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Binomial difference sequence spaces of order m

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Abstract

In this paper, we introduce the binomial sequence spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$ by combining the binomial transformation and mth order difference operator. We prove the BK-property and some inclusion relations. Also, we obtain the Schauder bases and compute the α-, β- and γ-duals of these sequence spaces.

Keywords: sequence space; matrix domain; Schauder basis; α-, β- and γ-duals

1 Introduction and preliminaries

Let w denote the space of all sequences. By ℓ_∞, c and c_0, we denote the spaces of bounded, convergent and null sequences, respectively. We write bs, cs and ℓ_p for the spaces of all bounded, convergent and p-absolutely summable series, respectively; $1 \leq p < \infty$. A Banach sequence space Z is called a BK-space [1] provided each of the maps $p_n : Z \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$, which is of great importance in the characterization of matrix transformations between sequence spaces. It is well known that the sequence spaces ℓ_∞, c and c_0 are BK-spaces with their usual sup-norm.

Let Z be a sequence space, then Kizmaz [2] introduced the following difference sequence spaces:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z \in \{\ell_\infty, c, c_0\}$, where $\Delta x_k = x_k - x_{k+1}$ for each $k \in \mathbb{N}$. Et and Colak [3] defined the generalization of the difference sequence spaces

$$Z(\Delta^m) = \{(x_k) \in w : (\Delta^m x_k) \in Z\}$$

for $Z \in \{\ell_\infty, c, c_0\}$, where $m \in \mathbb{N}$, $\Delta^0 x_k = x_k$, $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ for each $k \in \mathbb{N}$, which is equivalent to the binomial representation $\Delta^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k+i}$. Since then, many authors have studied further generalization of the difference sequence spaces [4–8]. Moreover, Altay and Polat [9], Başarir [10], Başarir, Kara and Konca [11], Başarir and Kara [12–17], Başarir, Öztürk and Kara [18], Polat and Başarir [19] and many others have studied new sequence spaces from matrix point of view that represent difference operators.

For an infinite matrix $A = (a_{n,k})$ and $x = (x_k) \in w$, the A-transform of x is defined by $(Ax)_n = \sum_{k=0}^\infty a_{n,k} x_k$ and is supposed to be convergent for all $n \in \mathbb{N}$. For two sequence spaces X, Y and an infinite matrix $A = (a_{n,k})$, the sequence space X_A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}, \tag{1.1}$$

which is called the domain of matrix A. By $(X : Y)$, we denote the class of all matrices such that $X \subseteq Y_A$.

The Euler means E^r of order r is defined by the matrix $E^r = (e^r_{n,k})$, where $0 < r < 1$ and

$$e^r_{n,k} = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

The Euler sequence spaces e^r_0, e^r_c and e^r_∞ were defined by Altay and Başar [20] and Altay, Başar and Mursaleen [21] as follows:

$$e^r_0 = \left\{x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}(1-r)^{n-k}r^k x_k = 0\right\},$$

$$e^r_c = \left\{x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}(1-r)^{n-k}r^k x_k \text{ exists}\right\},$$

and

$$e^r_\infty = \left\{x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left|\sum_{k=0}^{n} \binom{n}{k}(1-r)^{n-k}r^k x_k\right| < \infty\right\}.$$

Altay and Polat [9] defined further generalization of the Euler sequence spaces $e^r_0(\nabla)$, $e^r_c(\nabla)$ and $e^r_\infty(\nabla)$ by

$$Z(\nabla) = \{x = (x_k) \in w : (\nabla x_k) \in Z\}$$

for $Z \in \{e^r_0, e^r_c, e^r_\infty\}$, where $\nabla x_k = x_k - x_{k-1}$ for each $k \in \mathbb{N}$. Here any term with negative subscript is equal to naught.

Polat and Başar [19] employed the technique matrix domain of triangle limitation method for obtaining the following sequence spaces:

$$Z(\nabla^{(m)}) = \{x = (x_k) \in w : (\nabla^{(m)} x_k) \in Z\}$$

for $Z \in \{e^r_0, e^r_c, e^r_\infty\}$, where $\nabla^{(m)} = (\delta^{(m)}_{n,k})$ is a triangle matrix defined by

$$\delta^{(m)}_{n,k} = \begin{cases} (-1)^{n-k}\binom{m}{n-k} & \text{if } \max\{0, n-m\} \le k \le n, \\ 0 & \text{if } 0 \le k < \max\{0, n-m\} \text{ or } k > n, \end{cases}$$

for all $k, n, m \in \mathbb{N}$.

Recently Bişgin [22, 23] defined another generalization of the Euler sequence spaces and introduced the binomial sequence spaces $b_0^{r,s}$, $b_c^{r,s}$, $b_\infty^{r,s}$ and $b_p^{r,s}$. Let $r,s \in \mathbb{R}$ and $r+s \neq 0$. Then the binomial matrix $B^{r,s} = (b_{n,k}^{r,s})$ is defined by

$$b_{n,k}^{r,s} = \begin{cases} \frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n, \end{cases}$$

for all $k,n \in \mathbb{N}$. For $sr > 0$ we have

(i) $\|B^{r,s}\| < \infty$,
(ii) $\lim_{n\to\infty} b_{n,k}^{r,s} = 0$ for each $k \in \mathbb{N}$,
(iii) $\lim_{n\to\infty} \sum_k b_{n,k}^{r,s} = 1$.

Thus, the binomial matrix $B^{r,s}$ is regular for $sr > 0$. Unless stated otherwise, we assume that $sr > 0$. If we take $s+r = 1$, we obtain the Euler matrix E^r. So the binomial matrix generalizes the Euler matrix. Bişgin defined the following spaces of binomial sequences:

$$b_0^{r,s} = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k = 0 \right\},$$

$$b_c^{r,s} = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \text{ exists} \right\},$$

and

$$b_\infty^{r,s} = \left\{ x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right| < \infty \right\}.$$

The purpose of the present paper is to study the difference spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$ of the binomial sequence whose $B^{r,s}(\nabla^{(m)})$-transforms are in the spaces c_0, c and ℓ_∞, respectively. These new sequence spaces are the generalization of the sequence spaces defined in [22, 23] and [19]. Also, we give some inclusion relations and compute the bases and α-, β- and γ-duals of these sequence spaces.

2 The binomial difference sequence spaces

In this section, we introduce the spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$, $b_\infty^{r,s}(\nabla^{(m)})$ and prove the BK-property and inclusion relations.

We first define the binomial difference sequence spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$ by

$$Z(\nabla^{(m)}) = \{ x = (x_k) \in w : (\nabla^{(m)} x_k) \in Z \}$$

for $Z \in \{b_0^{r,s}, b_c^{r,s}, b_\infty^{r,s}\}$. By using the notion of (1.1), the sequence spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$ can be redefined by

$$b_0^{r,s}(\nabla^{(m)}) = (b_0^{r,s})_{\nabla^{(m)}}, \qquad b_c^{r,s}(\nabla^{(m)}) = (b_c^{r,s})_{\nabla^{(m)}}, \qquad b_\infty^{r,s}(\nabla^{(m)}) = (b_\infty^{r,s})_{\nabla^{(m)}}. \tag{2.1}$$

It is obvious that the sequence spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$ may be reduced to some sequence spaces in the special cases of s, r and $m \in \mathbb{N}$. For instance, we take $m = 0$, then obtain the spaces $b_0^{r,s}$, $b_c^{r,s}$ and $b_\infty^{r,s}$ defined by Bişgin [22, 23]. On taking $s + r = 1$, we obtain the spaces $e_0^r(\nabla^{(m)})$, $e_c^r(\nabla^{(m)})$ and $e_\infty^r(\nabla^{(m)})$ defined by Polat and Başar [19].

Let us define the sequence $y = (y_n)$ as the $B^{r,s}(\nabla^{(m)})$-transform of a sequence $x = (x_k)$ by

$$y_n = \left[B^{r,s}\left(\nabla^{(m)}x_k\right)\right]_n = \frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k\left(\nabla^{(m)}x_k\right) \tag{2.2}$$

for each $n \in \mathbb{N}$, where

$$\nabla^{(m)}x_k = \sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k-i} = \sum_{i=\max\{0,k-m\}}^{m}(-1)^{k-i}\binom{m}{k-i}x_i.$$

Then the binomial difference sequence spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$ can be redefined by all sequences whose $B^{r,s}(\nabla^{(m)})$-transforms are in the spaces c_0, c and ℓ_∞.

Theorem 2.1 *Let $Z \in \{b_0^{r,s}, b_c^{r,s}, b_\infty^{r,s}\}$. Then $Z(\nabla^{(m)})$ is a BK-space with the norm $\|x\|_{Z(\nabla^{(m)})} = \|(\nabla^{(m)}x_k)\|_Z$.*

Proof The sequence spaces $b_0^{r,s}$, $b_c^{r,s}$ and $b_\infty^{r,s}$ are *BK*-spaces (see [22], Theorem 2.1 and [23], Theorem 2.1). Moreover, $\nabla^{(m)}$ is a triangle matrix and (2.1) holds. By using Theorem 4.3.12 of Wilansky [24], we deduce that the binomial sequence spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$ are *BK*-spaces. □

Theorem 2.2 *The sequence spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$ are linearly isomorphic to the spaces c_0, c and ℓ_∞, respectively.*

Proof Similarly, we prove the theorem only for the space $b_0^{r,s}(\nabla^{(m)})$. To prove $b_0^{r,s}(\nabla^{(m)}) \cong c_0$, we must show the existence of a linear bijection between the spaces $b_0^{r,s}(\nabla^{(m)})$ and c_0.

Consider $T : b_0^{r,s}(\nabla^{(m)}) \to c_0$ by $T(x) = B^{r,s}(\nabla^{(m)}x_k)$. The linearity of T is obvious and $x = 0$ whenever $T(x) = 0$. Therefore, T is injective.

Let $y = (y_n) \in c_0$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{i=0}^{k}(s+r)^i\sum_{j=i}^{k}\binom{m+k-j-1}{k-j}\binom{j}{i}r^{-j}(-s)^{j-i}y_i \tag{2.3}$$

for each $k \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty}\left[B^{r,s}\left(\nabla^{(m)}x_k\right)\right]_n = \lim_{n\to\infty}\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k\left(\nabla^{(m)}x_k\right) = \lim_{n\to\infty}y_n = 0,$$

which implies that $x \in b_0^{r,s}(\nabla^{(m)})$ and $T(x) = y$. Consequently, T is surjective and is norm preserving. Thus, $b_0^{r,s}(\nabla^{(m)}) \cong c_0$. □

The following theorems give some inclusion relations for this class of sequence spaces. We have the well-known inclusion $c_0 \subseteq c \subseteq \ell_\infty$, then the corresponding extended versions also preserve this inclusion.

Theorem 2.3 *The inclusion $b_0^{r,s}(\nabla^{(m)}) \subseteq b_c^{r,s}(\nabla^{(m)}) \subseteq b_\infty^{r,s}(\nabla^{(m)})$ holds.*

Theorem 2.4 *The inclusions $b_0^{r,s}(\nabla^{(m)}) \subseteq b_0^{r,s}(\nabla^{(m+1)})$, $b_c^{r,s}(\nabla^{(m)}) \subseteq b_c^{r,s}(\nabla^{(m+1)})$ and $b_\infty^{r,s}(\nabla^{(m)}) \subseteq b_\infty^{r,s}(\nabla^{(m+1)})$ hold.*

Proof Let $x = (x_k) \in b_0^{r,s}(\nabla^{(m)})$, then the inequality

$$\begin{aligned}\bigl|\bigl[B^{r,s}\bigl(\nabla^{(m+1)}x_k\bigr)\bigr]_n\bigr| &= \bigl|\bigl[B^{r,s}\bigl(\nabla^{(m)}(\nabla x_k)\bigr)\bigr]_n\bigr| \\ &= \bigl|\bigl[B^{r,s}\bigl(\nabla^{(m)}x_k\bigr)\bigr]_n - \bigl[B^{r,s}\bigl(\nabla^{(m)}x_k\bigr)\bigr]_{n-1}\bigr| \\ &\leq \bigl|\bigl[B^{r,s}\bigl(\nabla^{(m)}x_k\bigr)\bigr]_n\bigr| + \bigl|\bigl[B^{r,s}\bigl(\nabla^{(m)}x_k\bigr)\bigr]_{n-1}\bigr|\end{aligned}$$

holds and tends to 0 as $n \to \infty$, which implies that $x \in b_0^{r,s}(\nabla^{(m+1)})$. □

Theorem 2.5 *The inclusions $e_0^r(\nabla^{(m)}) \subseteq b_0^{r,s}(\nabla^{(m)})$, $e_c^r(\nabla^{(m)}) \subseteq b_c^{r,s}(\nabla^{(m)})$ and $e_\infty^r(\nabla^{(m)}) \subseteq b_\infty^{r,s}(\nabla^{(m)})$ strictly hold.*

Proof Similarly, we only prove the inclusion $e_0^r(\nabla^{(m)}) \subseteq b_0^{r,s}(\nabla^{(m)})$. If $r + s = 1$, we have $E^r = B^{r,s}$. So $e_0^r(\nabla^{(m)}) \subseteq b_0^{r,s}(\nabla^{(m)})$ holds. Take $0 < r < 1$ and $s = 4$. We define a sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k} \binom{m+k-j-1}{k-j}\left(-\frac{3}{r}\right)^j$$

for all $m, k \in \mathbb{N}$. It is clear that $[E^r(\nabla^{(m)}x_k)]_n = ((-2-r)^n) \notin c_0$ and $[B^{r,s}(\nabla^{(m)}x_k)]_n = ((\frac{1}{4+r})^n) \in c_0$. So, we have $x \in b_0^{r,s}(\nabla^{(m)}) \setminus e_0^r(\nabla^{(m)})$. This shows that the inclusion $e_0^r(\nabla^{(m)}) \subseteq b_0^{r,s}(\nabla^{(m)})$ strictly holds. □

3 The Schauder basis and α-, β- and γ-duals

For a normed space $(X, \|\cdot\|)$, a sequence $\{x_k : x_k \in X\}_{k\in\mathbb{N}}$ is called a *Schauder basis* [1] if for every $x \in X$, there is an unique scalar sequence (λ_k) such that $\|x - \sum_{k=0}^n \lambda_k x_k\| \to 0$ as $n \to \infty$. We shall construct the Schauder bases for the sequence spaces $b_0^{r,s}(\nabla^{(m)})$ and $b_c^{r,s}(\nabla^{(m)})$.

We define the sequence $g^{(k)}(r,s) = \{g_i^{(k)}(r,s)\}_{i\in\mathbb{N}}$ by

$$g_i^{(k)}(r,s) = \begin{cases} 0 & \text{if } 0 \leq i < k, \\ (s+r)^k \sum_{j=k}^{i} \binom{m+i-j-1}{i-j}\binom{j}{k} r^{-j}(-s)^{j-k} & \text{if } i \geq k, \end{cases}$$

for each $k \in \mathbb{N}$.

Theorem 3.1 *The sequence $(g^{(k)}(r,s))_{k\in\mathbb{N}}$ is a Schauder basis for the binomial sequence space $b_0^{r,s}(\nabla^{(m)})$ and every $x = (x_i) \in b_0^{r,s}(\nabla^{(m)})$ has an unique representation by*

$$x = \sum_k \lambda_k(r,s) g^{(k)}(r,s), \tag{3.1}$$

where $\lambda_k(r,s) = [B^{r,s}(\nabla^{(m)}x_i)]_k$ for each $k \in \mathbb{N}$.

Proof Obviously, $B^{r,s}(\nabla^{(m)}g_i^{(k)}(r,s)) = e_k \in c_0$, where e_k is the sequence with 1 in the kth place and zeros elsewhere for each $k \in \mathbb{N}$. This implies that $g^{(k)}(r,s) \in b_0^{r,s}(\nabla^{(m)})$ for each $k \in \mathbb{N}$.

For $x \in b_0^{r,s}(\nabla^{(m)})$ and $n \in \mathbb{N}$, we put

$$x^{(n)} = \sum_{k=0}^{n} \lambda_k(r,s)g^{(k)}(r,s).$$

By the linearity of $B^{r,s}(\nabla^{(m)})$, we have

$$B^{r,s}\big(\nabla^{(m)}x_i^{(n)}\big) = \sum_{k=0}^{n} \lambda_k(r,s)B^{r,s}\big(\nabla^{(m)}g_i^{(k)}(r,s)\big) = \sum_{k=0}^{n} \lambda_k(r,s)e_k$$

and

$$\big[B^{r,s}\big(\nabla^{(m)}\big(x_i - x_i^{(n)}\big)\big)\big]_k = \begin{cases} 0 & \text{if } 0 \le k < n, \\ [B^{r,s}(\nabla^{(m)}x_i)]_k & \text{if } k \ge n, \end{cases}$$

for each $k \in \mathbb{N}$.

For every $\varepsilon > 0$, there is a positive integer n_0 such that

$$\big|\big[B^{r,s}\big(\nabla^{(m)}x_i\big)\big]_k\big| < \frac{\varepsilon}{2}$$

for all $k \ge n_0$. Then we have

$$\big\|x - x^{(n)}\big\|_{b_0^{r,s}(\nabla^{(m)})} = \sup_{k \ge n}\big|\big[B^{r,s}\big(\nabla^{(m)}x_i\big)\big]_k\big| \le \sup_{k \ge n_0}\big|\big[B^{r,s}\big(\nabla^{(m)}x_i\big)\big]_k\big| < \frac{\varepsilon}{2} < \varepsilon,$$

which implies $x \in b_0^{r,s}(\nabla^{(m)})$ is represented as in (3.1).

To show the uniqueness of this representation, we assume that

$$x = \sum_k \mu_k(r,s)g^{(k)}(r,s).$$

Then we have

$$\big[B^{r,s}\big(\nabla^{(m)}x_i\big)\big]_k = \sum_k \mu_k(r,s)\big[B^{r,s}\big(\nabla^{(m)}g_i^{(k)}(r,s)\big)\big]_k = \sum_k \mu_k(r,s)(e_k)_k = \mu_k(r,s),$$

which is a contradiction with the assumption that $\lambda_k(r,s) = [B^{r,s}(\nabla^{(m)}x_i)]_k$ for each $k \in \mathbb{N}$. This shows the uniqueness of this representation. □

Theorem 3.2 *We define $g = (g_n)$ by*

$$g_n = \sum_{k=0}^{n}(s+r)^k \sum_{j=k}^{n} \binom{m+n-j-1}{n-j}\binom{j}{k} r^{-j}(-s)^{j-k}$$

for all $n \in \mathbb{N}$ *and* $\lim_{k\to\infty} \lambda_k(r,s) = l$. *The set* $\{g, g^{(0)}(r,s), g^{(1)}(r,s), \ldots, g^{(k)}(r,s), \ldots\}$ *is a Schauder basis for the space* $b_c^{r,s}(\nabla^{(m)})$ *and every* $x \in b_c^{r,s}(\nabla^{(m)})$ *has an unique representation by*

$$x = lg + \sum_k \bigl[\lambda_k(r,s) - l\bigr] g^{(k)}(r,s). \tag{3.2}$$

Proof Obviously, $B^{r,s}(\nabla^{(m)} g_i^k(r,s)) = e_k \in c_0 \subseteq c$ and $g \in b_c^{r,s}(\nabla^{(m)})$. For $x \in b_c^{r,s}(\nabla^{(m)})$, we put $y = x - lg$ and we have $y \in b_0^{r,s}(\nabla^{(m)})$. Hence, we deduce that y has an unique representation by (3.1), which implies that x has an unique representation by (3.2). Thus, we complete the proof. □

From Theorem 2.1, we know that $b_0^{r,s}(\nabla^{(m)})$ and $b_c^{r,s}(\nabla^{(m)})$ are Banach spaces. By combining this fact with Theorem 3.1 and Theorem 3.2, we can give the following corollary.

Corollary 3.3 *The sequence spaces* $b_0^{r,s}(\nabla^{(m)})$ *and* $b_c^{r,s}(\nabla^{(m)})$ *are separable.*

Köthe and Toeplitz [25] first computed the dual whose elements can be represented as sequences and defined the α-dual (or Köthe-Toeplitz dual). Chandra and Tripathy [26] generalized the notion of Köthe-Toeplitz dual of sequence spaces. Next, we compute the α-, β- and γ-duals of the sequence spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$.

For the sequence spaces X and Y, define multiplier space $M(X,Y)$ by

$$M(X,Y) = \bigl\{u = (u_k) \in w : ux = (u_k x_k) \in Y \text{ for all } x = (x_k) \in X\bigr\}.$$

Then the α-, β- and γ-duals of a sequence space X are defined by

$$X^\alpha = M(X, \ell_1), \qquad X^\beta = M(X, cs) \quad \text{and} \quad X^\gamma = M(X, bs),$$

respectively.

Let us give the following properties:

$$\sup_{K\in\Gamma} \sum_n \left| \sum_{k\in K} a_{n,k} \right| < \infty, \tag{3.3}$$

$$\sup_{n\in\mathbb{N}} \sum_k |a_{n,k}| < \infty, \tag{3.4}$$

$$\lim_{n\to\infty} a_{n,k} = a_k \quad \text{for each } k \in \mathbb{N}, \tag{3.5}$$

$$\lim_{n\to\infty} \sum_k a_{n,k} = a, \tag{3.6}$$

$$\lim_{n\to\infty} \sum_k |a_{n,k}| = \sum_k \Bigl| \lim_{n\to\infty} a_{n,k} \Bigr|, \tag{3.7}$$

where Γ is the collection of all finite subsets of $\mathbb{N}$.

Lemma 3.4 ([27]) *Let* $A = (a_{n,k})$ *be an infinite matrix, then*:

(i) $A \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$ *if and only if* (3.3) *holds.*

(ii) $A \in (c_0 : c)$ *if and only if* (3.4) *and* (3.5) *hold.*

(iii) $A \in (c : c)$ *if and only if* (3.4), (3.5) *and* (3.6) *hold.*
(iv) $A \in (\ell_\infty : c)$ *if and only if* (3.5) *and* (3.7) *hold.*
(v) $A \in (c_0 : \ell_\infty) = (c : \ell_\infty) = (\ell_\infty : \ell_\infty)$ *if and only if* (3.4) *holds.*

Theorem 3.5 *The* α*-dual of the spaces* $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ *and* $b_\infty^{r,s}(\nabla^{(m)})$ *is the set*

$$U_1^{r,s} = \left\{ u = (u_k) \in w : \sup_{K \in \Gamma} \sum_k \left| \sum_{i \in K} (s+r)^i \sum_{j=i}^{k} \binom{m+k-j-1}{k-j} \binom{j}{i} r^{-j} (-s)^{j-i} u_k \right| < \infty \right\}.$$

Proof Let $u = (u_k) \in w$ and $x = (x_k)$ be defined by (2.3), then we have

$$u_k x_k = \sum_{i=0}^{k} (s+r)^i \sum_{j=i}^{k} \binom{m+k-j-1}{k-j} \binom{j}{i} r^{-j} (-s)^{j-i} u_k y_i = \left(G^{r,s} y\right)_k$$

for each $k \in \mathbb{N}$, where $G^{r,s} = (g_{k,i}^{r,s})$ is defined by

$$g_{k,i}^{r,s} = \begin{cases} (s+r)^i \sum_{j=i}^{k} \binom{m+k-j-1}{k-j} \binom{j}{i} r^{-j} (-s)^{j-i} u_k & \text{if } 0 \le i \le k, \\ 0 & \text{if } i > k. \end{cases}$$

Therefore, we deduce that $ux = (u_k x_k) \in \ell_1$ whenever $x \in b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ or $b_\infty^{r,s}(\nabla^{(m)})$ if and only if $G^{r,s} y \in \ell_1$ whenever $y \in c_0, c$ or ℓ_∞, which implies that $u = (u_k) \in [b_0^{r,s}(\nabla^{(m)})]^\alpha$, $[b_c^{r,s}(\nabla^{(m)})]^\alpha$ or $[b_\infty^{r,s}(\nabla^{(m)})]^\alpha$ if and only if $G^{r,s} \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$. By Lemma 3.4(i), we obtain

$$u = (u_k) \in \left[b_0^{r,s}\left(\nabla^{(m)}\right)\right]^\alpha = \left[b_c^{r,s}\left(\nabla^{(m)}\right)\right]^\alpha = \left[b_\infty^{r,s}\left(\nabla^{(m)}\right)\right]^\alpha$$

if and only if

$$\sup_{K \in \Gamma} \sum_k \left| \sum_{i \in K} (s+r)^i \sum_{j=i}^{k} \binom{m+k-j-1}{k-j} \binom{j}{i} r^{-j} (-s)^{j-i} u_k \right| < \infty.$$

Thus, we have $[b_0^{r,s}(\nabla^{(m)})]^\alpha = [b_c^{r,s}(\nabla^{(m)})]^\alpha = [b_\infty^{r,s}(\nabla^{(m)})]^\alpha = U_1^{r,s}$. □

Now, we define the sets $U_2^{r,s}$, $U_3^{r,s}$, $U_4^{r,s}$ and $U_5^{r,s}$ by

$$U_2^{r,s} = \left\{ u = (u_k) \in w : \sup_{n \in \mathbb{N}} \sum_k |u_{n,k}| < \infty \right\},$$

$$U_3^{r,s} = \left\{ u = (u_k) \in w : \lim_{n \to \infty} u_{n,k} \text{ exists for each } k \in \mathbb{N} \right\},$$

$$U_4^{r,s} = \left\{ u = (u_k) \in w : \lim_{n \to \infty} \sum_k |u_{n,k}| = \sum_k \left| \lim_{n \to \infty} u_{n,k} \right| \right\},$$

and

$$U_5^{r,s} = \left\{ u = (u_k) \in w : \lim_{n \to \infty} \sum_k u_{n,k} \text{ exists} \right\},$$

where

$$u_{n,k} = (s+r)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{m+i-j-1}{i-j} \binom{j}{k} r^{-j}(-s)^{j-k} u_i.$$

Theorem 3.6 *The following equations hold*:

(i) $[b_0^{r,s}(\nabla^{(m)})]^\beta = U_2^{r,s} \cap U_3^{r,s}$,

(ii) $[b_c^{r,s}(\nabla^{(m)})]^\beta = U_2^{r,s} \cap U_3^{r,s} \cap U_5^{r,s}$,

(iii) $[b_\infty^{r,s}(\nabla^{(m)})]^\beta = U_3^{r,s} \cap U_4^{r,s}$.

Proof Since the proof may be obtained in the same way for (ii) and (iii), we only prove (i). Let $u = (u_k) \in w$ and $x = (x_k)$ be defined by (2.3), then we consider the following equation:

$$\begin{aligned} \sum_{k=0}^{n} u_k x_k &= \sum_{k=0}^{n} u_k \left[\sum_{i=0}^{k} (s+r)^i \sum_{j=i}^{k} \binom{m+k-j-1}{k-j} \binom{j}{i} r^{-j}(-s)^{j-i} y_i \right] \\ &= \sum_{k=0}^{n} \left[(s+r)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{m+i-j-1}{i-j} \binom{j}{k} r^{-j}(-s)^{j-k} u_i \right] y_k \\ &= \left(U^{r,s} y\right)_n, \end{aligned}$$

where $U^{r,s} = (u_{n,k}^{r,s})$ is defined by

$$u_{n,k} = \begin{cases} (s+r)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{m+i-j-1}{i-j} \binom{j}{k} r^{-j}(-s)^{j-k} u_i & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Therefore, we deduce that $ux = (u_k x_k) \in cs$ whenever $x \in b_0^{r,s}(\nabla^{(m)})$ if and only if $U^{r,s} y \in c$ whenever $y \in c_0$, which implies that $u = (u_k) \in [b_0^{r,s}(\nabla^{(m)})]^\beta$ if and only if $U^{r,s} \in (c_0 : c)$. By Lemma 3.4(ii), we obtain $[b_0^{r,s}(\nabla^{(m)})]^\beta = U_2^{r,s} \cap U_3^{r,s}$. □

Theorem 3.7 *The γ-dual of the spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$ is the set $U_2^{r,s}$.*

Proof Using Lemma 3.4(v) instead of (ii), the proof can be given in a similar way. So, we omit the details. □

4 Conclusion

By considering the definitions of the binomial matrix $B^{r,s} = (b_{n,k}^{r,s})$ and mth order difference operator, we introduce the sequence spaces $b_0^{r,s}(\nabla^{(m)})$, $b_c^{r,s}(\nabla^{(m)})$ and $b_\infty^{r,s}(\nabla^{(m)})$. These spaces are the natural continuation of [3, 19, 22, 23]. Our results are the generalization of the matrix domain of the Euler matrix of order r.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JM came up with the main ideas and drafted the manuscript. MS revised the paper. All authors read and approved the final manuscript.

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