau,s)f_2\big(s,u(s),v(s)\big)\,ds \\
&= \omega t^{\alpha_1-1}T_1(u,v)(\tau), \\
T_1(u,v)(t) &= \int_0^1 K_1(t,s)f_1\big(s,u(s),v(s)\big)\,ds + \int_0^1 H_1(t,s)f_2\big(s,u(s),v(s)\big)\,ds \\
&\geq \int_0^1 \omega t^{\alpha_1-1}H_2(\tau,s)f_1\big(s,u(s),v(s)\big)\,ds + \int_0^1 \omega t^{\alpha_1-1}K_2(\tau,s)f_2\big(s,u(s),v(s)\big)\,ds \\
&= \omega t^{\alpha_1-1}T_2(u,v)(\tau).
\end{aligned}$$

Then we have

$$\begin{aligned}
&T_1(u,v)(t) \geq \omega t^{\alpha_1-1}\big\|T_1(u,v)\big\|, \qquad T_1(u,v)(t) \geq \omega t^{\alpha_1-1}\big\|T_2(u,v)\big\|, \\
&T_1(u,v)(t) \geq \frac{1}{2}\omega t^{\alpha_1-1}\big\|T_1(u,v),T_2(u,v)\big\|.
\end{aligned}$$

In the same way, we can prove that

$$\begin{aligned}
&T_2(u,v)(t) \geq \omega t^{\alpha_2-1}\big\|T_2(u,v)\big\|, \qquad T_2(u,v)(t) \geq \omega t^{\alpha_2-1}\big\|T_1(u,v)\big\|, \\
&T_2(u,v)(t) \geq \frac{1}{2}\omega t^{\alpha_2-1}\big\|T_1(u,v),T_2(u,v)\big\|.
\end{aligned}$$

Therefore, we have $TP_{[r_1,r_2]} \subseteq TP$. Further $T : P_{[r_1,r_2]} \to P$ is completely continuous combining with Lemma 2.8. □

Theorem 4.2 *Assume that* (H_3)-(H_5) *hold. Then the boundary value problem* (1.1)-(1.2) *has at least one positive solution* (u^*,v^*), *and there exists a real number* $0 < \delta < 1$ *satisfying*

$$\delta t^{\alpha_1-1} \leq u^*(t) \leq \delta^{-1}t^{\alpha_1-1}, \qquad \delta t^{\alpha_2-1} \leq v^*(t) \leq \delta^{-1}t^{\alpha_2-1}, \quad t \in [0,1].$$

Proof First, we show that the boundary value problem (1.1)-(1.2) has at least one positive solution.

There exists a positive constant c satisfying $0 < c < 1 - c < 1$. Choose r and R such that

$$\begin{aligned}
&0 < r \leq \min\left\{\left(\frac{1}{2}\varrho c^{\alpha_1-1}\omega^{\theta_1}\int_0^1 \varphi_1(s)f_1\big(s,s^{\alpha_1-1},1\big)\,ds\right)^{\frac{1}{1-\theta_1}}, \frac{1}{4}\right\}, \\
&R \geq \max\left\{\left(\frac{\rho}{2}\int_0^1 \varphi_1(s)f_1\big(s,1,s^{\alpha_2-1}\big)\,ds\right.\right. \\
&\qquad \left.\left. + \frac{\rho}{2}\int_0^1 \varphi_2(s)f_2\big(s,s^{\alpha_1-1},1\big)\,ds\right)^{\frac{1}{1-\max\{\theta_1,\theta_2\}}}, \frac{1}{\omega}, 1\right\}.
\end{aligned}$$

For any $(u,v) \in \partial P_r$, we have

$$r\omega t^{\alpha_1-1} \leq u(t) \leq r, \qquad r\omega t^{\alpha_2-1} \leq v(t) \leq r, \quad t \in [0,1]. \tag{4.7}$$

By Lemma 2.5, Remark 4.1, (H_3) and (H_4), for any $(u,v) \in \partial P_r$, we get

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,s)f_1\big(s,u(s),v(s)\big)\,ds + \int_0^1 H_1(t,s)f_2\big(s,u(s),v(s)\big)\,ds \\ &\geq \int_0^1 K_1(t,s)f_1\big(s,u(s),v(s)\big)\,ds \geq \varrho t^{\alpha_1-1}\int_0^1 \varphi_1(s)f_1\big(s,r\omega s^{\alpha_1-1},r\big)\,ds \\ &\geq \varrho c^{\alpha_1-1}(r\omega)^{\theta_1}\int_0^1 \varphi_1(s)f_1\big(s,s^{\alpha_1-1},1\big)\,ds \\ &\geq 2r = \big\|(u,v)\big\|, \quad \text{for } t \in [c,1-c]. \end{aligned}$$

Similarly, we can get $T_2(u,v)(t) \geq \|u,v\|$, for $t \in [c,1-c]$. This guarantees that

$$\big\|T(u,v)\big\| \geq \big\|(u,v)\big\|, \quad \forall (u,v) \in \partial P_r. \tag{4.8}$$

On the other hand, for any $(u,v) \in \partial P_R$, we have

$$R\omega t^{\alpha_1-1} \leq u(t) \leq R, \qquad R\omega t^{\alpha_2-1} \leq v(t) \leq R, \quad t \in [0,1].$$

By Lemma 2.5, (H_3)-(H_5), (4.3) and (4.4), for any $(u,v) \in \partial P_R$, we get

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,s)f_1\big(s,u(s),v(s)\big)\,ds + \int_0^1 H_1(t,s)f_2\big(s,u(s),v(s)\big)\,ds \\ &\leq \rho\int_0^1 \varphi_1(s)f_1\big(s,R,R\omega s^{\alpha_2-1}\big)\,ds + \rho\int_0^1 \varphi_2(s)f_2\big(s,R\omega s^{\alpha_1-1},R\big)\,ds \\ &\leq \rho R^{\theta_1}\int_0^1 \varphi_1(s)f_1\big(s,1,s^{\alpha_2-1}\big)\,ds + \rho R^{\theta_2}\int_0^1 \varphi_2(s)f_2\big(s,s^{\alpha_1-1},1\big)\,ds \\ &\leq \rho R^{\max\{\theta_1,\theta_2\}}\bigg(\int_0^1 \varphi_1(s)f_1\big(s,1,s^{\alpha_2-1}\big)\,ds + \int_0^1 \varphi_2(s)f_2\big(s,s^{\alpha_1-1},1\big)\,ds\bigg) \\ &\leq 2R = \big\|(u,v)\big\|. \end{aligned}$$

In the same way, we have $T_2(u,v)(t) \leq 2R = \|(u,v)\|$, for all $(u,v) \in \partial P_R$. So we have

$$\big\|T(u,v)\big\| \leq \big\|(u,v)\big\|, \quad \forall (u,v) \in \partial P_R. \tag{4.9}$$

By the complete continuity of T, (4.8), (4.9), and Lemma 2.7, we find that T has at least a fixed point $(u^*,v^*) \in P_{[r,R]}$. Consequently, boundary value problem (1.1)-(1.2) has a positive solution $(u^*,v^*) \in P_{[r,R]}$.

Next, we show there exists a real number $0 < \delta < 1$ satisfying

$$\delta t^{\alpha_1-1} \leq u^*(t) \leq \delta^{-1}t^{\alpha_1-1}, \qquad \delta t^{\alpha_2-1} \leq v^*(t) \leq \delta^{-1}t^{\alpha_2-1}, \quad t \in [0,1].$$

From the Lemma 4.1, we know $(u^*,v^*) \in P \setminus \{(0,0)\}$. So, we have

$$\begin{aligned} &\omega t^{\alpha_1-1}\big\|\big(u^*,v^*\big)\big\| \leq u^*(t) \leq \big\|\big(u^*,v^*\big)\big\|, \quad t \in [0,1], \\ &\omega t^{\alpha_2-1}\big\|\big(u^*,v^*\big)\big\| \leq v^*(t) \leq \big\|\big(u^*,v^*\big)\big\|, \quad t \in [0,1]. \end{aligned}$$

Choose κ, such that $\|(u^*,v^*)\|/\kappa < 1$, $\kappa > 1/\omega$. By Lemma 2.5, (H_3) and (H_4), for $t \in [0,1]$, we have

$$\begin{aligned}
u^*(t) &= \int_0^1 K_1(t,s) f_1\big(s,u^*(s),v^*(s)\big)\,ds + \int_0^1 H_1(t,s) f_2\big(s,u^*(s),v^*(s)\big)\,ds \\
&\le \int_0^1 \rho t^{\alpha_1-1} f_1\big(s,\kappa,\omega s^{\alpha_2-1}\big\|\big(u^*,v^*\big)\big\|\big)\,ds + \int_0^1 \rho t^{\alpha_1-1} f_2\big(s,\omega s^{\alpha_1-1}\big\|\big(u^*,v^*\big)\big\|,\kappa\big)\,ds \\
&\le \rho t^{\alpha_1-1}\left(\int_0^1 f_1\left(s,\kappa,\frac{\omega\|(u^*,v^*)\|}{\kappa}s^{\alpha_2-1}\right)ds + \int_0^1 f_2\left(s,\frac{\omega\|(u^*,v^*)\|}{\kappa}s^{\alpha_1-1},\kappa\right)ds\right) \\
&\le \rho t^{\alpha_1-1}\left(\kappa^{\theta_1+\vartheta_1}\big(\omega\big\|\big(u^*,v^*\big)\big\|\big)^{-\vartheta_1}\int_0^1 f_1\big(s,1,s^{\alpha_2-1}\big)\,ds\right. \\
&\quad \left. + \kappa^{\theta_2+\vartheta_2}\big(\omega\big\|\big(u^*,v^*\big)\big\|\big)^{-\vartheta_2}\int_0^1 f_2\big(s,s^{\alpha_1-1},1\big)\,ds\right) \\
&\le \rho t^{\alpha_1-1}\left(\kappa^{\theta_1+\vartheta_1}(2\omega R)^{-\vartheta_1}\int_0^1 f_1\big(s,1,s^{\alpha_2-1}\big)\,ds\right. \\
&\quad \left. + \kappa^{\theta_2+\vartheta_2}(2\omega R)^{-\vartheta_2}\int_0^1 f_2\big(s,s^{\alpha_1-1},1\big)\,ds\right).
\end{aligned}$$

In the same way, for $t \in [0,1]$, we also have

$$\begin{aligned}
v^*(t) \le \rho t^{\alpha_2-1}&\left(\kappa^{\theta_1+\vartheta_1}(2\omega R)^{-\vartheta_1}\int_0^1 f_1\big(s,1,s^{\alpha_2-1}\big)\,ds\right. \\
&\left. + \kappa^{\theta_2+\vartheta_2}(2\omega R)^{-\vartheta_2}\int_0^1 f_2\big(s,s^{\alpha_1-1},1\big)\,ds\right).
\end{aligned}$$

Choose

$$\begin{aligned}
\delta = \min\Bigg\{\omega r,&\left(\rho\kappa^{\theta_1+\vartheta_1}(2\omega R)^{-\vartheta_1}\int_0^1 f_1\big(s,1,s^{\alpha_2-1}\big)\,ds\right. \\
&\left. + \rho\kappa^{\theta_2+\vartheta_2}(2\omega R)^{-\vartheta_2}\int_0^1 f_2\big(s,s^{\alpha_1-1},1\big)\,ds\right)^{-1},\frac{1}{2}\Bigg\},
\end{aligned}$$

combining with (4.7), we have

$$\delta t^{\alpha_1-1} \le u^*(t) \le \delta^{-1}t^{\alpha_1-1}, \qquad \delta t^{\alpha_2-1} \le v^*(t) \le \delta^{-1}t^{\alpha_2-1}, \quad t \in [0,1].$$

This completes the proof of Theorem 4.2. □

Theorem 4.3 *Assume that* (H_3)-(H_5) *hold. Furthermore, assume* $\theta_1+\vartheta_1 < 1$ *and* $\theta_2+\vartheta_2 < 1$. *Then boundary value problem* (1.1)-(1.2) *has a unique positive solution on* $[0,1]$.

Proof Assume that the coupled boundary value problem (1.1)-(1.2) has two different positive solutions (u_1,v_1) and (u_2,v_2). By Theorem 4.2, there exist $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$ such that

$$\begin{aligned}
&\delta_1 t^{\alpha_1-1} \le u_1(t) \le \delta_1^{-1}t^{\alpha_1-1}, \qquad \delta_1 t^{\alpha_2-1} \le v_1(t) \le \delta_1^{-1}t^{\alpha_2-1}, \quad t \in [0,1], \\
&\delta_2 t^{\alpha_1-1} \le u_2(t) \le \delta_2^{-1}t^{\alpha_1-1}, \qquad \delta_2 t^{\alpha_2-1} \le v_2(t) \le \delta_2^{-1}t^{\alpha_2-1}, \quad t \in [0,1].
\end{aligned}$$

Further, we have

$$\delta_1\delta_2 u_2(t) \le u_1(t) \le (\delta_1\delta_2)^{-1}u_2(t), \quad t \in [0,1],$$
$$\delta_1\delta_2 v_2(t) \le v_1(t) \le (\delta_1\delta_2)^{-1}v_2(t), \quad t \in [0,1].$$

Obviously, one has $\delta_1\delta_2 \neq 1$. Put

$$\Delta = \sup\bigl\{\delta : \delta u_2(t) \le u_1(t) \le \delta^{-1}u_2(t), \delta v_2(t) \le v_1(t) \le \delta^{-1}v_2(t), t \in [0,1]\bigr\}.$$

It is easy to see that $0 < \delta_1\delta_2 < \Delta < 1$, and

$$\Delta u_2(t) \le u_1(t) \le \Delta^{-1}u_2(t), \qquad \Delta v_2(t) \le v_1(t) \le \Delta^{-1}v_2(t), \quad t \in [0,1]. \tag{4.10}$$

By (H_3), (H_4) and (4.10), we get

$$\begin{aligned} f_1\bigl(t,u_1(t),v_1(t)\bigr) &\ge f_1\bigl(t,\Delta u_2(t),\Delta^{-1}v_2(t)\bigr) \ge \Delta^{\theta_1+\vartheta_1} f_1\bigl(t,u_2(t),v_2(t)\bigr) \\ &\ge \Delta^{\sigma} f_1\bigl(t,u_2(t),v_2(t)\bigr), \\ f_2\bigl(t,u_1(t),v_1(t)\bigr) &\ge f_2\bigl(t,\Delta^{-1}u_2(t),\Delta v_2(t)\bigr) \ge \Delta^{\theta_2+\vartheta_2} f_2\bigl(t,u_2(t),v_2(t)\bigr) \\ &\ge \Delta^{\sigma} f_2\bigl(t,u_2(t),v_2(t)\bigr), \end{aligned} \tag{4.11}$$

where $\sigma = \max\{\theta_1+\vartheta_1, \theta_2+\vartheta_2\}$ such that $\sigma < 1$. Similarly, by (H_3), (H_4) and (4.10), we have

$$\begin{aligned} f_1\bigl(t,u_2(t),v_2(t)\bigr) &\ge f_1\bigl(t,\Delta u_1(t),\Delta^{-1}v_1(t)\bigr) \ge \Delta^{\theta_1+\vartheta_1} f_1\bigl(t,u_1(t),v_1(t)\bigr) \\ &\ge \Delta^{\sigma} f_1\bigl(t,u_1(t),v_1(t)\bigr), \\ f_2\bigl(t,u_2(t),v_2(t)\bigr) &\ge f_2\bigl(t,\Delta^{-1}u_1(t),\Delta v_1(t)\bigr) \ge \Delta^{\theta_2+\vartheta_2} f_2\bigl(t,u_1(t),v_1(t)\bigr) \\ &\ge \Delta^{\sigma} f_2\bigl(t,u_1(t),v_1(t)\bigr). \end{aligned} \tag{4.12}$$

From (4.11), for $t \in [0,1]$, we have

$$\begin{aligned} u_1(t) &= T_1(u_1,v_1)(t) \\ &= \int_0^1 K_1(t,s) f_1\bigl(s,u_1(s),v_1(s)\bigr)\,ds + \int_0^1 H_1(t,s) f_2\bigl(s,u_1(s),v_1(s)\bigr)\,ds \\ &\ge \int_0^1 K_1(t,s)\Delta^{\sigma} f_1\bigl(s,u_2(s),v_2(s)\bigr)\,ds + \int_0^1 H_1(t,s)\Delta^{\sigma} f_2\bigl(s,u_2(s),v_2(s)\bigr)\,ds \\ &= \Delta^{\sigma} T_1(u_2,v_2)(t) = \Delta^{\sigma} u_2(t), \\ v_1(t) &= T_2(u_1,v_1)(t) \\ &= \int_0^1 K_2(t,s) f_2\bigl(s,u_1(s),v_1(s)\bigr)\,ds + \int_0^1 H_2(t,s) f_1\bigl(s,u_1(s),v_1(s)\bigr)\,ds \\ &\ge \int_0^1 K_2(t,s)\Delta^{\sigma} f_2\bigl(s,u_2(s),v_2(s)\bigr)\,ds + \int_0^1 H_2(t,s)\Delta^{\sigma} f_1\bigl(s,u_2(s),v_2(s)\bigr)\,ds \\ &= \Delta^{\sigma} T_2(u_2,v_2)(t) = \Delta^{\sigma} v_2(t). \end{aligned} \tag{4.13}$$

Similarly, from (4.12), for $t \in [0,1]$, we have

$$\begin{aligned} u_2(t) &= T_1(u_2,v_2)(t) \\ &= \int_0^1 K_1(t,s)f_1\big(s,u_2(s),v_2(s)\big)\,ds + \int_0^1 H_1(t,s)f_2\big(s,u_2(s),v_2(s)\big)\,ds \\ &\geq \int_0^1 K_1(t,s)\Delta^\sigma f_1\big(s,u_1(s),v_1(s)\big)\,ds + \int_0^1 H_1(t,s)\Delta^\sigma f_2\big(s,u_1(s),v_1(s)\big)\,ds \\ &= \Delta^\sigma T_1(u_1,v_1)(t) = \Delta^\sigma u_1(t), \\ v_2(t) &= T_2(u_2,v_2)(t) \\ &= \int_0^1 K_2(t,s)f_2\big(s,u_2(s),v_2(s)\big)\,ds + \int_0^1 H_2(t,s)f_1\big(s,u_2(s),v_2(s)\big)\,ds \\ &\geq \int_0^1 K_2(t,s)\Delta^\sigma f_2\big(s,u_1(s),v_1(s)\big)\,ds + \int_0^1 H_2(t,s)\Delta^\sigma f_1\big(s,u_1(s),v_1(s)\big)\,ds \\ &= \Delta^\sigma T_2(u_1,v_1)(t) = \Delta^\sigma v_1(t). \end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14), we can obtain

$$\Delta^\sigma u_2(t) \leq u_1(t) \leq \big(\Delta^\sigma\big)^{-1}u_2(t), \qquad \Delta^\sigma v_2(t) \leq v_1(t) \leq \big(\Delta^\sigma\big)^{-1}v_2(t), \quad t \in [0,1].$$

Noticing that $0 < \Delta, \sigma < 1$, we get to a contradiction with the maximality of Δ. Thus, the boundary value problem (1.1)-(1.2) has a unique positive solution (u^*, v^*). This completes the proof of Theorem 4.3. □

Example 4.1 Consider the following fractional boundary value problem:

$$\begin{cases} (D_{0+}^{2.5}u)(t) + \frac{\sqrt{u(t)}+1}{(2-t)(\sqrt[3]{v(t)}+1)} = 0, \\ (D_{0+}^{2.5}v)(t) + \frac{\sqrt[3]{v(t)}+1}{(2-t)(\sqrt{u(t)}+1)} = 0, \\ u(0) = u'(0) = 0, \qquad u(1) = \sum_{i=1}^2 \int_{\xi_i}^{\eta_i} v(s)\,dA_i(s) + \sum_{i=1}^2 b_i v(\sigma_i), \\ v(0) = v'(0) = 0, \qquad v(1) = \sum_{i=1}^2 \int_{\xi_i}^{\eta_i} u(s)\,dA_i(s) + \sum_{i=1}^2 b_i u(\sigma_i), \end{cases} \tag{4.15}$$

where

$$\begin{aligned} &A_1(t) = 0.5t, \qquad \xi_1 = \frac{1}{8}, \qquad \eta_1 = \frac{3}{8}, \qquad b_1 = \frac{1}{3}, \qquad \sigma_1 = \frac{1}{3}, \\ &A_2(t) = 0.8t, \qquad \xi_2 = \frac{5}{8}, \qquad \eta_2 = \frac{7}{8}, \qquad b_2 = \frac{2}{3}, \qquad \sigma_2 = \frac{2}{3}. \end{aligned}$$

Obviously, we have $\alpha_1, \alpha_2 = 2.5$. We have

$$f_1(t,u,v) = \frac{\sqrt{u(t)}+1}{(2-t)(\sqrt[3]{v(t)}+1)}, \qquad f_2(t,u,v) = \frac{\sqrt[3]{v(t)}+1}{(2-t)(\sqrt{u(t)}+1)}.$$

It is easy to see that $f_1 : [0,1] \times [0,\infty) \times [0,\infty)$ is continuous, $f_1(t,u,v)$ is nondecreasing in u and nonincreasing in v, $f_2 : [0,1] \times [0,\infty) \times [0,\infty)$ is continuous, $f_2(t,u,v)$ is nonincreasing in u and nondecreasing in v. Take

$$\theta_1 = \frac{11}{20}, \qquad \vartheta_1 = \frac{2}{5}, \qquad \theta_2 = \frac{3}{5}, \qquad \vartheta_2 = \frac{1}{5}.$$

Then we know that condition (H_3) and (H_4) holds. As

$$\begin{aligned}
\int_0^1 \varphi_1(s) f_1\bigl(s,1,s^{\alpha_2-1}\bigr)\,ds &= \int_0^1 \frac{2(1-s)^{1.5}s}{(2-s)(s^{0.5}+1)}\,ds \\
&\le \int_0^1 \frac{2}{2-s}\,ds = 2\ln 2 \approx 1.3863 < +\infty, \\
\int_0^1 \varphi_2(s) f_2\bigl(s,s^{\alpha_1-1},1\bigr)\,ds &= \int_0^1 \frac{2(1-s)^{1.5}s}{(2-s)(s^{0.75}+1)}\,ds \\
&\le \int_0^1 \frac{2}{2-s}\,ds = 2\ln 2 \approx 1.3863 < +\infty.
\end{aligned}$$

The condition (H_5) is also satisfied. Therefore, by Theorem 4.2, we see that the coupled boundary value problem (4.15) has at least one positive solution (u^*, v^*). Furthermore,

$$\theta_1 + \vartheta_1 = \frac{19}{20} < 1, \qquad \theta_2 + \vartheta_2 = \frac{4}{5} < 1.$$

By Theorem 4.3, we see that (u^*, v^*) is the unique positive solution of the coupled boundary value problem (4.15).

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References
1. Podlubny, I: Fractional Differential Equations. Mathematics in Science and Engineering. Academic Press, New York (1999)
2. Wang, JR, Zhou, Y: Analysis of nonlinear fractional control systems in Banach spaces. Nonlinear Anal. **74**, 5929-5942 (2011)
3. Wang, JR, Zhou, Y, Wei, W, Xu, HL: Nonlocal problems for fractional integrodifferential equations via fractional operators and optimal controls. Comput. Math. Appl. **62**, 1427-1441 (2011)
4. Wang, JR, Zhou, Y, Fečkan, M: On recent developments in the theory of boundary value problems for impulsive fractional differential equations. Comput. Math. Appl. **64**, 3008-3020 (2012)
5. Henderson, J, Kosmatov, N: Eigenvalue comparison for fractional boundary value problems with the Caputo derivative. Fract. Calc. Appl. Anal. **17**, 872-880 (2014)
6. Wang, G: Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval. Appl. Math. Lett. **47**, 1-7 (2015)
7. Jia, M, Zhang, H, Chen, Q: Existence of positive solutions for fractional differential equation with integral boundary conditions on the half-line. Bound. Value Probl., **2016**, 104 (2016)
8. Bai, ZB, Zhang, S, Sun, SJ, Yin, C: Monotone iterative method for fractional differential equations. Electron. J. Differ. Equ. **2016** (2016)

9. Dong, XY, Bai, ZB, Zhang, SQ: Positive solutions to boundary value problems of p-Laplacian with fractional derivative. Bound. Value Probl. **2017**, 5 (2017). doi:10.1186/s13661-016-0735-2
10. Ge, ZM, Ou, CY: Chaos synchronization of fractional order modified Duffing systems with parameters excited by a chaotic signal. Chaos Solitons Fractals **35**, 705-717 (2008)
11. Faieghi, M, Kuntanapreeda, S, Delavari, H, Baleanu, D: LMI-based stabilization of a class of fractional-order chaotic systems. Nonlinear Dyn. **72**, 301-309 (2013)
12. Zhang, F, Chen, G, Li, C, Kurths, J: Chaos synchronization in fractional differential systems. Philos. Trans. R. Soc. A, Math. Phys. Eng. Sci. **371**, 20120155 (2013)
13. Sokolov, IM, Klafter, J, Blumen, A: Fractional kinetics. Phys. Today **55**, 48-54 (2002)
14. Carvalho, A, Pinto, CMA: A delay fractional order model for the co-infection of malaria and HIV/AIDS. Int. J. Dyn. Control **5**, 168-186 (2017). doi:10.1007/s40435-016-0224-3
15. Nyamoradi, N, Javidi, M, Ahmad, B: Dynamics of SVEIS epidemic model with distinct incidence. Int. J. Biomath. **08**(6), 99-117 (2015)
16. Javidi, M, Ahmad, B: Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system. Ecol. Model. **318**, 8-18 (2015)
17. Agarwal, RP, Ahmad, B, Garout, D, Alsaedi, A: Existence results for coupled nonlinear fractional differential equations equipped with nonlocal coupled flux and multi-point boundary conditions. Chaos Solitons Fractals **102**, 149-161 (2017). doi:10.1016/j.chaos.2017.03.025
18. Qarout, D, Ahmad, B, Alsaedi, A: Existence theorems for semi-linear Caputo fractional differential equations with nonlocal discrete and integral boundary conditions. Fract. Calc. Appl. Anal. **19**, 463-479 (2016). doi:10.1515/fca-2016-0024
19. Ahmad, B, Ntouyas, SK: Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. Appl. Math. Comput. **266**, 615-622 (2015)
20. Zhao, QB, Yang, WG: Positive solutions for singular coupled integral boundary value problem of nonlinear higher-order fractional q-difference equations. Adv. Differ. Equ. **2015**, 290 (2015)
21. Wang, Y, Zhang, JW: Positive solutions for higher-order singular fractional differential system with coupled integral boundary conditions. Adv. Differ. Equ. **2016**, 117 (2016)
22. Guo, LM, Liu, LS, Wu, YH: Uniqueness of iterative positive solutions for the singular fractional differential equations with integral boundary conditions. Bound. Value Probl. **2016**, 147 (2016)
23. Tan, JJ: Positive solutions of singular fractional order differential system with Riemann-Stieltjes integral boundary condition. Adv. Differ. Equ. **2016**, 293 (2016)
24. Alsaedi, A, Ntouyas, SK, Garout, D, Ahmad, B: Coupled fractional-order systems with nonlocal coupled integral and discrete boundary conditions. Bull. Malays. Math. Sci. Soc. **10**, 1007 (2017)
25. Alsaedi, A, Aljoudi, S, Ahmad, B: Existence of solutions for Riemann-Liouville type coupled systems of fractional integro-differential equations and boundary conditions. Electron. J. Differ. Equ. **2016**, 211 (2016)
26. Ahmad, B, Alsaedi, A, Aljoudi, S, Ntouyas, SK: A six-point nonlocal boundary value problem of nonlinear coupled sequential fractional integro-differential equations and coupled integral boundary conditions. J. Appl. Math. Comput. (2016). doi:10.1007/s12190-016-1078-8
27. Ahmad, B, Alsaedi, A, Aljoudi, S, Ntouyas, SK: On a coupled system of sequential fractional differential equations with variable coefficients and coupled integral boundary conditions. Bull. Math. Soc. Sci. Math. Roum. **60(108)**(1), 3-18 (2017)