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Asymptotic behavior of a diffusive eco-epidemiological model with an infected prey population

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Abstract

We study a diffusive predator-prey system with a ratio-dependent functional response when a prey population is infected under homogeneous Neumann boundary condition. All non-negative and positive equilibria are investigated, and the conditions that give rise to asymptotic behavior of these equilibria are examined. In particular, we present a biological interpretation of disease-free and total extinction states. A comparison principle and the stability analysis for the parabolic problem are employed.

MSC: 35B35; 35K57; 92D25

Keywords: predator-prey model; ratio-dependent functional response; locally/globally asymptotical stability; disease free

1 Introduction

We focus on the diffusive predator-prey system with a ratio-dependent functional response and disease in the prey; specifically,

$$\begin{cases} u_t - d\Delta u = u[r - \frac{r}{K}u - \frac{\alpha w}{mw+u+v} - bv], \\ v_t - d\Delta v = v[bu - d_1 - \frac{\beta w}{mw+u+v}], \\ w_t - D\Delta w = w[-d_2 + \frac{c\alpha u}{mw+u+v} + \frac{c\beta v}{mw+u+v}] \quad \text{in } (0,\infty)\times\Omega, \\ \frac{\partial u}{\partial\eta} = \frac{\partial v}{\partial\eta} = \frac{\partial w}{\partial\eta} = 0 \quad \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), \qquad v(0,x) = v_0(x), \qquad w(0,x) = w_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded region with smooth boundary $\partial\Omega$, and r, m, K, b, d_i, D_i, c, α, and β are positive constants; a, b_1, b_2, l and k are positive constants as well. The initial functions u_0, v_0, and w_0 are not identically zero in Ω; u, v, and w represent the densities of the susceptible prey, infected prey, and predator, respectively, and η is the outward directional derivative normal to $\partial\Omega$. Furthermore, α and β are the searching efficiency constants of the predation rate for the susceptible and infective prey, respectively. $\frac{\alpha}{m}$ and $\frac{\beta}{m}$ are the maximum per capita capturing rates of the predator for the susceptible prey and infected prey, respectively. m is the predation rate for the susceptible prey and infected prey. Finally, b is the force of infection, d_1 and d_2 are the death rates of the infected prey and predator, respectively, and c is a conversion rate. The homogeneous Neumann boundary condition describes an environment with no flux at the boundary of the region.

During the last three decades, various types of predator-prey models are studied extensively by many researchers. Many models have a functional response in which ignoring an effect of predator density, *i.e.*, the function that describe a density of prey which is consumed by its predator depends only on prey. However, there is explicit biological and physiological evidence [1–4] that in many situations, when predator have to search for food, a more suitable general predator-prey model in heterogeneous situations should be had the ratio-dependent functional response, which the per capita predator growth rate should be a function of the ratio of prey to predator abundance. Ratio-dependent models have been mathematically studied for both the spatially homogeneous case [5–8] and the spatially inhomogeneous case [9–11]. In [8, 12], one examined the model of Arditi and Ginzburg [13]. One showed that under some conditions, the whole population can be extinct.

On the other hand, epidemic models have also received a lot of attention since Kermack-McKendrick's model. Among them, we are interested in eco-epidemiological systems with predator-prey interactions. Considerable research has been done on the spatially homogeneous case [14–20].

In the real application, diffusive system in this study can be used to describe the interaction between marine viruses in aquatic ecosystems and the species [21, 22], since there is evidence that viral infection might accelerate the termination of phytoplankton blooms [23]. In fact, in [24], the authors showed experimentally that viral disease can infect bacteria and phytoplankton in coastal water. In [14, 25, 26], they observed oscillations and waves in a phytoplankton-zooplankton system with Holling-type II and III grazing under lysogenic viral infection and frequency-dependent transmission. Hilker [27] also investigated the local dynamics of phytoplankton with lytic infection and frequency-dependent transmission as well as zooplankton with Holling-type II grazing.

Arino *et al.* [28] suggested the non-dimensionalized model, which is a non-spatial version of (1.1). There, the authors obtained the conditions for which no trajectory can reach the origin following any fixed direction or spirally. Also the criteria of persistence were found. The above studies have been done mostly for the non-spatial case.

In this paper, we investigate the conditions of the asymptotic behavior of a unique positive constant solution and the non-negative equilibria of (1.1), which is a spatially dependent model with diffusion.

Model (1.1) is based on the following assumptions:

(a) In the absence of disease, the prey population grows according to logistic law with carrying capacity $K > 0$ and an intrinsic growth rate $r > 0$.
(b) In the presence of disease, the prey consists of two classes: susceptible prey and infected prey.
(c) Only susceptible prey can reproduce themselves logistically and contribute to its carrying capacity. Infected prey do not grow, recover, or reproduce.
(d) Disease can only be spread among the prey, and it is not inherited. Disease transmission follows the simple law of mass action.

From the literature [28], the assumption (c) can be justified in many cases: the experiment on dinoflagellate Noctiluca scintillans in the German Bight by Uhlig and Sahling [29] indicated that the cells become damaged, and they neither feed anymore nor reproduce.

The model of Hamilton *et al.* [30] showed that infected individuals do not contribute in the reproduction process; infection reduces the remaining capacity due to the inability to compete for resources. Thus, we may assume that the growth term of the susceptible population follows only the law of logistic growth.

For additional background information pertaining to (1.1), we refer to [28] and the references therein.

The remainder of this paper is organized as follows. In Section 2, we investigate the large time behavior of non-negative constant solutions and the asymptotic stability of a positive constant solution. Finally, the results obtained are analyzed in terms of biological interpretations in Section 3.

2 Asymptotical behavior of constant solutions

In this section, the asymptotic behavior of non-negative and positive constant solutions to (1.1) is examined.

For convenience, we denote the growth rate terms as follows:

$$\begin{aligned} f_1(u,v,w) &:= r - \frac{r}{K}u - \frac{\alpha w}{mw+u+v} - bv, \\ f_2(u,v,w) &:= bu - d_1 - \frac{\beta w}{mw+u+v}, \\ f_3(u,v,w) &:= -d_2 + \frac{c\alpha u}{mw+u+v} + \frac{c\beta v}{mw+u+v}. \end{aligned}$$

Using the uniform bound of u, v and w, one can show that (uf_1, vf_2, wf_3) satisfies the Lipschitz condition. Using the upper and lower solution method in [31], it can also be shown that (1.1) has a non-negative solution.

The next theorem states that the solution to (1.1) is uniformly bounded [32].

Theorem 2.1 *The solution* (u,v,w) *of* (1.1) *is uniformly bounded*; *specifically,*

$$0 \le u(t,x) \le B_1, \qquad 0 \le v(t,x) \le B_2, \qquad 0 \le w(t,x) \le B_3,$$

where B_i *is defined by*

$$\begin{aligned} B_1 &:= \max\{K, \|u_0\|_\infty\}, \\ B_2 &:= \max\left\{\frac{1}{d_1}\frac{K}{r}\left(\frac{r+d_1}{2}\right)^2, \|u_0\|_\infty + \|v_0\|_\infty\right\}, \\ B_3 &:= \max\left\{\|w_0\|_\infty, \frac{c(\alpha+\beta)-d_2}{d_2 m}B_2\right\}. \end{aligned}$$

The dissipation and persistence of the parabolic system (1.1) can be found in [32].

Theorem 2.2 *For a solution* $\mathbf{u} = (u(t,x), v(t,x), w(t,x))$ *to the parabolic system* (1.1),

$$\limsup_{t\to\infty} \mathbf{u} \le \left(K, \frac{1}{d_1}\frac{K}{r}\left(\frac{r+d_1}{2}\right)^2, \frac{c(\alpha+\beta)-d_2}{d_2 m}\frac{1}{d_1}\frac{K}{r}\left(\frac{r+d_1}{2}\right)^2\right)$$

if $c(\alpha+\beta) > d_2$.

Theorem 2.3 *Assume that $\beta \geq \alpha > \frac{d_2}{c}$, $r > \min\{\frac{b}{d_1}\frac{K}{r}(\frac{r+d_1}{2})^2 + \frac{\alpha}{m}, \frac{1}{b}\frac{r}{K}(d_1 + \frac{\beta}{m}) + \frac{\alpha}{m}\}$. Then*

$$\liminf_{t\to\infty} \mathbf{u} \geq (\Theta_1, \Theta_2, \Theta_3),$$

where $\Theta_1 := (r - \frac{b}{d_1}\frac{K}{r}(\frac{r+d_1}{2})^2 - \frac{\alpha}{m})\frac{K}{r}$, $\Theta_2 := \frac{1}{b}(r - \frac{1}{b}\frac{r}{K}(d_1 + \frac{\beta}{m}) - \frac{\alpha}{m})$, and $\Theta_3 := \frac{c\alpha-d_2}{d_2 m}\Theta_1$ for $\frac{\alpha}{m^2\Theta_3} \leq b$.

2.1 Equilibria

System (1.1) has the following non-negative equilibria:

$$\begin{cases} \mathbf{e_0} = (0,0,0), \\ \mathbf{e_1} = (K,0,0), \\ \mathbf{e_2} = (K(1 - \frac{c\alpha-d_2}{cmr}), 0, \frac{c\alpha-d_2}{d_2 m}K(1 - \frac{c\alpha-d_2}{cmr})) \quad \text{if } 0 < c\alpha - d_2 < cmr, \\ \mathbf{e_3} = (\frac{d_1}{b}, \frac{r}{b}(1 - \frac{d_1}{bK}), 0) \quad \text{if } bK > d_1. \end{cases} \tag{2.1}$$

Note that the given growth rates in (1.1) are not defined at $(u, v, w) = (0, 0, 0)$. Since

$$\lim_{(u,v,w)\to(0,0,0)} \frac{uw}{mw + u + v} = \lim_{(u,v,w)\to(0,0,0)} \frac{vw}{mw + u + v} = 0,$$

the domain of $\frac{uw}{mw+u+v}$ and $\frac{vw}{mw+u+v}$ may be extended to $\{(u, v, w) : u \geq 0, v \geq 0, w \geq 0\}$ so that $(0, 0, 0)$ becomes a *trivial solution* to (1.1) [8].

Furthermore, if the following conditions are satisfied:

$$AS^2 + BS + C < 0 \quad \text{and} \quad d_2 < c\beta, \tag{2.2}$$

where

$$\begin{aligned} A &= cmr\beta b, \\ B &= -c\bigl[r\beta(\beta + md_1) + mk(r\beta + \alpha d_1)b\bigr] - d_2\bigl(-r\beta + Kb(\beta - \alpha)\bigr), \\ C &= K(r\beta + \alpha d_1)\bigl[c(\beta + md_1) - d_2\bigr], \\ S &= \frac{(r\beta + \alpha d_1)K}{r\beta + \alpha bK}, \end{aligned}$$

then there exists a unique positive equilibrium point $\mathbf{u}_* = (u_*, v_*, w_*)$, where

$$\begin{aligned} u_* &= \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \\ v_* &= -\left(\frac{r}{bK} + \frac{\alpha}{\beta}\right)u_* + \left(\frac{r}{b} + \frac{\alpha d_1}{b\beta}\right), \\ w_* &= \frac{(c\alpha - d_2)u_* + (c\beta - d_2)v_*}{d_2 m}. \end{aligned}$$

2.2 Asymptotic stability of equilibria

In this subsection, we investigate the non-negative equilibria $\mathbf{e_0}$, $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$ defined in (2.1) and the positive equilibrium point $\mathbf{u}_*$.

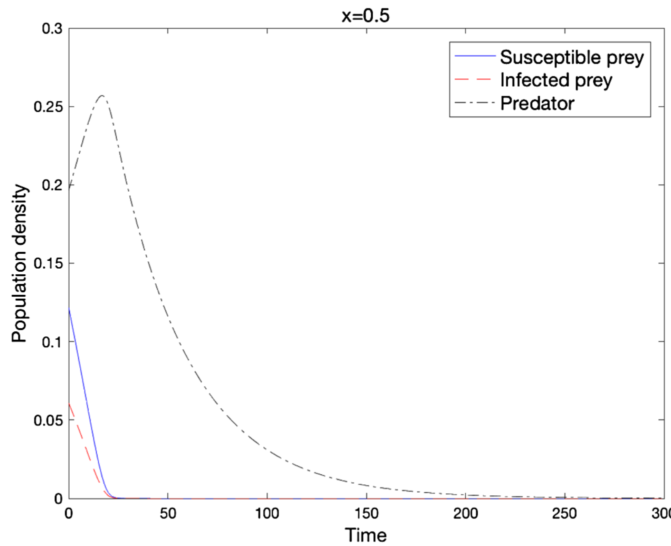

Figure 1 Local stability of e_0 when the condition of Theorem 2.4 holds ($d = 0.01, D = 0.01, r = 0.05, b = 0.430, d_1 = 0.03, d_2 = 0.6, K = 1.0, \alpha = 1.2, \beta = \alpha, c = 1.0, m = 0.08$).

2.2.1 Asymptotic stability of $\mathbf{e_0}$

We investigate the stability at $(0,0,0)$. For the stability of $\mathbf{e_0}$, we assume $d = D$.

Figure 1 shows that under some conditions, all three species become extinct. We point out that the corresponding non-spatial model had the same asymptotic behavior under the same condition in the following theorem.

Theorem 2.4 *Assume that* $mc \le 1$, $\beta \ge \alpha$, $\min\{c\alpha - d_2, \frac{\alpha}{1+m}\} \ge r$. *If the initial data satisfies* $w_0 \ge u_0 + v_0$, *then* $\lim_{t\to\infty} \mathbf{u} = \mathbf{e_0}$.

Proof Subtracting the first and second equations from the third equation in (1.1) yields

$$\begin{aligned}(w-u-v)_t - d\Delta(w-u-v) &= wf_3 - uf_1 - vf_2 \\ &= (w-u-v)f_3 + (f_3 - f_2)v + (f_3 - f_1)u. \end{aligned} \tag{2.3}$$

Also note that

$$\begin{aligned} f_3 - f_2 &= \frac{c\alpha u + c\beta v + \beta w}{mw + u + v} - d_2 - bu + d_1 \\ &= c\alpha \frac{u + \frac{\beta}{\alpha} v + \frac{\beta}{c\alpha} w}{mw + u + v} - d_2 - bu + d_1 \\ &\ge c\alpha - d_2 + d_1 - bu, \\ f_3 - f_1 &= \frac{c\alpha u + c\beta v + \alpha w}{mw + u + v} - d_2 - r + \frac{r}{K} u + bv \\ &= c\alpha \frac{u + \frac{\beta}{\alpha} v + \frac{1}{c} w}{mw + u + v} - d_2 - r + \frac{r}{K} u + bv \\ &\ge c\alpha - d_2 - r + \frac{r}{K} u + bv \end{aligned}$$

hold under the assumptions that $mc \leq 1$ and $\beta \geq \alpha$. As a result, $(f_3 - f_2)v + (f_3 - f_1)u \geq (c\alpha - d_2 + d_1)v + (c\alpha - d_2 - r + \frac{r}{K}u)u \geq 0$, since $c\alpha - d_2 \geq r$. Thus, applying the positivity lemma [31] to (2.3), $w \geq u + v$ holds if $w_0 \geq u_0 + v_0$. In the light of these facts, the main result is satisfied; specifically,

$$\begin{aligned}
u_t - d\Delta u &= u\left[r - \frac{r}{K}u - \frac{\alpha w}{mw + u + v} - bv\right] \\
&\leq u\left[r - \frac{r}{K}u - \frac{\alpha w}{mw + w}\right] \\
&= u\left[r - \frac{r}{K}u - \frac{\alpha}{m + 1}\right] \\
&\leq 0,
\end{aligned}$$

since $\frac{\alpha}{1+m} \geq r$. Thus, $\lim_{t\to\infty} w = 0$ on $\overline{\Omega}$. Consequently, $\lim_{t\to\infty} u = 0$ and $\lim_{t\to\infty} v = 0$ on $\overline{\Omega}$ since $w \geq u + v$. □

Theorem 2.5

(i) *If there exists a positive constant θ such that*

$$\begin{aligned}
&rm - \alpha + d_2 m \leq (c\alpha - r - d_2)\theta, \\
&d_2 m - d_1 m - \beta \leq (c\beta + d_1 - d_2)\theta,
\end{aligned} \tag{2.4}$$

holds, then the region $\Sigma = \{(u, v, w) : u, v, w \geq 0, u + v \leq \theta w\}$ is an invariant set for (1.1).

(ii) *In addition to* (2.4), *if $-m + \frac{\alpha}{r} \geq \theta$, $\lim_{t\to\infty} \mathbf{u} = e_0$ for the initial function $(u_0, v_0, w_0) \in \Sigma$.*

(iii) *In addition to* (2.4), *if $c\max\{\alpha, \beta\} \leq d_2$, $\lim_{t\to\infty} \mathbf{u} = e_0$ for the initial function $(u_0, v_0, w_0) \in \Sigma$.*

(iv) *In addition to* (2.4), *if $c\beta \geq c\alpha > d_2$ and $\theta < \frac{d_2 m}{c\alpha - d_2}$, $\lim_{t\to\infty} \mathbf{u} = e_0$ for the initial function $(u_0, v_0, w_0) \in \Sigma$.*

Proof (i) Let $G(u, v, w) = u + v - \theta w$. To achieve the desired result, Corollary 14.8 of [33] is used; in particular, we will show that (uf_1, vf_2, wf_3) points into Σ on $\partial\Sigma$. On the boundary of Σ (except for the boundary $u + v = \theta w$), $dG \cdot (uf_1, vf_2, wf_3) \leq 0$ can easily be verified.

It is straightforward to show that $dG \cdot (uf_1, vf_2, wf_3) \leq 0$ on the boundary $u + v = \theta w$. In fact,

$$\begin{aligned}
&dG \cdot (uf_1, vf_2, wf_3) \\
&\quad = (1, 1, -\theta) \cdot (uf_1, vf_2, wf_3) \\
&\quad = uf_1 + vf_2 - \theta wf_3 \\
&\quad = u\left[r - \frac{r}{K}u - \frac{\alpha w}{mw + \theta w} - bv\right] + v\left[bu - d_1 - \frac{\beta w}{mw + \theta w}\right] \\
&\qquad - \theta w\left[-d_2 + \frac{c\alpha u}{mw + \theta w} + \frac{c\beta v}{mw + \theta w}\right]
\end{aligned}$$

$$
\begin{aligned}
&= u\left(r - \frac{\alpha}{m+\theta}\right) - \frac{r}{K}u^2 + d_2\theta w - v\left(d_1 + \frac{\beta}{m+\theta}\right) - \frac{c\alpha\theta u}{m+\theta} - \frac{c\beta\theta v}{m+\theta} \\
&= u\left(r - \frac{\alpha}{m+\theta} + d_2 - \frac{c\alpha\theta}{m+\theta}\right) + v\left(d_2 - d_1 - \frac{\beta}{m+\theta} - \frac{c\beta\theta}{m+\theta}\right) - \frac{r}{K}u^2 \\
&\leq 0.
\end{aligned}
$$

The last inequality holds by assumption (2.4).

(ii) Since Σ is an invariant region under assumption (2.4), $u + v \leq \theta w$ holds for $(u_0, v_0, w_0) \in \Sigma$. Thus, the following inequality is satisfied if $u + v \leq \theta w$ and $-m + \frac{\alpha}{r} \geq \theta$:

$$
\begin{aligned}
u_t - d\Delta u &= u\left[r - \frac{r}{K}u - \frac{\alpha w}{mw + u + v} - bv\right] \\
&\leq u\left[r - \frac{r}{K}u - \frac{\alpha w}{mw + \theta w}\right] \\
&= u\left[r - \frac{r}{K}u - \frac{\alpha}{m+\theta}\right] \\
&\leq 0.
\end{aligned}
$$

Therefore, $\lim_{t\to\infty} u = 0$ on $\overline{\Omega}$. Consequently, v and w go to zero as $t \to \infty$.

(iii) By assumption, w goes to zero as $t \to \infty$. Also, u and v go to zero as $t \to \infty$ since (u, v, w) is contained in Σ.

(iv) Adding the first and second equations in (1.1) and using the facts that $u + v \leq \theta w$ and $\beta \geq \alpha$ imply

$$
\begin{aligned}
(u+v)_t - d\Delta(u+v) &= uf_1(u,v,w) + vf_2(u,v,w) \\
&\leq u\left(r - \frac{r}{K}u\right) - \frac{\alpha}{m+\theta}u - \frac{\beta}{m+\theta}v - d_1 v \\
&\leq u\left(r + d_1 - \frac{r}{K}u\right) - \left(\frac{\alpha}{m+\theta} + d_1\right)(u+v) \\
&\leq \frac{K}{r}\left(\frac{r+d_1}{2}\right)^2 - \left(\frac{\alpha}{m+\theta} + d_1\right)(u+v).
\end{aligned}
$$

Thus, $\limsup_{t\to\infty}(u+v) \leq \frac{m+\theta}{d_1(m+\theta)+\alpha}\frac{K}{r}(\frac{r+d_1}{2})^2 := \rho$, as in Theorem 2.2. Hence, there exists a T_0 such that $u(t,x) + v(t,x) \leq \rho + \varepsilon$ on $\overline{\Omega}$ for time $t \geq T_0$.

Consider the third equation in (1.1):

$$
\begin{aligned}
w_t - D\Delta w &= wf_3(u,v,w) \\
&\leq w\left[\frac{c\beta(u+v)}{mw + u + v} - d_2\right] \\
&\leq w\left[\frac{(c\beta - d_2)(\rho+\varepsilon) - d_2 mw}{mw + \rho + \varepsilon}\right].
\end{aligned}
\tag{2.5}
$$

Then there exists a $T_1 \geq T_0$ such that $w(t,x) \leq \frac{c\beta - d_2}{d_2 m}(\rho + \varepsilon) + \varepsilon$ on $\overline{\Omega}$ for time $t \geq T_1$. Since $u + v \leq \theta w$ holds, $u(t,x) + v(t,x) \leq \theta[\frac{c\beta - d_2}{d_2 m}(\rho + \varepsilon) + \varepsilon] := \rho(\varepsilon)$ is satisfied on $\overline{\Omega}$ for time $t \geq T_1$.

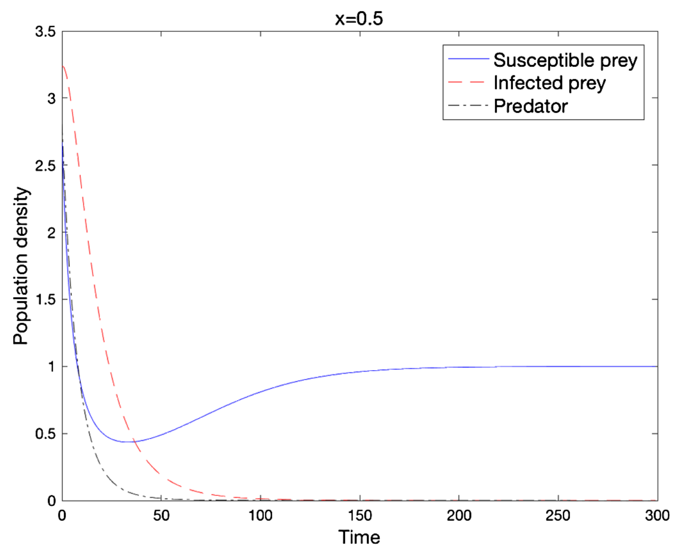

Figure 2 Local stability of e_1 when the condition of Theorem 2.6 holds **($d = 0.01, D = 0.01, r = 0.5, b = 0.430, d_1 = 1.0, d_2 = 2.0, K = 1.0, \alpha = 1.2, \beta = \alpha, c = 1.0, m = 10.0$).**

Let $\tau = \frac{1}{2}[1 + \theta\frac{c\beta-d_2}{d_2m}]$. Under the assumption that $\theta < \frac{d_2m}{c\alpha-d_2}$, $\tau < 1$ is satisfied. Since $\rho(0) = \theta\frac{c\beta-d_2}{d_2m}\rho < \tau\rho$, if a sufficiently small $\varepsilon > 0$ is chosen such that $\rho(\varepsilon) < \tau\rho$, $u(t,x) + v(t,x) \leq \theta[\frac{c\beta-d_2}{d_2m}(\rho+\varepsilon)+\varepsilon] < \tau\rho$ on $\overline{\Omega}$ for $t \geq T_1$.

Now, consider (2.5) under the restriction that $u(t,x) + v(t,x) \leq \tau\rho$. Then $\limsup_{t\to\infty} w \leq \frac{c\beta-d_2}{d_2m}\tau\rho$. Thus, there exists a $T_2 \geq T_1$ such that $w(t,x) \leq \frac{c\beta-d_2}{d_2m}\tau\rho + \varepsilon$ on $\overline{\Omega}$ for time $t \geq T_2$. Again, $u(t,x) + v(t,x) \leq \theta[\frac{c\beta-d_2}{d_2m}(\tau\rho)+\varepsilon] \leq \tau^2\rho$ on $\overline{\Omega}$ for $t \geq T_2$ and for a sufficiently small $\varepsilon > 0$.

Inductively, there exists a sequence T_n with $T_n \to \infty$ such that $u(t,x) + v(t,x) \leq \tau^n\rho$ on $\overline{\Omega}$ for $t \geq T_n$. Moreover, since $\tau < 1$, $u + v \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Consequently, w goes to zero as $t \to \infty$ as well. □

2.2.2 *Asymptotic stability of* $\mathbf{e_1}$

In this subsection, we investigate the stability at $(K, 0, 0)$ under the following conditions:

$$d_1 \geq bK, \qquad d_2 \geq c\alpha \quad \text{and} \quad r > \frac{\alpha}{m}. \tag{2.6}$$

The next result implies that only the susceptible prey can survives (Figure 2).

Theorem 2.6 *Under assumption* (2.6), $\lim_{t\to\infty} \mathbf{u} = \mathbf{e_1}$ *uniformly on* $\overline{\Omega}$.

Proof From Theorem 2.2, we already know $\limsup_{t\to\infty} u \leq K$. Furthermore, since $d_1 \geq bK$, $v_t - d\Delta v = vf_2(u,v,w) \leq 0$ implies $v \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Thus there exists a $T_1 > 0$ such that $v \leq \varepsilon$ for an arbitrary $\varepsilon > 0$ and $t \geq T_1$. Since ε is arbitrary, the assumption that $d_2 \geq c\alpha$ and the comparison principle imply $w \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Therefore, there exists $T_2 \geq T_1$ such that $w \leq \varepsilon$ for $t \geq T_2$.

Note that $\liminf_{t\to\infty} u \geq (r - b\varepsilon - \frac{\alpha}{m})\frac{K}{r} := \Theta$ can also be obtained using the methods from Theorem 2.3. Since $uf_1(u,v,w) \geq u[r - \frac{r}{K}u - \frac{\alpha\varepsilon}{m\varepsilon+u} - b\varepsilon] \geq u[r - \frac{r}{K}u - \frac{\alpha\varepsilon}{m\varepsilon+\Theta-\varepsilon} - b\varepsilon]$

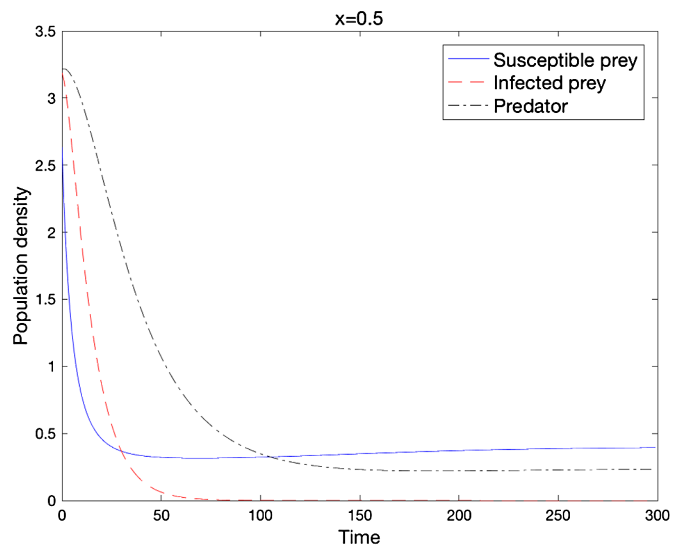

Figure 3 Local stability of e_2 when the condition of Theorem 2.7 holds ($d = 0.01, D = 0.01, r = 0.5, b = 0.430, d_1 = 1.0, d_2 = 0.5, K = 1.0, \alpha = 3.5, \beta = \alpha, c = 1.0, m = 10.0$).

for $t \geq T_2$, $\liminf_{t\to\infty} u \geq (r - b\varepsilon - \frac{\alpha\varepsilon}{(m-1)\varepsilon+\Theta})\frac{K}{r} > \Theta$ follows from the comparison principle. Therefore, since ε is arbitrary, $\lim_{t\to\infty} \mathbf{u} = e_1$ uniformly on $\overline{\Omega}$. □

2.2.3 *Asymptotic stability of* $\mathbf{e_2}$

We investigate the stability at $(K(1 - \frac{c\alpha-d_2}{cmr}), 0, \frac{c\alpha-d_2}{d_2m}K(1 - \frac{c\alpha-d_2}{cmr}))$ under the following condition:

$$rmd_2 \leq (rm - \alpha)(c\alpha - d_2), \qquad bK < d_1 \quad \text{and} \quad 0 < c\alpha - d_2 < cmr. \tag{2.7}$$

For simplicity, let $u_2^* = K(1 - \frac{c\alpha-d_2}{cmr})$ and $w_2^* = \frac{c\alpha-d_2}{d_2m}u_2^*$.

The following theorem indicates that one can control the infected prey, namely, only the infected prey can be removed out under some conditions (Figure 3).

Theorem 2.7 *Under assumption* (2.7), $\lim_{t\to\infty} \mathbf{u} = \mathbf{e_2}$ *uniformly on* $\overline{\Omega}$.

Proof We prove this theorem by induction. First, consider the following parabolic problem:

$$\begin{cases} \overline{u}_{1t} - d\Delta\overline{u}_1 = \overline{u}_1(r - \frac{r}{K}\overline{u}_1) & \text{in } (0,\infty) \times \Omega, \\ \frac{\partial\overline{u}_1}{\partial\eta} = 0 \quad \text{on } (0,\infty) \times \partial\Omega, \\ \overline{u}_1(0,x) = u_0(x) \quad \text{in } \Omega. \end{cases}$$

Then there exists a $T_1^1 > 0$ such that $u \leq \overline{u}_1^*(\equiv K) + \varepsilon$ on $\overline{\Omega}$ for $t \geq T_1^1$ and a sufficiently small ϵ such that $\frac{d_1}{b} - K > \varepsilon > 0$.

Next, consider the following problem under the condition that $bK < d_1$:

$$\begin{cases} \overline{v}_t - d\Delta\overline{v} = \overline{v}(b(\overline{u}_1^* + \varepsilon) - d_1) & \text{in } (T_1^1, \infty) \times \Omega, \\ \frac{\partial \overline{v}}{\partial \eta} = 0 & \text{on } (T_1^1, \infty) \times \partial\Omega, \\ \overline{v}(0,x) = v(T_1^1, x) & \text{in } \Omega. \end{cases}$$

Then there exists a $T_2^1 \geq T_1^1$ such that $v \leq \varepsilon$ on $\overline{\Omega}$ for $t \geq T_2^1$.

Consider the following problem:

$$\begin{cases} \underline{u}_{1t} - d\Delta\underline{u}_1 = \underline{u}_1(r - \frac{\alpha}{m} - b\varepsilon - \frac{r}{K}\underline{u}_1) & \text{in } (T_2^1, \infty) \times \Omega, \\ \frac{\partial \underline{u}_1}{\partial \eta} = 0 & \text{on } (T_2^1, \infty) \times \partial\Omega, \\ \underline{u}_1(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Then there exists a $T_3^1 \geq T_2^1$ such that $u \geq \underline{u}_1^* (\equiv \frac{K}{r}(r - \frac{\alpha}{m})) - (1 + \frac{K}{r})\varepsilon$ on $\overline{\Omega}$ for $t \geq T_3^1$. For simplicity, and since the choice of ε does not affect our proof, redefine $(1 + \frac{K}{r})\varepsilon$ by $\varepsilon > 0$.

Consider the following problem:

$$\begin{cases} \underline{w}_{1t} - D\Delta\underline{w}_1 = \underline{w}_1(\frac{c\alpha(\underline{u}_1^* - \varepsilon)}{m\underline{w}_1 + \underline{u}_1^*} - d_2) & \text{in } (T_3^1, \infty) \times \Omega, \\ \frac{\partial \underline{w}_1}{\partial \eta} = 0 & \text{on } (T_3^1, \infty) \times \partial\Omega, \\ \underline{w}_1(0,x) = w(T_3^1, x) & \text{in } \Omega. \end{cases}$$

Then there exists a $T_4^1 \geq T_3^1$ such that $w \geq \underline{w}_1^* (\equiv \frac{c\alpha - d_2}{d_2 m}\underline{u}_1^*) - \varepsilon$ on $\overline{\Omega}$ for $t \geq T_3^1$.

Consider the following problem:

$$\begin{cases} \overline{w}_{1t} - D\Delta\overline{w}_1 = \overline{w}_1(\frac{c\alpha(\overline{u}_1^* + \varepsilon)}{m\overline{w}_1 + \overline{u}_1^* + \varepsilon} - d_2 + \frac{c\beta\varepsilon}{m(\underline{w}_1^* - \varepsilon) + \varepsilon}) & \text{in } (T_4^1, \infty) \times \Omega, \\ \frac{\partial \overline{w}_1}{\partial \eta} = 0 & \text{on } (T_4^1, \infty) \times \partial\Omega, \\ \overline{w}_1(0,x) = w(T_4^1, x) & \text{in } \Omega. \end{cases}$$

Then there exists a $T_5^1 \geq T_4^1$ such that $w \leq \overline{w}_1^* (\equiv \frac{c\alpha - d_2}{d_2 m}\overline{u}_1^*) + \varepsilon$ on $\overline{\Omega}$ for $t \geq T_5^1$.

Consequently, for $t \geq T^1 \equiv T_5^1$ and $x \in \overline{\Omega}$, the relation

$$\begin{aligned} &\underline{u}_1^* - \varepsilon \leq u \leq \overline{u}_1^* + \varepsilon, \\ &0 \leq v \leq \varepsilon, \\ &\underline{w}_1^* - \varepsilon \leq w \leq \overline{w}_1^* + \varepsilon \end{aligned}$$

are satisfied.

For induction, consider the following problems for $T^{n-1} \leq T_1^{n-1} \leq T_2^{n-1} \leq T_3^{n-1} \leq T_4^{n-1}$ and $n \geq 2$:

$$\begin{cases} \overline{u}_{nt} - d\Delta\overline{u}_n = \overline{u}_n(r - \frac{\alpha(\underline{w}_{n-1}^* - \varepsilon)}{m(\underline{w}_{n-1}^* - \varepsilon) + \overline{u}_{n-1}^* + \varepsilon} - \frac{r}{K}\overline{u}_n) & \text{in } (T^{n-1}, \infty) \times \Omega, \\ \frac{\partial \overline{u}_n}{\partial \eta} = 0 & \text{on } (T^{n-1}, \infty) \times \partial\Omega, \\ \overline{u}_n(0,x) = u(T^{n-1}, x) & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \overline{v}_{nt} - d\Delta\overline{v}_n = \overline{v}_n(b(\overline{u}_n^* + \varepsilon) - d_1) & \text{in } (T_1^{n-1}, \infty) \times \Omega, \\ \frac{\partial \overline{v}_n}{\partial \eta} = 0 & \text{on } (T_1^{n-1}, \infty) \times \partial\Omega, \\ \overline{v}_n(0,x) = v(T_1^{n-1}, x) & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \overline{w}_{nt} - D\Delta\overline{w}_n = \overline{w}_n(\frac{c\alpha(\overline{u}_n^*+\varepsilon)}{m\overline{w}_n+\overline{u}_n^*+\varepsilon} - d_2 + \frac{c\beta\varepsilon}{m(\underline{w}_1^*-\varepsilon)+\varepsilon}) & \text{in } (T_2^{n-1}, \infty) \times \Omega, \\ \frac{\partial \overline{w}_n}{\partial \eta} = 0 & \text{on } (T_2^{n-1}, \infty) \times \partial\Omega, \\ \overline{w}_n(0,x) = w(T_2^{n-1}, x) & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \underline{u}_{nt} - d\Delta\underline{u}_n = \underline{u}_n(r - \frac{\alpha(\overline{w}_n^*+\varepsilon)}{m(\overline{w}_n^*+\varepsilon)+\underline{u}_{n-1}^*} - b\varepsilon - \frac{r}{K}\underline{u}_n) & \text{in } (T_3^{n-1}, \infty) \times \Omega, \\ \frac{\partial \underline{u}_n}{\partial \eta} = 0 & \text{on } (T_3^{n-1}, \infty) \times \partial\Omega, \\ \underline{u}_n(0,x) = u(T_3^{n-1}, x) & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \underline{w}_{nt} - D\Delta\underline{w}_n = \underline{w}_n(\frac{c\alpha(\underline{u}_n^*-\varepsilon)}{m\underline{w}_n+(\underline{u}_n^*-\varepsilon)} - d_2) & \text{in } (T_4^{n-1}, \infty) \times \Omega, \\ \frac{\partial \underline{w}_1}{\partial \eta} = 0 & \text{on } (T_4^{n-1}, \infty) \times \partial\Omega, \\ \underline{w}_1(0,x) = w(T_4^{n-1}, x) & \text{in } \Omega. \end{cases}$$

Therefore, for $t \geq T^n \equiv T_4^{n-1}$, $n \geq 2$ and $x \in \overline{\Omega}$,

$$\underline{u}_n^* - \varepsilon \leq u \leq \overline{u}_n^* + \varepsilon,$$
$$0 \leq v \leq \varepsilon,$$
$$\underline{w}_n^* - \varepsilon \leq w \leq \overline{w}_n^* + \varepsilon,$$

where

$$\overline{u}_n^* = \frac{K}{r}\left(r - \frac{\alpha \underline{w}_{n-1}^*}{m\underline{w}_{n-1}^* + \overline{u}_{n-1}^*}\right),$$
$$\overline{w}_n^* = \frac{c\alpha - d_2}{d_2 m}\overline{u}_n^*,$$
$$\underline{u}_n^* = \frac{K}{r}\left(r - \frac{\alpha \overline{w}_n^*}{m\overline{w}_n^* + \underline{u}_{n-1}^*}\right),$$
$$\underline{w}_n^* = \frac{c\alpha - d_2}{d_2 m}\underline{u}_n^*.$$

Note that $\overline{u}_n^*$, $\overline{w}_n^*$, $\underline{u}_n^*$, and $\underline{w}_n^*$ are all positive constants. Moreover, the following monotonicity holds:

$$\underline{u}_1^* \leq \underline{u}_2^* \leq \cdots \leq \underline{u}_n^* \leq \cdots \leq u_2{}^* \leq \cdots \overline{u}_n^* \leq \cdots \overline{u}_2^* \leq \overline{u}_1^*,$$
$$\underline{w}_1^* \leq \underline{w}_2^* \leq \cdots \leq \underline{w}_n^* \leq \cdots \leq w_2{}^* \leq \cdots \overline{w}_n^* \leq \cdots \overline{w}_2^* \leq \overline{w}_1^*,$$

since $\frac{\alpha \underline{w}_n^*}{m\underline{w}_n^*+\underline{u}_n^*} \geq \frac{\alpha \underline{w}_{n-1}^*}{m\underline{w}_{n-1}^*+\overline{u}_{n-1}^*}$ and $\frac{\alpha \overline{w}_n^*}{m\overline{w}_n^*+\underline{u}_{n-1}^*} \leq \frac{\alpha \overline{w}_{n-1}^*}{m\overline{w}_{n-1}^*+\underline{u}_{n-2}^*}$ for all n by induction. Also $\underline{u}_n^* \leq u_2^* \leq \overline{u}_n^*$ holds for all n, since $\frac{\alpha \underline{w}_{n-1}^*}{m\underline{w}_{n-1}^*+\overline{u}_{n-1}^*} \leq \frac{\alpha\frac{c\alpha-d_2}{d_2 m}}{m\frac{c\alpha-d_2}{d_2 m}+1} = \frac{c\alpha-d_2}{cm} \leq \frac{\alpha \overline{w}_n^*}{m\overline{w}_n^*+\underline{u}_{n-1}^*}$ for $\frac{\overline{u}_n^*}{\underline{u}_{n-1}^*} \geq 1$ and by the definitions of $\overline{w}_n^*$ and $\underline{w}_n^*$. It follows that $\underline{w}_n^* \leq w_2^* \leq \overline{w}_n^*$ for all n.

Thus, since the constant sequences $\{\overline{u}_n^*\}$ and $\{\overline{w}_n^*\}$ are monotone nonincreasing, and bounded from below, and the sequences $\{\underline{u}_n^*\}$ and $\{\underline{w}_n^*\}$ are monotone nondecreasing, and bounded from above, the limits of these sequences exist. Denote these limits by $\overline{u}$, $\overline{w}$, $\underline{u}$, and $\underline{w}$, respectively. Consequently, $\underline{u} \le u_2^* \le \overline{u}$ and $\underline{w} \le w_2^* \le \overline{w}$. The following also holds:

$$\begin{cases} \overline{u} = \frac{K}{r}(r - \frac{\alpha \underline{w}}{m\underline{w}+\overline{u}}), \\ \overline{w} = \frac{c\alpha - d_2}{d_2 m}\overline{u}, \\ \underline{u} = \frac{K}{r}(r - \frac{\alpha \overline{w}}{m\overline{w}+\underline{u}}), \\ \underline{w} = \frac{c\alpha - d_2}{d_2 m}\underline{u}. \end{cases} \tag{2.8}$$

Suppose to the contrary that $\overline{u} \neq \underline{u}$. The first and third equations in (2.8) can be rewritten as

$$r - \frac{r}{K}\overline{u} - \frac{\alpha \frac{c\alpha-d_2}{d_2 m}\underline{u}}{m\frac{c\alpha-d_2}{d_2 m}\underline{u} + \overline{u}} = 0,$$

$$r - \frac{r}{K}\underline{u} - \frac{\alpha \frac{c\alpha-d_2}{d_2 m}\overline{u}}{m\frac{c\alpha-d_2}{d_2 m}\overline{u} + \underline{u}} = 0,$$

respectively. These two equations imply

$$\begin{cases} (rm-\alpha)\frac{c\alpha-d_2}{d_2 m}\underline{u} + r\overline{u} - \frac{r}{K}m\frac{c\alpha-d_2}{d_2 m}\overline{u}\underline{u} - \frac{r}{K}\overline{u}\overline{u} & = 0, \\ (rm-\alpha)\frac{c\alpha-d_2}{d_2 m}\overline{u} + r\underline{u} - \frac{r}{K}m\frac{c\alpha-d_2}{d_2 m}\overline{u}\underline{u} - \frac{r}{K}\underline{u}\underline{u} & = 0. \end{cases} \tag{2.9}$$

Subtracting the second equation from the first equation in (2.9) yields

$$(\overline{u} - \underline{u})\left(r - \frac{r}{K}(\overline{u} + \underline{u}) - (rm-\alpha)\frac{c\alpha - d_2}{d_2 m}\right) = 0.$$

By assumption, since $\overline{u} \neq \underline{u}$ (*i.e.*, $\overline{u} > \underline{u}$), $A := r - \frac{r}{K}(\overline{u} + \underline{u}) - (rm-\alpha)\frac{c\alpha-d_2}{d_2 m}$ must be zero. But $A < r - (rm-\alpha)\frac{c\alpha-d_2}{d_2 m} \le 0$ from (2.7). Hence, $\overline{u} = \underline{u} = u_2^*$; likewise, $\overline{w} = \underline{w} = w_2^*$. Consequently, as time t goes to infinity (*i.e.*, $n \to \infty$),

$$u_2^* - \varepsilon \le u \le u_2^* + \varepsilon,$$

$$0 \le v \le \varepsilon,$$

$$w_2^* - \varepsilon \le w \le w_2^* + \varepsilon,$$

are satisfied for an arbitrary $\varepsilon > 0$. Therefore, the desired result is achieved. □

In the following theorem, we modify the condition that $bK < d_1$ in (2.7) by reversing the inequality, *i.e.*, $bK > d_1$, since $bK < d_1$ causes v to converge to zero automatically.

Theorem 2.8 *If the following conditions hold*:

$$rmd_2 \le (rm-\alpha)(c\alpha - d_2),$$

$$d_1 < bK < d_1 + \frac{\beta\sigma}{m\sigma + \theta},$$

$$0 < c\alpha - d_2 < cmr,$$
$$r > \frac{\alpha}{m} + b\theta,$$

where $\theta := \frac{1}{d_1}\frac{K}{r}(\frac{r+d_1}{2})^2$ *and* $\sigma := \frac{c\alpha-d_2}{d_2 m}(r - \frac{\alpha}{m} - b\theta)\frac{K}{r}$, *then* $\lim_{t\to\infty} \mathbf{u} = \mathbf{e_2}$ *uniformly on* $\overline{\Omega}$.

Proof First, note that there exists a $T_1 > 0$ such that $u \le K + \varepsilon$ and $u + v \le \theta + \varepsilon$ for $t \ge T_1$ and $x \in \overline{\Omega}$, as in Theorem 2.2.

Consider the following parabolic problem:

$$\begin{cases} U_t - d\Delta U = U(r - \frac{\alpha}{m} - b(\theta + \varepsilon) - \frac{r}{K}U) & \text{in } (T_1, \infty) \times \Omega, \\ \frac{\partial U}{\partial \eta} = 0 \quad \text{on } (T_1, \infty) \times \partial\Omega, \\ U(0,x) = u(T_1,x) \quad \text{in } \Omega. \end{cases}$$

Then there exists $T_2 \ge T_1$ such that $u \ge (r - \frac{\alpha}{m} - b\theta)\frac{K}{r} - \varepsilon$ on $\overline{\Omega}$ for $t \ge T_2$. It follows that $w \ge \sigma - \varepsilon$ for $t \ge T_3$ and $x \in \overline{\Omega}$ where $T_3 \ge T_2$. Now, we are ready to prove that $\lim_{t\to\infty} v = 0$ uniformly on $\overline{\Omega}$.

Consider the following problem:

$$\begin{cases} V_t - d\Delta V = V(b(K + \varepsilon) - \frac{\beta(\sigma-\varepsilon)}{m(\sigma-\varepsilon)+\theta+\varepsilon} - d_1) & \text{in } (T_3, \infty) \times \Omega, \\ \frac{\partial V}{\partial \eta} = 0 \quad \text{on } (T_3, \infty) \times \partial\Omega, \\ V(0,x) = v(T_3,x) \quad \text{in } \Omega. \end{cases} \tag{2.10}$$

For a sufficiently small $\varepsilon > 0$, the right hand side of the first equation in (2.10) is negative because $bK < d_1 + \frac{\beta\sigma}{m\sigma+\theta}$.

Hence, similar to Theorem 2.7, there exists $T_4 \ge T_3$ such that $0 \le v \le \varepsilon$ for $t \ge T_4$. The remainder of this proof follows using the same argument as Theorem 2.7. □

2.2.4 Asymptotic stability of $\mathbf{e_3}$

In this subsection, we investigate the stability at $(\frac{d_1}{b}, \frac{1}{b}(r - \frac{r}{K}\frac{d_1}{b}), 0)$. Before developing our argument, we define the following notation, which is similar to the notation defined in [34, 35].

Notation 2.9

(i) μ_i: Eigenvalue of $-\Delta$ on Ω under Neumann boundary condition.
(ii) $E(\mu_i)$: The eigenspace corresponding to μ_i.
(iii) $\{\varphi_{ij} : j = 1, \dots, \dim E(\mu_i)\}$: An orthonormal basis of $E(\mu_i)$.
(iv) $\mathbf{X_{ij}} = \{\mathbf{c} \cdot \varphi_{ij} | \mathbf{c} \in \mathbb{R}^3\}$.
(v) $\mathbf{X} = \{\mathbf{u} = (u, v, w) \in [C^1(\overline{\Omega})]^3 | \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \text{ on } \partial\Omega\}$.

The eigenvalues in (i) satisfy $0 = \mu_1 < \mu_2 < \mu_3 < \cdots \to \infty$. Also, $\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X_i}$, where $\mathbf{X_i} = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbf{X_{ij}}$. Now, we show the local stability at $\mathbf{e_3}$.

The susceptible prey and the infected prey may survive together without the predator (Figure 4).

Theorem 2.10 *If* $\max\{\alpha, \beta\} < \frac{d_2}{c}$ *and* $d_1(r + bK) > b^2K^2 > d_1^2$, *then equilibrium* $\mathbf{e_3}$ *of* (1.1) *is locally asymptotically stable.*

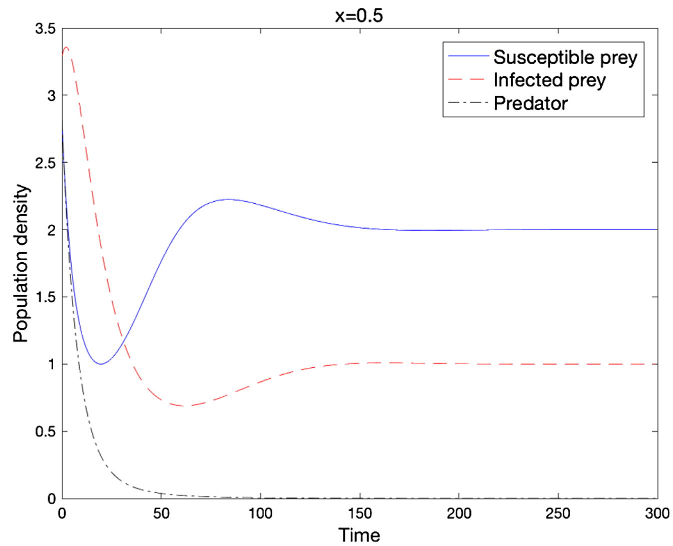

Figure 4 Local stability of e_3 when the condition of Theorem 2.10 holds ($d = 0.01, D = 0.01, r = 1.5, b = 0.5, d_1 = 1.0, d_2 = 2.0, K = 3.0, \alpha = 1.5, \beta = \alpha, c = 1.0, m = 10.0$).

Proof First, note that the above assumptions guarantee the positiveness of $\mathbf{e_3}$. For simplicity, let $u_3^* = \frac{d_1}{b}$ and $v_3^* = \frac{1}{b}(r - \frac{r}{K}u_3^*)$. The assumptions that $\max\{\alpha, \beta\} < \frac{d_2}{c}$ and $d_1(r + bK) > b^2K^2$ also guarantee that the positivity of $\Theta := \frac{(d_2-c\alpha)u_3^*+(d_2-c\beta)v_3^*}{u_3^*+v_3^*}$ and $(\frac{r}{K})^2u_3^* - b^2v_3^*$.

The linearization of (1.1) is $\mathbf{u}_t = (\mathbf{D}\Delta + \mathbf{F_u}(\mathbf{e_3}))\mathbf{u}$ at the constant solution $\mathbf{e_3}$, where $\mathbf{u} = (u(t,x), v(t,x), w(t,x))^T$, $\mathbf{F} = (uf_1, vf_2, wf_3)$,

$$\mathbf{D} = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & D \end{pmatrix} \quad \text{and} \quad \mathbf{F_u}(\mathbf{e_3}) = \begin{pmatrix} -\frac{r}{K}u_3^* & -bu_3^* & -\frac{\alpha u_3^*}{u_3^*+v_3^*} \\ bv_3^* & 0 & -\frac{\alpha u_3^*}{u_3^*+v_3^*} \\ 0 & 0 & -\Theta \end{pmatrix}.$$

For $i \geq 1$, $\mathbf{X_i}$ is invariant under the operator $\mathbf{D}\Delta + \mathbf{F_u}(\mathbf{e_3})$. Note that λ is an eigenvalue of this operator on $\mathbf{X_i}$ if and only if it is an eigenvalue of the matrix $-\mu_i\mathbf{D} + \mathbf{F_u}(\mathbf{e_3})$. The coefficients of the characteristic polynomial $\det(\lambda\mathbf{I} + \mu_i\mathbf{D} - \mathbf{F_u}(\mathbf{e_3}))$ are given by $\lambda^3 + A_i\lambda^2 + B_i\lambda + C_i$, where

$$A_i = (2d + D)\mu_i + \frac{r}{K}u_3^* + \Theta > 0,$$

$$B_i = (d + 2D)d\mu_i^2 + \left(\left(\frac{r}{K}u_3^* + 2\Theta\right)d + \frac{r}{K}u_3^*D\right)\mu_i + b^2u_3^*v_3^* + \frac{r}{K}u_3^*\Theta > 0,$$

$$C_i = d^2D\mu_i^3 + \left(\frac{r}{K}u_3^*dD + d^2\Theta\right)\mu_i^2 + \left(2b^2u_3^*v_3^*D + \frac{r}{K}u_3^*d\Theta\right)\mu_i + 2b^2u_3^*v_3^*\Theta > 0.$$

It is easy to verify that A_i, B_i and C_i are all positive.

Finally, we obtain $A_iB_i - C_i = \tau_i^1\mu_i^3 + \tau_i^2\mu_i^2 + \tau_i^3\mu_i + \tau_i^4$, where

$$\tau_i^1 = (2d + D)(d^2 + 2dD) - d^2D = 2d(d + D)^2 > 0,$$

$$\tau_i^2 = (2d + D)\left(\left(\frac{r}{K}u_3^* + 2\Theta\right)d + \frac{r}{K}u_3^*D\right) + d^2\frac{r}{K}u_3^* + dD\left(\frac{r}{K}u_3^* + 2\Theta\right) > 0,$$

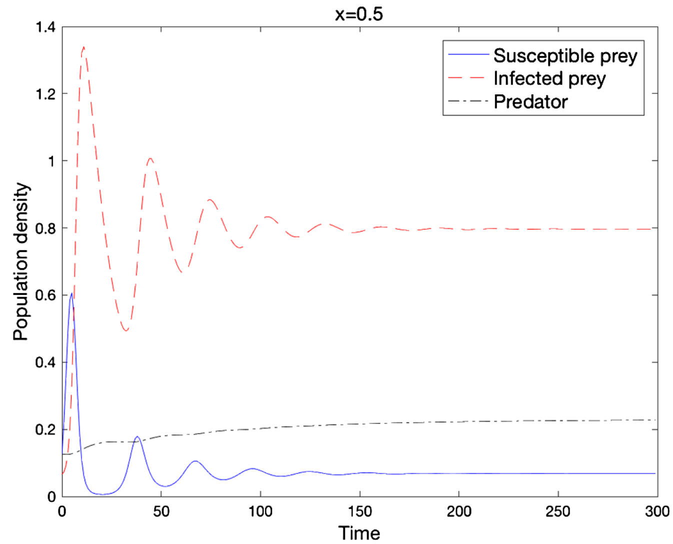

Figure 5 Local stability of $\mathbf{u}_*$ when the condition of Theorem 2.11 holds ($d = 0.01, D = 0.01, r = 0.3699, b = 0.430, d_1 = 0.0291, d_2 = 0.0069, K = 0.9299, \alpha = 0.0122, \beta = \alpha, c = 8.0999, m = 50.3$).

$$\begin{aligned}\tau_i^3 &= b^2u_3^*v_3^*2d + \frac{r}{K}u_3^*d\Theta + 2\frac{r}{K}u_3^*\Theta D \\ &\quad + \left(\frac{r}{K}u_3^* + \Theta\right)\left(\frac{r}{K} + 2\Theta\right)d + \left(\left(\frac{r}{K}\right)^2 u_3^*u_3^* - b^2u_3^*v_3^*\right)D > 0, \\ \tau_i^4 &= \frac{r}{K}b^2u_3^*v_3^* + \frac{r}{K}u_3^*\Theta^2 + \left(\left(\frac{r}{K}\right)^2 u_3^*u_3^* - b^2u_3^*v_3^*\right)\Theta > 0.\end{aligned}$$

Hence, $A_iB_i - C_i > 0$ for all $i \geq 1$. From the Routh-Hurwitz criterion for each i, the three roots of $\lambda^3 + A_i\lambda^2 + B_i\lambda + C_i = 0$ have negative real parts since A_i, C_i, and $A_iB_i - C_i > 0$. The remainder of this proof follows from Theorem 5.1.1 in [36]. □

2.2.5 *Asymptotic stability of* $\mathbf{u}_*$

We investigate the asymptotic stability of the positive equilibrium point under (2.2) and the following conditions:

$$\begin{cases} mc \geq 1, \qquad \alpha = \beta, \\ d_2 < c\alpha, \\ \max\{\frac{d_2 b}{c\alpha}, \alpha\frac{c\alpha - d_2}{d_2 m}\frac{b}{d_1}\} \leq \frac{r}{K} \leq \frac{bd_1}{\alpha}\frac{d_2 m}{c\alpha - d_2}. \end{cases} \tag{2.11}$$

Here, we can choose numerical values that satisfy condition (2.2) and (2.11), for example, $r = 0.3699$, $b = 0.430$, $d_1 = 0.0291$, $K = 0.9299$, $\alpha = \beta = 0.0122$, $m = 50.3$, $c = 8.0999$, and $d_2 = 0.0069$.

The final result says that all three species can survives together under specific conditions (Figure 5).

Theorem 2.11 *If* (2.2) *and* (2.11) *hold, then the equilibrium solution* $\mathbf{u}_*$ *of* (1.1) *is locally asymptotically stable.*

Proof The linearization of (1.1) is $\mathbf{u}_t = (\mathbf{D}\Delta + \mathbf{F_u}(\mathbf{u}_*))\mathbf{u}$ at the constant solution $\mathbf{u}_*$, where $\mathbf{u} = (u(t,x), v(t,x), w(t,x))^T$, $\mathbf{F} = (uf_1, vf_2, wf_3)$,

$$\mathbf{D} = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & D \end{pmatrix}$$

and

$$\mathbf{F_u}(\mathbf{u}_*) = \begin{pmatrix} u_*(-\frac{r}{K} + \frac{\alpha w_*}{(mw_*+u_*+v_*)^2}) & u_*(\frac{\alpha w_*}{(mw_*+u_*+v_*)^2} - b) & -\alpha u_*(\frac{u_*+v_*}{(mw_*+u_*+v_*)^2}) \\ v_*(\frac{\beta w_*}{(mw_*+u_*+v_*)^2} + b) & (\frac{\beta v_* w_*}{(mw_*+u_*+v_*)^2}) & -\beta v_*(\frac{u_*+v_*}{(mw_*+u_*+v_*)^2}) \\ w_*(\frac{c\alpha m w_* + c(\alpha-\beta)v_*}{(mw_*+u_*+v_*)^2}) & w_*(\frac{c\beta m w_* + c(\beta-\alpha)u_*}{(mw_*+u_*+v_*)^2}) & -mw_*(\frac{c(\alpha u_* + \beta v_*)}{(mw_*+u_*+v_*)^2}) \end{pmatrix}.$$

The following notation is adopted for simplicity:

$$\mathbf{F_u}(\mathbf{u}_*) = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}. \tag{2.12}$$

For $i \geq 1$, $\mathbf{X_i}$ is invariant under the operator $\mathbf{D}\Delta + \mathbf{F_u}(\mathbf{u}_*)$, and λ is an eigenvalue of this operator on $\mathbf{X_i}$, if and only if it is an eigenvalue of the matrix $-\mu_i\mathbf{D} + \mathbf{F_u}(\mathbf{u}_*)$. The coefficients of the characteristic polynomial $\det(\lambda\mathbf{I} + \mu_i\mathbf{D} - \mathbf{F_u}(\mathbf{u}_*))$ are given by $\lambda^3 + A_i\lambda^2 + B_i\lambda + C_i$, where

$$\begin{aligned} A_i &= (2d + D)\mu_i - L_{11} - L_{22} - L_{33}, \\ B_i &= d(d + 2D)\mu_i^2 + \bigl(-(L_{11} + L_{22} + 2L_{33})d - (L_{11} + L_{22})D\bigr)\mu_i + L_{11}L_{22} \\ &\quad + L_{22}L_{33} + L_{11}L_{33} - L_{13}L_{31} - L_{32}L_{23} - L_{21}L_{12}, \\ C_i &= d^2D\mu_i^3 + \bigl(-(L_{11} + L_{22})dD - d^2L_{33}\bigr)\mu_i^2 + \bigl((L_{11}L_{33} + L_{22}L_{33} - L_{13}L_{31} - L_{32}L_{23})d \\ &\quad + (L_{11}L_{22} - L_{21}L_{12})D\bigr)\mu_i - L_{11}L_{22}L_{33} - L_{21}L_{13}L_{32} - L_{31}L_{12}L_{23} + L_{13}L_{31}L_{22} \\ &\quad + L_{32}L_{23}L_{11} + L_{21}L_{12}L_{33} - L_{22}L_{33}L_{33} + L_{13}L_{21}L_{32} + L_{31}L_{12}L_{23}. \end{aligned}$$

We now verify that the coefficients A_i, B_i and C_i are positive under assumption (2.11). In particular,

$$\begin{aligned} -L_{11} - L_{22} - L_{33} &= \frac{r}{K}u_* + \frac{(mc-1)\alpha w_*(u_* + v_*)}{(mw_* + u_* + v_*)^2} > 0, \\ -L_{11} - L_{22} &= \frac{r}{K}u_* - \frac{\alpha u_* w_*}{(mw_* + u_* + v_*)^2} - \frac{\beta v_* w_*}{(mw_* + u_* + v_*)^2} \\ &= \frac{r}{K}u_* - \frac{\alpha(u_* + v_*)}{mw_* + u_* + v_*}\frac{w_*}{mw_* + u_* + v_*} \\ &= \frac{r}{K}u_* - \frac{d_2}{c}\frac{bu_* - d_1}{\alpha} \\ &= \left(\frac{r}{K} - \frac{d_2 b}{c\alpha}\right)u_* + \frac{d_2 d_1}{c\alpha} > 0, \end{aligned}$$

since $\frac{\alpha(u_*+v_*)}{mw_*+u_*+v_*} = \frac{d_2}{c}$ and $\frac{w_*}{mw_*+u_*+v_*} = \frac{bu_*-d_1}{\alpha}$. Also, $-L_{11} - L_{22} - 2L_{33} > 0$ since $L_{33} < 0$.

Since $w_* = \frac{c\alpha - d_2}{d_2 m}(u_* + v_*)$, $b(u_* + v_*) = r - \frac{r}{K}u_* + d_1$ and $r - \frac{r}{K}u_* > 0$,

$$\begin{aligned}
&L_{11}L_{22} + L_{22}L_{33} + L_{11}L_{33} - L_{13}L_{31} - L_{32}L_{23} - L_{21}L_{12} \\
&\quad = b^2 u_* v_* + \frac{r}{K}\alpha \frac{u_* w_*}{(mw_* + u_* + v_*)^2}\bigl(mcw_* + (mc-1)v_*\bigr) > 0, \\
&L_{11}L_{33} + L_{22}L_{33} - L_{13}L_{31} - L_{32}L_{23} = \frac{r}{K} mcu_* \alpha \frac{(u_* + v_*)w_*}{(mw_* + u_* + v_*)^2} > 0, \\
&L_{11}L_{22} - L_{21}L_{12} \\
&\quad = u_* v_* \left(-\alpha \frac{r}{K} \frac{w_*}{(mw_* + u_* + v_*)^2} + b^2\right) \\
&\quad = u_* v_* \left(-\alpha \frac{r}{K} \frac{c\alpha - d_2}{d_2 m} \frac{u_* + v_*}{(mw_* + u_* + v_*)^2} + b^2\right) \\
&\quad \geq u_* v_* \left(-\alpha \frac{r}{K} \frac{c\alpha - d_2}{d_2 m} \frac{1}{u_* + v_*} + b^2\right) \\
&\quad = u_* v_* \left(-\alpha \frac{r}{K} \frac{c\alpha - d_2}{d_2 m} \frac{b}{r - \frac{r}{K}u_* + d_1} + b^2\right) \\
&\quad > u_* v_* b \left(-\alpha \frac{r}{K} \frac{c\alpha - d_2}{d_2 m} \frac{1}{d_1} + b\right) > 0
\end{aligned}$$

and

$$\begin{aligned}
&-L_{11}L_{22}L_{33} - L_{21}L_{13}L_{32} - L_{31}L_{12}L_{23} + L_{13}L_{31}L_{22} + L_{32}L_{23}L_{11} + L_{21}L_{12}L_{33} \\
&\quad = c\alpha b^2 m u_* v_* w_* \frac{u_* + v_*}{(mw_* + u_* + v_*)^2} > 0.
\end{aligned}$$

It follows that $A_i B_i - C_i = \tau_i^1 \mu_i^3 + \tau_i^2 \mu_i^2 + \tau_i^3 \mu_i + \tau_i^4$, where

$$\begin{aligned}
\tau_i^1 &= 2d\bigl((2d + D)D + d^2\bigr), \\
\tau_i^2 &= d^2(-3L_{11} - 3L_{22} - 4L_{33}) + D^2(-L_{11} - L_{22}) + 4dD(-L_{11} - L_{22} - L_{33}) > 0, \\
\tau_i^3 &= d\bigl[(L_{11}L_{22} + L_{22}L_{33} + L_{11}L_{33} - L_{13}L_{31} - L_{32}L_{23} - L_{21}L_{12}) + (L_{11}L_{22} - L_{21}L_{12})\bigr] \\
&\quad + d(-L_{11} - L_{22} - L_{33})(-L_{11} - L_{22} - 2L_{33}) + D(-L_{11} - L_{22} - L_{33})(-L_{11} - L_{22}) \\
&\quad + D\bigl[(-L_{11} - L_{22})(-L_{33}) + (-L_{32}L_{23}) + (-L_{13}L_{31})\bigr] > 0, \\
\tau_i^4 &= -L_{11}L_{11}L_{22} - L_{11}L_{11}L_{33} + L_{11}L_{13}L_{31} + L_{11}L_{21}L_{12} - L_{11}L_{22}L_{22} - L_{22}L_{22}L_{33} \\
&\quad - L_{11}L_{22}L_{33} + L_{22}L_{32}L_{23} + L_{21}L_{22}L_{12} - L_{11}L_{22}L_{33} - L_{11}L_{33}L_{33} + L_{13}L_{31}L_{33} \\
&\quad + L_{32}L_{23}L_{33} - L_{22}L_{33}L_{33} + L_{13}L_{21}L_{32} + L_{31}L_{12}L_{23}.
\end{aligned}$$

The positivity of τ_i^2 and τ_i^3 follows directly from the above calculations. Now, we investigate the sign of τ_i^4 for $L_{23}L_{32} = L_{22}L_{33}$. Note that

$$\begin{aligned}
-L_{11} &= u_* \left(\frac{r}{K} - \frac{\alpha w_*}{(mw_* + u_* + v_*)^2}\right) \\
&= u_* \left(\frac{r}{K} - \alpha \frac{c\alpha - d_2}{d_2 m} \frac{u_* + v_*}{(mw_* + u_* + v_*)^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&> u_*\left(\frac{r}{K} - \alpha\frac{c\alpha - d_2}{d_2 m}\frac{1}{u_* + v_*}\right)\\
&= u_*\left(\frac{r}{K} - \alpha\frac{c\alpha - d_2}{d_2 m}\frac{b}{r - \frac{r}{K}u_* + d_1}\right)\\
&> u_*\left(\frac{r}{K} - \alpha\frac{c\alpha - d_2}{d_2 m}\frac{b}{d_1}\right) \geq 0,
\end{aligned}
$$

since $w_* = \frac{c\alpha - d_2}{d_2 m}(u_* + v_*)$, $b(u_* + v_*) = r - \frac{r}{K}u_* + d_1$, and $r - \frac{r}{K}u_* > 0$. Finally, since $L_{31} = L_{32}$, we obtain the positivity of

$$
\begin{aligned}
\tau_i^4 &= (-L_{11} - L_{22})(L_{11}L_{22} - L_{12}L_{21}) + (-L_{11})(-L_{33})(-L_{11} - L_{22})\\
&\quad + (-L_{11})L_{33}(-L_{22} - L_{33}) + \Upsilon > 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\Upsilon &= L_{13}L_{31}L_{11} + L_{13}L_{31}L_{33} + L_{13}L_{21}L_{32} + L_{31}L_{12}L_{23}\\
&= L_{31}\Bigg((mc-1)\alpha^2(u_* + v_*)\frac{u_* v_* w_*}{(mw_* + u_* + v_*)^4} + (mc-1)\alpha^2(u_* + v_*)\frac{u_*^2 w_*}{(mw_* + u_* + v_*)^4}\\
&\quad + \frac{\alpha(u_* + v_*)}{(mw_* + u_* + v_*)^2}(-L_{11})\Bigg) > 0.
\end{aligned}
$$

Hence, $A_iB_i - C_i > 0$ for all $i \geq 1$. From the Routh-Hurwitz criterion for each i, the three roots of $\lambda^3 + A_i\lambda^2 + B_i\lambda + C_i = 0$ have negative real parts since A_i, C_i, and $A_iB_i - C_i > 0$. The remainder of this proof follows from Theorem 5.1.1 in [36]. □

3 Conclusion

A diffusive predator-prey model with a ratio-dependent functional response and infected prey population was investigated under homogeneous Neumann boundary conditions. We showed that depending on initial data, all species can become extinct if the predation rate is small and the searching efficiency constant of the predation rate of the predator for the susceptible prey is large; in other words, the predator overeats the susceptible prey. On the other hand, we showed that the infected prey becomes extinct if the death rate of the infected prey is sufficiently large without respect to the initial data. Furthermore, the same conclusion holds even if the death rate of the infected prey is relatively small. In [37], the authors proposed a model by considering that the encounter infection rate is meaningful only in the case that it follows the law of ratio-dependence and not the law of simple mass action. They showed that the model exhibits parasite-induced host extinction. Such an extinction is similar to that induced by a ratio-dependent predator-prey functional response. The stability of the disease-free equilibrium $\mathbf{e_2}$ implies that under certain conditions, total extinction is not possible and the introduction of infected prey into the system may act as a biological control to save the ecosystem from extinction.

In the case that the searching efficiency constants of the predation rate for the susceptible and infected prey are the same, if the maximum per capita capturing rate of the predator for the susceptible prey is small, *i.e.*, the predation rate is sufficiently large, then the positive equilibrium point is locally asymptotically stable.

As regards application, the model with ratio-dependent functional response in this study can be used and improved to describe the interaction among the diseased-species in ecosystems, the susceptible species, and additional species with a certain biological property.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors equally made contributions and approved the final version of this manuscript.

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