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Stability and Hopf bifurcation of a modified predator-prey model with a time delay and square root response function

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Abstract

In this paper, we consider a two-dimensional predator-prey model with a time delay and square root response function. We analyze the stability of equilibria with the delay τ increasing and the critical value of τ when Hopf bifurcation occurs. Because the model has the term of square root, the zero point is a singularity. In order to clearly study the stability of the zero point, we rescale the variable $x(t)$, say $x(t) = X^2(t)$. The conclusion is that the zero point is not stable and the instability is not affected by the delay τ. We apply the normal form method and center manifold theorem to obtain the direction and stability of the Hopf bifurcation. Finally, we make several numerical simulations which is consistent with the conclusion of theoretical analysis.

Keywords: predator-prey model; time delay; square root response function; Hopf bifurcation

1 Introduction

Dynamics of predator-prey models are one of the important subjects in ecology and mathematical ecology. Many researchers have studied predator-prey models with delay and derived some important results [1–9]. A few researchers have studied the model with the term of square root [10–12]. Braza [11] analyzed the following predator-prey model with square root response function:

$$\begin{cases} \dot{x}(t) = x(t) - x^2(t) - \sqrt{x(t)}y(t), \\ \dot{y}(t) = -sy(t) + c\sqrt{x(t)}y(t), \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ denote the population (or density) of the prey and predator, respectively. s is the death rate of the predator, and c is the biomass conversion or consumption rate.

In system (1.1), $x(t)$ stands for the number of prey without competition and predation, $x^2(t)$ stands for the reduced number of competition of prey and prey, the term of $\sqrt{x(t)}y(t)$ stands for the amount of assimilation by predator via competition of prey and predator [13], and $sy(t)$ is the dead quantity of predator. Because the c is the consumption rate, and the term of $\sqrt{x(t)}y(t)$ stands for the amount of assimilation by predator, the $c\sqrt{x(t)}y(t)$ stands for the growth of predator via competition of prey and predator. In [10], Salman *et al.* have investigated the nonlinear dynamics of a discrete predator-prey model with square root functional response, obtained by applying a forward Euler discretization method to system (1.1). In [11], the dynamics of the square root system (1.1) are compared and contrasted with the dynamics of predator-prey systems that use a typical Lotka-Volterra interaction term. The process of growth needs time to finish, so in order to accurately express the model, we should add a time delay to the model, and system (1.1) becomes system (1.2) as follows:

$$\begin{cases} \dot{x}(t) = x(t) - x^2(t) - \sqrt{x(t)}y(t), \\ \dot{y}(t) = -sy(t) + c\sqrt{x(t-\tau)}y(t). \end{cases} \tag{1.2}$$

Here, we assume the predator takes time τ to convert the food into its growth. By choosing τ as bifurcation parameter, we get the condition under which Hopf bifurcation occurs. At last we will give an example showing that the stability of the positive equilibrium will be changed with τ increasing.

This paper is organized as follows: In Section 2, we first focus on the stability of the equilibria and Hopf bifurcation by analyzing the eigenvalues. In Section 3, we derive the direction and stability of the Hopf bifurcation by using normal form method and central manifold theorem. In Section 4, a numerical simulation is made to examine the discussion of the previous section. A brief discussion is given in the last section.

2 Stability analysis and Hopf bifurcation

In this section, we research the stability of the equilibria and Hopf bifurcation by analyzing the eigenvalues of the system (1.2).

There are at most three nonnegative equilibria for system (1.2):

$$E_1^* = (0,0), \qquad E_2^* = (1,0), \qquad E_3^* = \bigl(s^2/c^2, s\bigl(c^2 - s^2\bigr)/c^3\bigr).$$

E_3^* is a unique positive equilibrium if and only if the following condition is true:

(H_1) $c > s$.

Let $E^* = (x^*, y^*)$ be the arbitrary equilibrium, then the linearized system of (1.2) at E^* is

$$\dot{u}(t) = Au(t) + Bu(t-\tau), \tag{2.1}$$

where $u(t) = (x(t), y(t))^T$,

$$A = \begin{bmatrix} 1 - 2x^* - \frac{y^*}{2\sqrt{x^*}} & -\sqrt{x^*} \\ 0 & -s + c\sqrt{x^*} \end{bmatrix} \triangleq \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \tag{2.2}$$

$$B = \begin{bmatrix} 0 & 0 \\ \frac{cy^*}{2\sqrt{x^*}} & 0 \end{bmatrix} \triangleq \begin{bmatrix} 0 & 0 \\ b_{21} & 0 \end{bmatrix}. \tag{2.3}$$

The characteristic determinant of system (2.1) is

$$\bigl|\lambda E - \bigl(A + Be^{-\lambda\tau}\bigr)\bigr| = 0. \tag{2.4}$$

The characteristic equation of system (2.1) is

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}b_{21}e^{-\lambda\tau} = 0. \tag{2.5}$$

In the following, we discuss the stability of the equilibria and Hopf bifurcation:

(1) $E^* = E_1^* = (0,0)$.

It is obvious that matrices (2.2) and (2.3) are indeterminate at E_1^*. We rescale the variable $x(t)$, say $x(t) = X^2(t)$, so system (1.2) becomes the following form:

$$\begin{cases} \dot{X}(t) = \frac{1}{2}X(t) - \frac{1}{2}X^3(t) - \frac{1}{2}y(t), \\ \dot{y}(t) = -sy(t) + cX(t-\tau)y(t). \end{cases} \tag{2.6}$$

The characteristic equation of system (2.6) at E_1^* is

$$\left(\lambda - \frac{1}{2}\right)(\lambda + s) = 0. \tag{2.7}$$

The eigenvalues of the characteristic equation (2.7) are $\frac{1}{2}$ and $-s$; it is obvious that E_1^* is unstable. From equation (2.7), it is seen that the delay τ has no effect on the stability of E_1^*.

(2) $E^* = E_2^* = (1,0)$.

The characteristic equation of system (2.1) is

$$(\lambda + 1)(\lambda + s - c) = 0. \tag{2.8}$$

The eigenvalues of the characteristic equation (2.8) are -1 and $c - s$, so E_2^* is stable when $s > c$ and unstable when $s < c$. However, if the predator death rate s is larger than its consumption rate c, the predator will die out, leaving the prey to flourish. From equation (2.8), it can be seen that the delay τ has also no effect on the stability of E_2^*.

(3) $E^* = E_3^* = (s^2/c^2, s(c^2 - s^2)/c^3)$.

In the following, we study the local stability of the positive equilibrium E_3^* by analyzing the distribution of the roots of equation (2.5). We consider two cases.

Case 1. When $\tau = 0$, the characteristic equation of system (2.1) is

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}b_{21} = 0. \tag{2.9}$$

From equation (2.9), we can see that when $\frac{a_{11}+a_{22}}{2} < 0$, $\Delta = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}b_{21}) < 0$, the two roots of equation (2.9) have always negative real parts; when $\frac{a_{11}+a_{22}}{2} > 0$, and $\Delta < 0$, the two roots of equation (2.9) have always negative real parts; when $\frac{a_{11}+a_{22}}{2} > 0$, and $\Delta \geq 0$, equation (2.9) has at least one positive root.

Let

(H_2) $\frac{a_{11}+a_{22}}{2} < 0$, $\Delta < 0$;

(H_3) $\frac{a_{11}+a_{22}}{2} > 0$.

Then it is not difficult to verify that the following result holds.

Lemma 2.1 *For $\tau = 0$, the following statements are true*:

(1) *If the conditions (H_1) and (H_2) hold, then E_3^* is stable.*

(2) *If the conditions (H_1) and (H_3) hold, then E_3^* is unstable.*

Case 2. When $\tau > 0$, the characteristic equation of system (2.1) is equation (2.5).

The equilibrium E_3^* is stable if all roots of (2.5) have negative real parts, thus we should observe the distribution of roots of equation (2.5). If $i\omega$ ($\omega > 0$) is a root of equation (2.5), ω should satisfy

$$-\omega^2 - i(a_{11} + a_{22})\omega + a_{11}a_{22} - a_{12}b_{21}(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{2.10}$$

Separating the real and imaginary parts, we have

$$\begin{cases} -\omega^2 + a_{11}a_{22} - a_{12}b_{21}\cos\omega\tau = 0, \\ -(a_{11} + a_{22})\omega + a_{12}b_{21}\sin\omega\tau = 0, \end{cases} \tag{2.11}$$

which implies

$$\omega^4 + \left(a_{11}^2 + a_{22}^2\right)\omega^2 + a_{11}^2a_{22}^2 - a_{12}^2b_{21}^2 = 0, \tag{2.12}$$

$$\begin{aligned} a_{11} &= 1 - 2x^* - \frac{y^*}{2\sqrt{x^*}} = \frac{c^2 - 3s^2}{2c^2}, \\ a_{12} &= -\sqrt{x^*} = -\frac{s}{c}, \\ a_{22} &= -s + c\sqrt{x^*} = -s + c\frac{s}{c} = 0, \\ b_{21} &= \frac{cy^*}{2\sqrt{x^*}} = \frac{c^2 - s^2}{2c}. \end{aligned} \tag{2.13}$$

Let $z = \omega^2$ and denote

$$p = a_{11}^2 + a_{22}^2, \qquad q = a_{11}^2a_{22}^2 - a_{12}^2b_{21}^2. \tag{2.14}$$

Then equation (2.12) becomes

$$z^2 + pz + q = 0. \tag{2.15}$$

Denote

$$h(z) = z^2 + pz + q$$

we have

$$h'(z) = 2z + p.$$

Assume that (H_1) holds, from (2.13) and (2.14), $-\frac{p}{2} \leq 0$ and $q < 0$, we know that equation (2.15) has one positive root for $z \in (0, \infty)$. The positive root is denoted by z_*. Equation (2.12) has a positive root denoted by ω_*, then $\omega_* = \sqrt{z_*}$.

By (2.11), we have $\cos\omega_*\tau_* = \frac{-\omega_*^2 + a_{11}a_{22}}{a_{12}b_{21}}$.

Thus, if we denote

$$\tau_*^j = \frac{1}{\omega_*}\left\{\cos^{-1}\left(\frac{-\omega_*^2 + a_{11}a_{22}}{a_{12}b_{21}}\right) + 2\pi j\right\}, \tag{2.16}$$

where $j = 0,1,2,\ldots$, then $\pm i\omega_*$ is a pair of purely imaginary roots of equation (2.5) with $\tau_*^{(j)}$. Define

$$\tau_0 = \tau_{*0}^{(0)} = \min\{\tau_*^{(j)}\}, \qquad \omega_0 = \omega_{*0} \quad (j = 0,1,2,\ldots). \tag{2.17}$$

In order to investigate the distribution of the roots of equation (2.5), we need to introduce the following Lemma 2.2 from Ruan and Wei [14].

Lemma 2.2 *Consider the exponential polynomial*

$$\begin{aligned} P(\lambda, e^{-\lambda\tau_1},\ldots,e^{-\lambda\tau_m}) = {}& \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ & + [p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} \\ & + \cdots + [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m}, \end{aligned}$$

where $\tau_i \geq 0$ $(i = 1,2,\ldots,m)$ *and* $p_j^{(i)}$ $(i = 0,1,\ldots m; j = 1,2,\ldots,n)$ *are constants. As* $(\tau_1,\tau_2,\ldots,\tau_m)$ *vary, the sum of the order of the zeros of* $p(\lambda, e^{-\lambda\tau_1},\ldots,e^{-\lambda\tau_m})$ *on the open right half plane can change only if a zero appears on the imaginary axis.*

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of equation (2.5) near $\tau = \tau_*^{(j)}$ satisfying

$$\alpha(\tau_*^{(j)}) = 0, \qquad \omega(\tau_*^{(j)}) = \omega_*.$$

Then the following transversality condition holds.

Lemma 2.3 *Suppose that* $z_* = \omega_*^2$, *and* $h'(z_*) \neq 0$, *then* $\frac{d(\operatorname{Re}\lambda(\tau_*^{(j)}))}{d\tau} \neq 0$, *and the sign of* $\frac{d(\operatorname{Re}\lambda(\tau_*^{(j)}))}{d\tau}$ *is consistent with that of* $h'(z_*)$.

Proof Substituting $\lambda(\tau)$ into equation (2.5) and differentiating the resulting equation in τ, we obtain

$$[2\lambda - (a_{11} + a_{22}) + a_{12}b_{21}e^{-\lambda\tau}(\tau)]\frac{d\lambda}{d\tau} = -a_{12}b_{21}e^{-\lambda\tau}\lambda,$$

thus

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{[2\lambda - (a_{11} + a_{22})]e^{\lambda\tau}}{-a_{12}b_{21}\lambda} + \frac{\tau}{-\lambda}, \tag{2.18}$$

where $\tau = \tau_*^{(j)}$, $\lambda = i\omega_*$,

$$\begin{aligned} &\{[2\lambda - (a_{11} + a_{22})]e^{\lambda\tau}\}_{\tau=\tau_*^{(j)}} = [2i\omega_* - (a_{11} + a_{22})](\cos\omega_*\tau_*^{(j)} + i\sin\omega_*\tau_*^{(j)}), \\ &\{-a_{12}b_{21}\lambda\}_{\tau=\tau_*^{(j)}} = -i\omega_* a_{12}b_{21}. \end{aligned} \tag{2.19}$$

From (2.18), (2.19) and (2.11), we obtain

$$\begin{aligned}
\left[\frac{\operatorname{Re} d\lambda(\tau)}{d\tau}\right]^{-1}_{\tau=\tau_*^{(j)}} &= \operatorname{Re}\left[\frac{[2\lambda-(a_{11}+a_{22})]e^{\lambda\tau}}{-a_{12}b_{21}\lambda}\right]_{\tau=\tau_*^{(j)}} + \operatorname{Re}\left[-\frac{\tau}{\lambda}\right]_{\tau=\tau_*^{(j)}} \\
&= \operatorname{Re}\left[\frac{[2\lambda-(a_{11}+a_{22})]e^{\lambda\tau}}{-a_{12}b_{21}\lambda}\right]_{\tau=\tau_*^{(j)}} \\
&= \operatorname{Re}\left[\frac{[2i\omega_*-(a_{11}+a_{22})](\cos\omega_*\tau_*^{(j)} + i\sin\omega_*\tau_*^{(j)})}{-a_{12}b_{21}i\omega_*}\right] \\
&= \operatorname{Re}\left[\frac{[2i\omega_*-(a_{11}+a_{22})](\frac{-\omega_*^2+a_{11}a_{22}}{a_{12}b_{21}} + i\frac{(a_{11}+a_{22})\omega_*}{a_{12}b_{21}})}{-a_{12}b_{21}i\omega_*}\right] \\
&= \operatorname{Re}\left[\frac{[2i\omega_*-(a_{11}+a_{22})][-\omega_*^2+a_{11}a_{22}+i(a_{11}+a_{22})\omega_*]i\omega_*}{a_{12}^2b_{21}^2\omega_*^2}\right] \\
&= \frac{2(\omega_*^2-a_{11}a_{22})+(a_{11}+a_{22})^2}{a_{12}^2b_{21}^2} \\
&= \frac{2\omega_*^2+a_{11}^2+a_{22}^2}{a_{12}^2b_{21}^2} \\
&= \frac{1}{\Lambda}\{2z_*+p\} \\
&= \frac{h'(z_*)}{\Lambda},
\end{aligned}$$

where $\Lambda = a_{12}^2b_{21}^2 > 0$. Thus, we have

$$\operatorname{sign}\left[\frac{\operatorname{Re} d\lambda(\tau)}{d\tau}\right]_{\tau=\tau_*^{(j)}} = \operatorname{sign}\left[\frac{\operatorname{Re} d\lambda(\tau)}{d\tau}\right]^{-1}_{\tau=\tau_*^{(j)}} = \operatorname{sign}\left[\frac{h'(z_*)}{\Lambda}\right].$$

Notice that $\Lambda > 0$, we conclude that the sign of $[\frac{\operatorname{Re} d\lambda(\tau)}{d\tau}]_{\tau=\tau_*^{(j)}}$ is determined by $h'(z_*)$. The proof is complete. □

Applying Lemmas 2.1-2.3, we get the following stability and bifurcation results for system (1.2).

Theorem 2.1 *Suppose that* (H_1) *and* (H_2) *hold, then all roots of equation* (2.5) *have negative real parts for* $\tau \in [0, \tau_*^{(0)})$, *the positive equilibrium* E_3^* *is asymptotically stable for* $\tau \in [0, \tau_*^{(0)})$ *and the system* (1.2) *undergoes a Hopf bifurcation at* E_3^* *when* $\tau = \tau_*^{(j)}$ $(j = 0, 1, 2, \ldots)$.

Theorem 2.2 *Suppose that* (H_1) *and* (H_3) *hold, then equation* (2.5) *has at least one root with positive real parts for* $\tau \in [0, \tau_*^{(0)})$, *the positive equilibrium* E_3^* *is unstable for* $\tau \in [0, \tau_*^{(0)})$ *and system* (1.2) *undergoes a Hopf bifurcation at* E_3^* *when* $\tau = \tau_*^{(j)}$ $(j = 0, 1, 2, \ldots)$.

3 Direction and stability of the Hopf bifurcation

In the second section, we got the condition of the system (1.2) appear as a Hopf bifurcation at $\tau = \tau^{(j)}$ $(j = 0, 1, 2, \ldots)$. We determine the Hopf bifurcation direction and the properties of these bifurcating periodic solutions by using the normal form method and center manifold theorem.

Let $x_1 = x - x^*$, $x_2 = y - y^*$, $\bar{x}_i(t) = x_i(\tau t)$, $\tau = \tau_*^{(j)} + \mu$, where $\tau_*^{(j)}$ is defined by (2.16). For convenience, drop the bar and let $p(x) = \sqrt{x}$, then the system (1.2) can be written as an FDE in $C = C([-1,0], R^2)$ as

$$\dot{x}(t) = L_\mu(x_t) + f(\mu, x_t), \tag{3.1}$$

where $x(t) = (x_1(t), x_2(t))^T \in R^2$ and $L_\mu : C \to R^2, f : R \times C \to R^2$ are given, respectively, by

$$L_\mu(\phi) = \left(\tau^{(j)} + \mu\right)A\phi(0) + \left(\tau^{(j)} + \mu\right)B\phi(-1) \tag{3.2}$$

and

$$f(\mu, \phi) = \left(\tau^{(j)} + \mu\right)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{3.3}$$

where $f_1 = -\phi_1^2(0) - l_1\phi_1(0)\phi_2(0) - l_2\phi_1^2(0)\phi_2(0) - l_3\phi_2^2(0) + \cdots$, $f_2 = l_4\phi_1(-1)\phi_2(0) + l_5\phi_1^2(-1)\phi_2(0) + l_6\phi_1^2(-1) + \cdots$, and $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in R^2$, $l_1 = p'(x^*)$, $l_2 = \frac{p''(x^*)}{2}$, $l_3 = \frac{p''(x^*)y^*}{2}$, $l_4 = cp'(x^*)$, $l_5 = \frac{cp''(x^*)}{2}$, $l_6 = \frac{cp''(x^*)y^*}{2}$.

In the previous section, we found that, in system (3.1), the Hopf bifurcation appears at $\tau = \tau_*^{(0)}$. Next we will apply the Riesz representation theorem to analyze a function $\eta(\theta, \mu)$.

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, 0)\phi(\theta), \quad \text{for } \phi \in C, \theta \in [-1, 0]. \tag{3.4}$$

In fact, we can choose

$$\eta(\theta, \mu) = \left(\tau^{(j)} + \mu\right)A\delta(\theta) - \left(\tau^{(j)} + \mu\right)B\delta(\theta + 1), \tag{3.5}$$

where δ is a Dirac-delta function. For $\phi \in C^1([-1,0], R^2)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(s, \mu)\phi(s), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Then, when $\theta = 0$, system (3.1) is equivalent to

$$\dot{x_t} = A(\mu)x_t + R(\mu)x_t, \tag{3.6}$$

where $x_t(\theta) = x(t + \theta)$ for $\theta \in [-1, 0)$. For $\psi \in C^1([0,1], (R^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta^T(t, 0)\psi(-t), & s = 0, \end{cases}$$

and a bilinear inner product

$$\langle\psi(s),\phi(\theta)\rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi, \tag{3.7}$$

where $\eta(\theta) = \eta(\theta,0)$. Denote $A = A(0)$, then A and A^* are adjoint operators. By Theorem 2.1, we know that $\pm i\omega_0\tau^{(j)}$ are eigenvalues of A.

$$\tau^{(j)}\begin{pmatrix} i\omega_0 - a_{11} & -a_{12} \\ -b_{21}e^{-i\omega_0\tau^{(j)}} & i\omega_0 - a_{22}\end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which yields

$$q(0) = (1,\alpha)^T = \left(1, \frac{i\omega_0 - a_{11}}{a_{12}}\right)^T.$$

Similarly, it can be verified that $q^*(s) = D(1,\alpha^*)e^{is\omega_0\tau^{(j)}}$ is the eigenvector of A^* corresponding to $-i\omega_0\tau^{(j)}$, where

$$q^*(0) = (1,\alpha^*)^T = \left(1, -\frac{a_{12}}{i\omega_0 + a_{22}}\right)^T.$$

By (3.7), we get

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle &= \overline{D}(1,\overline{\alpha^*})(1,\alpha)^T - \int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{D}(1,\overline{\alpha^*})e^{-i(\xi-\theta)\omega_0\tau^{(j)}}\,d\eta(\theta)(1,\alpha)^T e^{i\xi\omega_0\tau^{(j)}}\,d\xi \\ &= \overline{D}\left[1 + \alpha\overline{\alpha^*} - \int_{-1}^{0}(1,\overline{\alpha^*})\theta e^{i\theta\omega_0\tau^{(j)}}\,d\eta(\theta)(1,\alpha)^T\right] \\ &= \overline{D}\left[1 + \alpha\overline{\alpha^*} + b_{21}\overline{\alpha^*}\tau^{(j)}e^{-i\omega_0\tau^{(j)}}\right].\end{aligned}$$

Thus, we obtain

$$\overline{D} = \frac{1}{1 + \alpha\overline{\alpha^*} + b_{21}\overline{\alpha^*}\tau^{(j)}e^{-i\omega_0\tau^{(j)}}},$$

such that $\langle q^*(s), q(\theta)\rangle = 1$. In the following, we will compute the coordinates which describe the center manifold C_0 at $\mu = 0$ by using the same notations as the ideas in Hassard *et al.* [15]. Let x_t be the solution of equation (3.1) when $\mu = 0$. Define

$$z(t) = \langle q^*, x_t\rangle, \qquad W(t,\theta) = x_t(\theta) - 2\,\mathrm{Re}\big[z(t)q(\theta)\big]. \tag{3.8}$$

On the center manifold C_0, we have

$$W(t,\theta) = W\big(z(t),\overline{z}(t),\theta\big) = W_{20}(0)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots,$$

where z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q and $\overline{q}$. Note that W is real if x_t is real. We consider only real solutions. For the solution $x_t \in C_0$ of (3.1), since $\mu = 0$, we have

$$\begin{aligned}
\dot{z} &= i\omega_0\tau^{(j)}z + \langle q^*(\theta), f\bigl(0, W\bigl(z(t),\overline{z}(t),\theta\bigr) + 2\operatorname{Re}\bigl[z(t)q(\theta)\bigr]\bigr)\rangle \\
&= i\omega_0\tau^{(j)}z + \overline{q}^*(0)f\bigl(0, W\bigl(z(t),\overline{z}(t),0\bigr) + 2\operatorname{Re}\bigl[z(t)q(0)\bigr]\bigr) \\
&= i\omega_0\tau^{(j)}z + q^*(0)f_0(z,\overline{z}) \triangleq i\omega_0\tau^{(j)} + g(z,\overline{z}),
\end{aligned} \tag{3.9}$$

where

$$g(z,\overline{z}) = \overline{q^*}(0)f_0(z,\overline{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\overline{z} + g_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{3.10}$$

By (3.8), we have $x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta))^T = W(t,\theta) + zq(\theta) + \overline{zq}(\theta)$, and then

$$\begin{aligned}
x_{1t}(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{\overline{z}}^2}{2} + \cdots, \\
x_{2t}(0) &= z\alpha + \overline{z\alpha} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{\overline{z}}^2}{2} + \cdots, \\
x_{1t}(-1) &= ze^{-i\omega_0\tau^{(j)}} + \overline{z}e^{i\omega_0\tau^{(j)}} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{\overline{z}}^2}{2} + \cdots, \\
x_{2t}(-1) &= z\alpha e^{-i\omega_0\tau^{(j)}} + \overline{z\alpha}e^{i\omega_0\tau^{(j)}} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{\overline{z}}^2}{2} + \cdots.
\end{aligned}$$

It follows together with (3.3) that

$$\begin{aligned}
g(z,\overline{z}) &= \overline{q^*}(0)f_0(z,\overline{z}) = \overline{D}\tau^{(j)}\bigl(1,\overline{\alpha^*}\bigr)\begin{pmatrix} f_1^{(0)} \\ f_2^{(0)} \end{pmatrix} \\
&= \overline{D}\tau^{(j)}\bigl[-x_{1t}^2(0) - l_1x_{1t}(0)x_{2t}(0) - l_2x_{1t}^2(0)x_{2t}(0) - l_3x_{2t}^2(0) + \cdots \\
&\quad + \overline{\alpha^*}\bigl(l_4x_{1t}(-1)x_{2t}(0) + l_5x_{1t}^2(-1)x_{2t}(0) + l_6x_{1t}^2(-1) + \cdots\bigr)\bigr].
\end{aligned}$$

Comparing the coefficients with (3.12), we have

$$\begin{aligned}
g_{20} &= 2\overline{D}\tau^{(j)}\bigl[-1 - l_1\alpha - l_3\alpha^2 + \overline{\alpha^*}\bigl(l_4\alpha e^{-i\omega_0\tau^{(j)}} + l_6e^{-2i\omega_0\tau^{(j)}}\bigr)\bigr], \\
g_{11} &= \overline{D}\tau^{(j)}\bigl\{-2 - l_1(\overline{\alpha} + \alpha) - 2l_3\alpha\overline{\alpha} + \overline{\alpha^*}\bigl[l_4\bigl(\alpha e^{i\omega_0\tau^{(j)}} + \overline{\alpha}e^{-i\omega_0\tau^{(j)}}\bigr) + 2l_6\bigr]\bigr\}, \\
g_{02} &= 2\overline{D}\tau^{(j)}\bigl[-1 - l_1\overline{\alpha} - l_3\overline{\alpha}^2 + \overline{\alpha^*}\bigl(l_4\overline{\alpha}e^{i\omega_0\tau^{(j)}} + l_6e^{2i\omega_0\tau^{(j)}}\bigr)\bigr], \\
g_{21} &= 2\overline{D}\tau^{(j)}\Biggl\{-\bigl(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\bigr) - l_1\biggl(\frac{W_{20}^{(1)}(0)}{2}\overline{\alpha} + \frac{W_{20}^{(2)}(0)}{2} + W_{11}^{(1)}(0)\alpha + W_{11}^{(2)}(0)\biggr) \\
&\quad - l_2(\overline{\alpha} + 2\alpha) - l_3\bigl(W_{20}^{(2)}(0)\overline{\alpha} + 2\alpha W_{11}^{(2)}(0)\bigr) \\
&\quad + \overline{\alpha^*}\biggl[l_4\biggl(\frac{W_{20}^{(2)}(0)}{2}e^{i\omega_0\tau^{(j)}} + \frac{W_{20}^{(1)}(-1)}{2}\overline{\alpha} + W_{11}^{(2)}(0)e^{-i\omega_0\tau^{(j)}} + W_{11}^{(1)}(-1)\alpha\biggr) \\
&\quad + l_5\bigl(e^{-2i\omega_0\tau^{(j)}}\overline{\alpha} + 2\alpha\bigr) + l_6\bigl(2W_{11}^{(1)}(-1)e^{-i\omega_0\tau^{(j)}} + W_{20}^{(1)}(-1)e^{i\omega_0\tau^{(j)}}\bigr)\biggr]\Biggr\}.
\end{aligned}$$

In order to determine g_{21}, in the sequel, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (3.6) and (3.8), we have

$$\dot{W} = \dot{x}_t - \dot{z}q - \dot{\overline{z}}\overline{q} = \begin{cases} AW - 2\operatorname{Re}\{\overline{q^*}(0)f_0 q(\theta)\}, & \theta \in [-1,0), \\ AW - 2\operatorname{Re}\{\overline{q^*}(0)f_0 q(\theta)\} + f_0, & \theta = 0 \end{cases}$$
$$\triangleq AW + H(z,\overline{z},\theta), \tag{3.11}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{3.12}$$

Notice that near the origin on the center manifold C_0, we have

$$\dot{W} = W_z\dot{z} + W_{\overline{z}}\dot{\overline{z}}, \tag{3.13}$$

thus

$$\bigl(A - 2i\tau^{(j)}\omega_0 I\bigr)W_{20}(\theta) = -H_{20}(\theta), \qquad AW_{11}(\theta) = -H_{11}(\theta). \tag{3.14}$$

Since (3.11) holds, for $\theta \in [-1,0)$, we have

$$H(z,\overline{z},\theta) = -\overline{q^*}(0)f_0 q(\theta) - q^*(0)\overline{f_0}\overline{q}(\theta) = -gq(\theta) - \overline{gq}(\theta). \tag{3.15}$$

Comparing the coefficients with (3.12) gives

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \qquad H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{3.16}$$

From (3.14), (3.15) and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2i\tau^{(j)}\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta).$$

Notice that $q(\theta) = q(0)e^{i\tau^{(j)}\omega_0\theta}$. We have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau^{(j)}}q(0)e^{i\tau^{(j)}\omega_0\theta} + \frac{i\overline{g}_{02}}{3\omega_0\tau^{(j)}}\overline{q}(0)e^{-i\tau^{(j)}\omega_0\theta} + E_1e^{2i\tau^{(j)}\omega_0\theta}, \tag{3.17}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in R^2$ is a constant vector. In the same way, we also obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau^{(j)}}q(0)e^{i\tau^{(j)}\omega_0\theta} + \frac{i\overline{g}_{11}}{\omega_0\tau^{(j)}}\overline{q}(0)e^{-i\tau^{(j)}\omega_0\theta} + E_2, \tag{3.18}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}) \in R^2$ is also a constant vector. In what follows, we will seek appropriate E_1 and E_2. From the definition of A and (3.14), we obtain

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\tau^{(j)}\omega_0 W_{20}(0) - H_{20}(0) \tag{3.19}$$

and

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \tag{3.20}$$

where $\eta(\theta) = \eta(0,\theta)$. From (3.11) and (3.12), we have

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\tau^{(j)} \begin{pmatrix} -1 - l_1\alpha - l_3\alpha^2 \\ l_4\alpha e^{-i\omega_0\tau^{(j)}} + l_6 e^{-2i\omega_0\tau^{(j)}} \end{pmatrix} \tag{3.21}$$

and

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + 2\tau^{(j)} \begin{pmatrix} -2 - l_1(\overline{\alpha} + \alpha) - 2l_3\alpha\overline{\alpha} \\ l_4(\alpha e^{i\omega_0\tau^{(j)}} + \overline{\alpha} e^{-i\omega_0\tau^{(j)}}) + 2l_6 \end{pmatrix}. \tag{3.22}$$

Substituting (3.17) and (3.21) into (3.19) and noticing that

$$\left(i\omega_0\tau^{(j)} I - \int_{-1}^{0} e^{i\omega_0\tau^{(j)}\theta}\, d\eta(\theta)\right) q(0) = 0$$

and

$$\left(-i\omega_0\tau^{(j)} I - \int_{-1}^{0} e^{-i\omega_0\tau^{(j)}\theta}\, d\eta(\theta)\right) \overline{q}(0) = 0,$$

we obtain

$$\left(2i\omega_0\tau^{(j)} I - \int_{-1}^{0} e^{2i\omega_0\tau^{(j)}\theta}\, d\eta(\theta)\right) E_1 = 2\tau^{(j)} \begin{pmatrix} -1 - l_1\alpha - l_3\alpha^2 \\ l_4\alpha e^{-i\omega_0\tau^{(j)}} + l_6 e^{-2i\omega_0\tau^{(j)}} \end{pmatrix},$$

which leads to

$$\begin{pmatrix} 2i\omega_0 - a_{11} & -a_{12} \\ -b_{21}e^{-2i\omega_0\tau^{(j)}} & 2i\omega_0 - a_{22} \end{pmatrix} E_1 = 2 \begin{pmatrix} -1 - l_1\alpha - l_3\alpha^2 \\ l_4\alpha e^{-i\omega_0\tau^{(j)}} + l_6 e^{-2i\omega_0\tau^{(j)}} \end{pmatrix}.$$

It follows that

$$E_1^{(1)} = \frac{2}{D_1} \begin{vmatrix} -1 - l_1\alpha - l_3\alpha^2 & -a_{12} \\ l_4\alpha e^{-i\omega_0\tau^{(j)}} + l_6 e^{-2i\omega_0\tau^{(j)}} & 2i\omega_0 - a_{22} \end{vmatrix},$$

$$E_1^{(2)} = \frac{2}{D_1} \begin{vmatrix} 2i\omega_0 - a_{11} & -1 - l_1\alpha - {}_3\alpha^2 \\ -b_{21}e^{-2i\omega_0\tau^{(j)}} & l_4\alpha e^{-i\omega_0\tau^{(j)}} + l_6 e^{-2i\omega_0\tau^{(j)}} \end{vmatrix},$$

where

$$D_1 = \begin{vmatrix} 2i\omega_0 - a_{11} & -a_{12} \\ -b_{21}e^{-2i\omega_0\tau^{(j)}} & 2i\omega_0 - a_{22} \end{vmatrix}.$$

Similarly, substituting (3.18) and (3.22) into (3.20), we get

$$\begin{pmatrix} -a_{11} & -a_{12} \\ -b_{21} & -a_{22} \end{pmatrix} E_2 = 2 \begin{pmatrix} -2 - l_1(\overline{\alpha} + \alpha) - 2l_3\alpha\overline{\alpha} \\ l_4(\alpha e^{i\omega_0\tau^{(j)}} + \overline{\alpha} e^{-i\omega_0\tau^{(j)}}) + 2l_6 \end{pmatrix}.$$

Thus, we have

$$E_2^{(1)} = \frac{2}{D_2}\begin{vmatrix} -2 - l_1(\overline{\alpha} + \alpha) - 2l_3\alpha\overline{\alpha} & -a_{12} \\ l_4(\alpha e^{i\omega_0\tau^{(j)}} + \overline{\alpha}e^{-i\omega_0\tau^{(j)}}) + 2l_6 & -a_{22} \end{vmatrix},$$

$$E_2^{(2)} = \frac{2}{D_2}\begin{vmatrix} -a_{11} & -2 - l_1(\overline{\alpha} + \alpha) - 2l_3\alpha\overline{\alpha} \\ -b_{21} & l_4(\alpha e^{i\omega_0\tau^{(j)}} + \overline{\alpha}e^{-i\omega_0\tau^{(j)}}) + 2l_6 \end{vmatrix},$$

where

$$D_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -b_{21} & -a_{22} \end{vmatrix}.$$

Firstly, we determine g_{20}, g_{11}, g_{02}, then we can determine E_1 and E_2. Next, we can determine W_{20} and W_{11}. Finally, we can determine g_{21}. Thus it is easy to compute the following values:

$$c_1(0) = \frac{i}{2\omega_0\tau^{(j)}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \qquad \mu_2 = -\frac{\operatorname{Re}[c_1(0)]}{\operatorname{Re}[\lambda'(\tau^{(j)})]},$$

$$T_2 = -\frac{\operatorname{Im}[c_1(0)] + \mu_2 \operatorname{Im}[\lambda'(\tau^{(j)})]}{\omega_0\tau^{(j)}}, \qquad \beta_2 = 2\operatorname{Re}\bigl[c_1(0)\bigr],$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value $\tau^{(j)}$. Suppose $\operatorname{Re}\{\lambda'(\tau^{(j)})\} > 0$. μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0$ (< 0), then the Hopf bifurcation is supercritical (subcritical) and the bifurcation exists for $\tau > \tau^{(j)}$ ($< \tau^{(j)}$). β_2 determines the stability of the bifurcation periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ (> 0). And T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ (< 0).

4 Numerical simulation

In this section, we make numerical simulations to examine the conclusion of the previous sections. We might as well consider the following system:

$$\begin{cases} \dot{x}(t) = x(t) - x^2(t) - \sqrt{x(t)}y(t), \\ \dot{y}(t) = -0.3y(t) + 0.5\sqrt{x(t-\tau)}y(t). \end{cases} \tag{4.1}$$

System (4.1) has a positive equilibrium $E_3^*(0.36, 0.384)$, and the characteristic equation of system (4.1) at E_3^* is

$$\lambda^2 + 0.04\lambda + 0.096e^{-\lambda\tau} = 0. \tag{4.2}$$

Next, we analyze the stability of the positive equilibrium $E_3^*(0.36, 0.384)$ with τ increasing by several figures.

(1) When $\tau = 0$, from the Routh-Hurwitz criterion, it is obvious that $E_3^*(0.36, 0.384)$ is stable. This agrees with Figure 1.

Since $-\frac{p}{2}, q < 0$, from Theorem 2.1, it is obvious that the positive equilibrium E_3^* of equation (4.1) is stable for $\tau \in [0, \tau_*^{(0)})$.

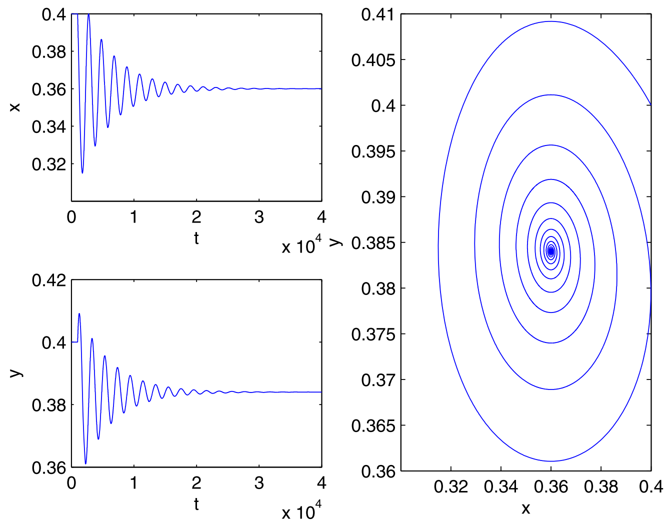

Figure 1 The trajectories of system (4.1) with $\tau = 0$. The positive equilibrium E_3^* is asymptotically stable.

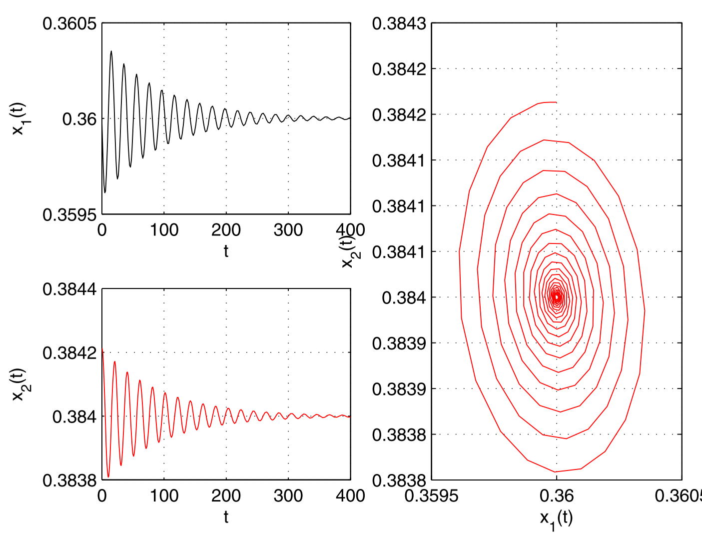

Figure 2 The trajectories of system (4.1) with $\tau = 0.2$. The positive equilibrium E_3^* is asymptotically stable.

Now, we will choose two appropriate values of τ to verify.

From the discussion of Section 2, we get

$$\begin{aligned} &p = 0.0016, \qquad q = -0.0092, \\ &z_* = 0.0951, \qquad \omega_* = 0.3084, \qquad \tau_*^{(0)} = 0.4417. \end{aligned} \tag{4.3}$$

(2) When $\tau = 0.2$, it is obvious that $E_3^*(0.36, 0.384)$ is asymptotically stable (see Figure 2).

(3) When $\tau = 0.44$, a periodic solution bifurcates from the equilibrium $E_3^*(0.36, 0.384)$ in Figure 3.

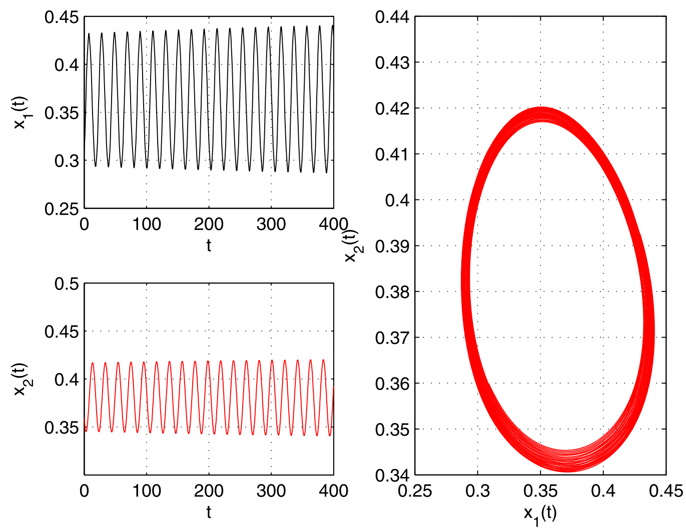

Figure 3 The trajectories of system (4.1) with $\tau = 0.44$. A periodic solution bifurcates from the equilibrium E_3^*.

From the discussion of Section 3, we also get a series of data:

$$c_1(0) = -0.6538 + 0.0939i,$$

$$\mu_2 = 0.0314, \qquad T_2 = -1.3628, \qquad \beta_2 = -1.3076.$$

Since $\mu_2 > 0$, $\beta_2 < 0$, the bifurcating periodic solution from the positive equilibrium E_3^* is supercritical and asymptotically stable at $\tau = \tau_*^{(0)}$.

5 Conclusion

In this paper, a predator-prey model with a time delay and square root response function is considered. Taking the time delay as bifurcating parameter, the stability of the equilibria and the existence of Hopf bifurcation is discussed. It is shown that the stability of $E_1^* = (0,0)$ and $E_2^* = (1,0)$ have not changed as the delay τ increases. We have also obtained the conditions at which system (1.2) undergoes a Hopf bifurcation at the positive equilibrium E_3^*. By using the center manifold theory and normal form method, the direction and stability of the Hopf bifurcation are determined. The results of our numerical simulations are in agreement with the theoretical findings.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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