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# Existence and uniqueness of solutions to the second order fuzzy dynamic equations on time scales

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**Abstract**
In this paper, we introduce a new metric space to study the existence and uniqueness of solutions to second order fuzzy dynamic equations on time scales. In this regard, we use Banach's fixed point theorem to prove this result. Also, we see that this metric guarantees an elegant and easier proof for the existence of solutions to second order fuzzy dynamic equations on time scales.

**MSC:** Primary 34A05; secondary 34N07

**Keywords:** second order fuzzy dynamic equations; Banach's fixed point theorem; metric space; time scales

## 1 Introduction

Recently, one of the most interesting and significant discussions in the field of differential equations is dynamic equations on time scales. The valuable applications of these equations in control theory, mathematical economics, mathematical biology, engineering and technology have made it more impressive, see [1–5].

The theory of dynamic equations and the essential concepts in the calculus of time scales were introduced by Bernd Aulbach and Stefan Hilger [5]. The dynamic systems on time scales have gained impetus since they demonstrate the interplay of two different theories, namely, the theories of continuous and discrete dynamic systems.

So far, many research papers have been done to investigate the existence of solutions for first and second order, boundary value and other types of dynamic equations, see [5–8].

Authors in [9] presented the definitions of delta derivative and delta integral for fuzzy-valued functions. So, as the second step, it is natural to investigate the existence and uniqueness of solutions to fuzzy dynamic equations on time scales. The main aim of this paper is to prove the existence and uniqueness of solutions to second order fuzzy dynamic equations on time scales, and we put off discussing this problem to Section 5. Before that, we introduce a new metric on the set of the fuzzy continuous functions on time scales and use it to define another new metric space.

This work is generalized as follows: Section 2 contains a brief summary of the theory of fuzzy sets and calculus of time scales. Section 3 deals with the fuzzy calculus on time scales. Then in Section 4 we restrict our attention to two new metric spaces. Finally in the last section the main results are stated and proved.

## 2 Preliminaries

In this section, we give some definitions and introduce necessary notations which will be used throughout the paper.

**Definition 2.1** ([10, 11]) Let $X$ be a nonempty set. A fuzzy set $u$ in $X$ is characterized by its membership function $u : X \to [0,1]$. Then $u(x)$ is interpreted as the degree of membership of an element $x$ in the fuzzy set $u$ for each $x \in X$.

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of the real axis (i.e., $u : \mathbb{R} \to [0,1]$), satisfying the following properties:

(i) $u$ is normal, i.e., there exists $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;

(ii) $u$ is fuzzy-convex set (i.e., $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0,1]$, $x, y \in \mathbb{R}$);

(iii) $u$ is upper semicontinuous on $\mathbb{R}$;

(iv) $cl\{x \in \mathbb{R}; u(x) > 0\}$ is compact, where $cl$ denotes the closure of a subset.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. Here $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$ is understood as $\mathbb{R} = \{\chi_{\{x\}}; x \text{ is a usual real number}\}$. For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in \mathbb{R}; u(x) \geq \alpha\}$ and $[u]^0 = cl\{x \in \mathbb{R}; u(x) > 0\}$.

Using the definition of fuzzy numbers, it follows that for any $\alpha \in [0,1]$, $[u]^\alpha$ is a bounded closed interval. The notation $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ denotes explicitly the $\alpha$-level set of $u$. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches on $u$, respectively.

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda \cdot u$ are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda \cdot u]^\alpha = \lambda[u]^\alpha$, $\forall \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha = \{x + y : x \in [u]^\alpha, y \in [v]^\alpha\}$ means the usual addition of two intervals of $\mathbb{R}$ and $\lambda[u]^\alpha = \{\lambda \cdot x : x \in [u]^\alpha\}$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^+ \cup \{0\}$, $D(u,v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}$, be the Hausdorff distance between fuzzy numbers, where $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$, $[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha]$.

The following properties are well known, see [12–14]:

- $D(u + w, v + w) = D(u,v)$, $\forall u, v, w \in \mathbb{R}_{\mathcal{F}}$,
- $D(k \cdot u, k \cdot v) = |k| D(u,v)$, $\forall k \in \mathbb{R}$, $u, v \in \mathbb{R}_{\mathcal{F}}$
- $D(u + v, w + e) \leq D(u,w) + D(v,e)$, $\forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$,
- $D(u \ominus w, u \ominus v) = D(w,v)$, $\forall u, v, w \in \mathbb{R}_{\mathcal{F}}$,

and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space. Also we define for each $x, y \in C(I, \mathbb{R}_{\mathcal{F}})$, $D(x,y) = \sup_{t \in I} D(x(t), y(t))$, where $C(I, \mathbb{R}_{\mathcal{F}})$ is a set of all fuzzy continuous functions on $I$.

**Theorem 2.2** ([13])

(i) *If we denote* $\tilde{0} = \chi_{\{0\}}$, *then* $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$ *is the neutral element with respect to* $+$, *i.e.,* $u + \tilde{0} = \tilde{0} + u = u$, *for all* $u \in \mathbb{R}_{\mathcal{F}}$;

(ii) *For any* $a, b \in \mathbb{R}$ *with* $a, b \leq 0$ *or* $a, b \geq 0$ *and any* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $(a + b) \cdot u = a \cdot u + b \cdot u$; *for general* $a, b \in \mathbb{R}$, *the above property does not hold*;

(iii) *For any* $\lambda \in \mathbb{R}$ *and any* $u, v \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$;

(iv) *For any* $\lambda, \mu \in \mathbb{R}$ *and any* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \cdot (\mu \cdot u) = (\lambda\mu) \cdot u$.

**Definition 2.3** ([13, 15]) Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then $z$ is called the *H-difference* of $x$ and $y$ and it is denoted by $x \ominus y$.

**Definition 2.4** ([16, 17]) Given $u, v \in \mathbb{R}_{\mathcal{F}}$, the *gH-difference* is the fuzzy number $w$, if it exists, such that

$$u \ominus_{gH} v = w \quad \Leftrightarrow \quad \begin{cases} \text{(i)} & u = v + w \\ \text{or} \quad \text{(ii)} & v = u + (-1) \cdot w. \end{cases} \tag{1}$$

If $u \ominus_{gH} v$ exists, its $\alpha$ cuts are given by

$$[u \ominus_{gH} v]^\alpha = \left[\min\left\{\underline{u}^\alpha - \underline{v}^\alpha, \overline{u}^\alpha - \overline{v}^\alpha\right\}, \max\left\{\underline{u}^\alpha - \underline{v}^\alpha, \overline{u}^\alpha - \overline{v}^\alpha\right\}\right]$$

and $u \ominus v = u \ominus_{gH} v$ if $u \ominus v$ exists. If (i) and (ii) are satisfied simultaneously, then $w$ is a crisp number.

**Remark 2.5** ([16, 17]) *In the fuzzy case, it is possible that the gH-difference of two fuzzy numbers does not exist. If* $u \ominus_{gH} v$ *exists, then* $v \ominus_{gH} u$ *exists and* $v \ominus_{gH} u = -(u \ominus_{gH} v)$.

**Proposition 2.6** [16, 17] *Let* $u, v \in \mathbb{R}_{\mathcal{F}}$ *be two fuzzy numbers; then*

(i) *if the gH-difference exists, it is unique;*
(ii) $u \ominus_{gH} v = u \ominus v$ *or* $u \ominus_{gH} v = -(u \ominus v)$ *whenever the expressions on the right exist; in particular,* $u \ominus_{gH} u = u \ominus u = \tilde{0}$;
(iii) *if* $u \ominus_{gH} v$ *exists in the sense* (i), *then* $v \ominus_{gH} u$ *exists in the sense* (ii) *and vice versa;*
(iv) $(u + v) \ominus_{gH} v = u$;
(v) $\tilde{0} \ominus_{gH} (u \ominus_{gH} v) = v \ominus_{gH} u$;
(vi) $u \ominus_{gH} v = v \ominus_{gH} u = w$ *if and only if* $w = -w$; *furthermore,* $w = \tilde{0}$ *if and only if* $u = v$.

**Definition 2.7** ([5]) A *time scale* $\mathbb{T}$ is a nonempty, closed subset of $\mathbb{R}$, equipped with the topology induced from the standard topology on $\mathbb{R}$.

**Definition 2.8** ([5]) The *forward (backward) jump operator* $\sigma(t)$ at $t$ for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ at $t$ for $t > \inf \mathbb{T}$) is given by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\} \qquad \big(\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}\big) \quad \text{for all } t \in \mathbb{T}. \tag{2}$$

Additionally, $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$. Furthermore, the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by $\mu(t) = \sigma(t) - t$, and also the *left-graininess function* $\nu : \mathbb{T} \to \mathbb{R}^+$ is defined by $\nu(t) = t - \rho(t)$.

**Definition 2.9** ([5]) If $\sigma(t) > t$, then the point $t$ is called *right-scattered*; while if $\rho(t) < t$, then $t$ is termed *left-scattered*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then the point $t$ is called *right-dense*; while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say that $t$ is *left-dense*.

**Definition 2.10** A mapping $f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ is rd-continuous if it is continuous at each right-dense point and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. We denote the set of rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}_{\mathcal{F}}$ by $C_{rd}[\mathbb{T}, \mathbb{R}_{\mathcal{F}}]$.

**Definition 2.11** ([5]) Fix $t \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$. Define $f^\Delta(t)$ to be the real number (provided it exists) with the property that given $\epsilon > 0$ there is a neighborhood $U_{\mathbb{T}}$ of $t$ (i.e., $U_{\mathbb{T}} =$

$(t-\delta, t+\delta) \cap \mathbb{T})$ such that

$$\left|\left(f\left(\sigma(t)\right) - f(s)\right) - f^{\Delta}(t)\left(\sigma(t) - s\right)\right| \leq \epsilon\left|\sigma(t) - s\right|$$

for all $s \in U_{\mathbb{T}}$. $f^{\Delta}(t)$ is called the $\Delta$-derivative of $f$ at $t$.

**Definition 2.12** ([18]) We say that a function $f : \mathbb{T} \to \mathbb{R}$ is right-increasing at a point $t_0 \in \mathbb{T} \setminus \{\min \mathbb{T}\}$ provided

(i) if $t_0$ is right-scattered, then $f(t_0) < f(\sigma(t_0))$;

(ii) if $t_0$ is right-dense, then there is a neighborhood $U_{\mathbb{T}} = (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$ of $t_0$ such that

$$f(t) > f(t_0) \quad \text{for all } t \in U_{\mathbb{T}} \text{ with } t > t_0.$$

Similarly, we say that $f$ is right-decreasing if above in (i), $f(\sigma(t_0)) < f(t_0)$ and in (ii), $f(t) < f(t_0)$.

**Theorem 2.13** ([18]) *Suppose $f : \mathbb{T} \to \mathbb{R}$ is differentiable at $t_0 \in \mathbb{T} \setminus \{\min \mathbb{T}\}$. If $f^{\Delta}(t_0) > 0$, then $f$ is right-increasing at the point $t_0$. If $f^{\Delta}(t_0) < 0$, then $f$ is right-decreasing at the point $t_0$.*

Here, we review some properties of the exponential function on time scales. For more details, we refer to Definition 2.30 [5].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$, and the function $p$ is called positively regressive if $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$. If $p : \mathbb{T} \to \mathbb{R}$ is a regressive function and $t_0 \in \mathbb{T}$, then (see Theorem 2.33 [5]) the exponential function $e_p(\cdot, t_0)$ is the unique solution of the initial value problem

$$y^{\Delta}(t) = p(t)y(t), \qquad y(t_0) = 1.$$

The following properties of the exponential function will be used in the last section.

(i) $e_0(t,s) = 1$, $e_p(t,t) = 1$;

(ii) $e_p(\sigma(t), s) = [1 + \mu(t)p(t)]e_p(t,s)$;

(iii) $e_p(t,r)e_p(r,s) = e_p(t,s)$;

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)}$;

(v) $e_p^{\Delta}(t, t_0) = p(t)e_p(t, t_0)$.

The set $\mathbb{T}^k$ is defined to be $\mathbb{T} \setminus \{m\}$ if $\mathbb{T}$ has a left-scattered maximum $m$, otherwise $\mathbb{T}^k = \mathbb{T}$.

## 3 Fuzzy delta derivative and integral on time scales

**Definition 3.1** ([9]) Assume that $f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ is a fuzzy function, and let $t \in \mathbb{T}^k$. Then $f$ is said to be right fuzzy delta differentiable at $t$ if there exists an element $\Delta_H^+ f(t)$ of $\mathbb{R}_{\mathcal{F}}$ with the property that given any $\epsilon > 0$, there exists a neighborhood $U_{\mathbb{T}}$ of $t$ [i.e., $U_{\mathbb{T}} = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$ such that for all $t + h \in U_{\mathbb{T}}$

$$D\left[f(t+h) \ominus_{gH} f\left(\sigma(t)\right), \Delta_H^+ f(t)\left(h - \mu(t)\right)\right] \leq \epsilon\left(h - \mu(t)\right),$$

with $0 \leq h < \delta$.

**Definition 3.2** ([9]) Assume that $f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ is a fuzzy function, and let $t \in \mathbb{T}^k$. Then $f$ is said to be left fuzzy delta differentiable at $t$ if there exists an element $\Delta_H^- f(t)$ of $\mathbb{R}_{\mathcal{F}}$ with the property that given any $\epsilon > 0$, there exists a neighborhood $U_{\mathbb{T}}$ of $t$ such that for all $t - h \in U_{\mathbb{T}}$

$$D\big[f\big(\sigma(t)\big) \ominus_{gH} f(t-h), \Delta_H^- f(t)\big(h + \mu(t)\big)\big] \le \epsilon\big(h + \mu(t)\big),$$

with $0 \le h < \delta$.

In the above definitions $\Delta_H^+ f(t)$ and $\Delta_H^- f(t)$ are termed, respectively, right fuzzy delta derivative and left fuzzy delta derivative at $t$.

**Definition 3.3** ([9]) Let $f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy function and $t \in \mathbb{T}^k$. Then $f$ is said to be $\Delta$-Hukuhara differentiable at $t$ if $f$ is both left and right fuzzy delta differentiable at $t \in \mathbb{T}^k$ and $\Delta_H^+ f(t) = \Delta_H^- f(t)$, and we will denote it by $\Delta_H f(t)$.

We call $\Delta_H f(t)$ the *$\Delta$-Hukuhara derivative* of $f$ at $t$. We say that $f$ is $\Delta_H$-differentiable at $t$ if its $\Delta_H$-derivative exists at $t$. Moreover, we say that $f$ is $\Delta_H$-differentiable on $\mathbb{T}^k$ if its $\Delta_H$-derivative exists at each $t \in \mathbb{T}^k$. The fuzzy function $\Delta_H f : \mathbb{T}^k \to \mathbb{R}_{\mathcal{F}}$ is then called the $\Delta_H$-derivative of $f$ on $\mathbb{T}^k$.

**Proposition 3.4** ([9]) *If the $\Delta_H$-derivative of $f$ at $t$ exists, then it is unique. Hence, the $\Delta_H$-derivative is well defined.*

**Lemma 3.5** ([9]) *Assume that $f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ is $\Delta_H$-differentiable at $t \in \mathbb{T}^k$, then $f$ is continuous at $t$.*

**Theorem 3.6** ([9]) *Assume that $f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ is a function, and let $t \in \mathbb{T}^k$, then we have the following*:

(i) *If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is $\Delta_H$-differentiable at $t$ with*

$$\Delta_H f(t) = \frac{f(\sigma(t)) \ominus_{gH} f(t)}{\mu(t)}; \tag{3}$$

(ii) *If $t$ is right-dense, then $f$ is $\Delta_H$-differentiable at $t$ iff the limits*

$$\lim_{h\to 0^+} \frac{f(t+h) \ominus_{gH} f(t)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(t) \ominus_{gH} f(t-h)}{h}$$

*exist and satisfy in this case*

$$\lim_{h\to 0^+} \frac{f(t+h) \ominus_{gH} f(t)}{h} = \lim_{h\to 0^+} \frac{f(t) \ominus_{gH} f(t-h)}{h} = \Delta_H f(t). \tag{4}$$

**Lemma 3.7** ([9]) *If $f$ is $\Delta_H$-differentiable at $t \in \mathbb{T}^k$, then $f(\sigma(t)) = f(t) + \mu(t)\Delta_H f(t)$ or $f(t) = f(\sigma(t)) + (-1)\mu(t)\Delta_H f(t)$.*

**Remark 3.8** *Assume that $f$ is $\Delta_H$-differentiable, we say that $f$ is $\Delta_H$-differentiable in the sense* (i) *or* (i)$\Delta_H$*-differentiable if, in the definition of $\Delta_H$-derivative, gH-difference is equivalent to H-difference, and we say that $f$ is $\Delta_H$-differentiable in the sense* (ii) *or* (ii)$\Delta_H$*-differentiable if gH-difference is equivalent to another case.*

**Definition 3.9** Let $f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ and $t \in \mathbb{T}$. We define the second order fuzzy delta derivative of $f$ as follows: We say that $f$ is fuzzy delta differentiable of the second order at $t$ if there exist elements $\Delta_H f(t)$ and $\Delta_H^2 f(t)$ such that given any $\epsilon > 0$, there exists a neighborhood $U_{\mathbb{T}}$ of $t$ [i.e., $U_{\mathbb{T}} = (t - \delta, t + \delta) \cap \mathbb{T}$] for some $\delta > 0$ such that for all $t + h \in U_{\mathbb{T}}$

$$D\big[\Delta_H f(t+h) \ominus_{gH} \Delta_H f\big(\sigma(t)\big), \Delta_H^2 f(t)\big(h - \mu(t)\big)\big] \le \epsilon\big(h - \mu(t)\big),$$

with $0 \le h < \delta$, and

$$D\big[f\big(\sigma(t)\big) \ominus_{gH} f(t-h), \Delta_H^2 f(t)\big(h + \mu(t)\big)\big] \le \epsilon\big(h + \mu(t)\big),$$

with $0 \le h < \delta$, where $\Delta_H^2(\cdot) = \Delta_H(\Delta_H(\cdot))$.

In fact, the second fuzzy delta derivative, or second order fuzzy delta derivative, is the fuzzy delta derivative of the fuzzy delta derivative of fuzzy function on time scales that we denote by $\Delta_H^2 f(t)$. Higher fuzzy delta derivative can also be defined.

**Lemma 3.10** *If $f, g : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ are $\Delta_H$-differentiable at $t \in \mathbb{T}^k$, in the same case of $\Delta_H$-differentiability* (*both are* (i)$\Delta_H$*-differentiable or* (ii)$\Delta_H$*-differentiable*), *then $f + g : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ is also $\Delta_H$-differentiable at t and*

$$\Delta_H(f+g)(t) = \Delta_H f(t) + \Delta_H g(t). \tag{5}$$

*Proof* It can be easily proved using Theorem 3.6. □

**Lemma 3.11** *If $f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ is $\Delta_H$-differentiable at $t \in \mathbb{T}^k$, then for any nonnegative constant $\lambda \in \mathbb{R}$, $\lambda f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ is $\Delta_H$-differentiable at t with*

$$\Delta_H(\lambda f)(t) = \lambda \Delta_H f(t).$$

*Proof* It follows easily from Theorem 3.6. □

Now, we present the definition of integral on time scales and give some properties of integrals on time scales for fuzzy-valued functions. Let $\mathbb{T}$ be a time scale, $a < b$ be points in $\mathbb{T}$, and $[a,b]_{\mathbb{T}}$ be the closed (and bounded) interval in $\mathbb{T}$. A partition of $[a,b]_{\mathbb{T}}$ is any finite ordered subset

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a,b]_{\mathbb{T}}, \quad \text{where } a = t_0 < t_1 < \cdots < t_n = b.$$

The number $n$ depends on the particular partition, so we have $n = n(P)$. The intervals $[t_{i-1}, t_i)$ for $1 \le i \le n$ are called the subintervals of the partition $P$. We denote the set of all partitions of $[a,b]_{\mathbb{T}}$ by $\mathcal{P} = \mathcal{P}(a,b)$.

**Lemma 3.12** ([19]) *For each $\delta > 0$, there exists a partition $P \in \mathcal{P}(a,b)$ given by $a = t_0 < t_1 < \cdots < t_n = b$ such that for each $i \in \{1, 2, \ldots, n\}$ either*

$$t_i - t_{i-1} < \delta$$

*or*

$$t_i - t_{i-1} > \delta \quad and \quad \rho(t_i) = t_{i-1}.$$

**Definition 3.13** ([9]) A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$ is called *Riemann* $\Delta$*-integrable* on $[a,b]_{\mathbb{T}}$ if there exists $I_R \in \mathbb{R}_{\mathcal{F}}$, with the property $\forall \epsilon > 0$, $\exists \delta > 0$, such that for any division of $[a,b]_{\mathbb{T}}$, $d : a = x_0 < \cdots < x_n = b$ with $x_i \in [a,b]_{\mathbb{T}}$, and for any points $\xi_i \in [x_i, x_{i+1})_{\mathbb{T}}$, $\overline{i = 0, n-1}$, we have

$$D\left[\sum_{i=0}^{n-1} f(\xi_i).(x_{i+1} - x_i), I_R\right] < \epsilon.$$

Then we denote by $I_R = \int_a^b f(x)\Delta x$ the fuzzy Riemann $\Delta$-integral.

**Definition 3.14** ([9]) Let $f : [0,t]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$. We define levelwise the $\Delta$-integral of $f$ in $[0,t]_{\mathbb{T}}$ (denoted by $\int_{[0,t]_{\mathbb{T}}} f(t)\Delta t$ or $\int_0^t f(t)\Delta t$) as the set of the integrals of the measurable selections for $[f]^\alpha$, for each $\alpha \in (0,1]$. We say that $f$ is $\Delta$-integrable over $[0,t]_{\mathbb{T}}$ if $\int_{[0,t]_{\mathbb{T}}} f(t)\Delta t \in \mathbb{R}_{\mathcal{F}}$, and we have

$$\left[\int_0^t f(t)\Delta t\right]^\alpha = \left[\int_0^t \underline{f}^\alpha(t)\Delta t, \int_0^t \overline{f}^\alpha(t)\Delta t\right] \tag{6}$$

for each $\alpha \in (0,1]$.

**Theorem 3.15** *If* $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$ *are* $\Delta$*-integrable on* $[a,b]_{\mathbb{T}}$, *then* $\alpha f + \beta g$, *where* $\alpha, \beta \in \mathbb{R}$ *is* $\Delta$*-integrable on* $[a,b]_{\mathbb{T}}$ *and*

$$\int_a^b \bigl(\alpha f(t) + \beta g(t)\bigr)\Delta t = \alpha\left(\int_a^b f(t)\Delta t\right) + \beta\left(\int_a^b g(t)\Delta t\right). \tag{7}$$

*Proof* It easily follows from Definition 3.13. □

**Theorem 3.16** *If* $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$ *is* $\Delta_H$*-differentiable on* $[a,b]_{\mathbb{T}}$ *and* $a \in \mathbb{T}$, *then* $\Delta_H f(t)$ *is* $\Delta$*-integrable over* $[a,b]_{\mathbb{T}}$ *and*

$$f(s) = f(a) + \int_a^s \Delta_H f(t)\Delta t,$$

*or*

$$f(a) = f(s) + (-1)\int_a^s \Delta_H f(t)\Delta t,$$

*for any* $s \in [a,b]_{\mathbb{T}}$.

*Proof* By setting the functions $\delta_L$ and $\delta_R$ defined in Definition 15 [9] as the same constant functions, the proof immediately follows from Theorem 18 [9]. □

**Theorem 3.17** *Let $f \in C_{rd}[\mathbb{T}, \mathbb{R}_{\mathcal{F}}]$, and let $t \in \mathbb{T}$. Then $f$ is $\Delta$-integrable from $t$ to $\sigma(t)$ and*

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t). \tag{8}$$

## 4 New metric space

Now, we are ready to define a new metric for the fuzzy continuous functions on time scales.

**Definition 4.1** Let $D$ denote the Hausdorff metric on the space $\mathbb{R}_{\mathcal{F}}$. Let $\gamma > 0$ be a constant. We define the space of all fuzzy continuous functions on time scales $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}})$ along with $\gamma$-metric $d_\gamma(x, y)$ which is defined as

$$d_\gamma(x, y) := \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \frac{D(x(t), y(t))}{e_\gamma(t, t_0)} \tag{9}$$

for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ and $x, y \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}})$;

Also, since $e_0(t, s) \equiv 1$, so $d_0$ is defined as

$$d_0(x, y) := \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} D\bigl(x(t), y(t)\bigr) \tag{10}$$

for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ and $x, y \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}})$, that is the same Hausdorff metric on a fuzzy continuous functions space.

In addition, we consider

$$\|x\|_\gamma := \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} \frac{D(x(t), \tilde{0})}{e_\gamma(t, t_0)}$$

for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ and $x \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}})$. Also, $\|x\|_0$ is defined as

$$\|x\|_0 := \sup_{t \in [t_0, t_0+a]_{\mathbb{T}}} D\bigl(x(t), \tilde{0}\bigr),$$

the $d_\gamma$ map is a new generalization of Bielecki's metric in [20]. The following two lemmas describe some important properties of $d_\gamma$ and $\|\cdot\|_\gamma$.

**Lemma 4.2** *If $\gamma > 0$ is constant then*:

(i) *$d_\gamma$ is a metric and it is equivalent to the sup-metric $d_0$;*

(ii) *$(C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}}), d_\gamma)$ is a complete metric space.*

*Proof* We note that $\gamma \in C_{rd}([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}})$ as any constant function is always *rd*-continuous. Since $\mu(t) \geq 0$, we have $1 + \mu(t)\gamma > 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$. Hence, $\gamma \in R^+$ (a set of positive regressive functions and *rd* continuous) and $e_\gamma(t, t_0) > 0$ for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$ (see [5]). It follows that for each $x, y \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}})$ we have

(i) Since $\gamma > 0$, $e_\gamma(t, t_0) > 0$, thus $d_\gamma \geq 0$. $d_\gamma(x, y) = 0$ if and only if $D(x, y) = 0$, and we know that $D(x, y) = 0$ if and only if $x = y$.

Since $D$ is a metric, so $d_\gamma(x, y) = d_\gamma(y, x)$.

Also we have

$$\begin{aligned} d_\gamma(x,z) &= \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \frac{D(x,z)}{e_\gamma(t,t_0)} \\ &\leq \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \frac{D(x,y)}{e_\gamma(t,t_0)} + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \frac{D(y,z)}{e_\gamma(t,t_0)} \\ &= d_\gamma(x,y) + d_\gamma(y,z). \end{aligned}$$

We know that $e_\gamma^\Delta(t,t_0) = \gamma e_\gamma(t,t_0) > 0$, then $e_\gamma(t,t_0)$ is right-increasing, so we have

$$\frac{1}{e_\gamma(t_0+a,t_0)} \leq \frac{1}{e_\gamma(t,t_0)} \leq 1,$$

it follows

$$\frac{1}{e_\gamma(t_0+a,t_0)} d_0(x,y) \leq d_\gamma(x,y) \leq d_0(x,y).$$

(ii) Now we show that $(C([t_0,t_0+a]_{\mathbb{T}},\mathbb{R}_{\mathcal{F}}),d_\gamma)$ is a complete metric space. We show that every Cauchy sequence in $(C([t_0,t_0+a]_{\mathbb{T}},\mathbb{R}_{\mathcal{F}}),d_\gamma)$ converges to a function in $C([t_0,t_0+a]_{\mathbb{T}},\mathbb{R}_{\mathcal{F}})$. Let $x_i(t)$ be a Cauchy sequence in $C([t_0,t_0+a]_{\mathbb{T}};\mathbb{R}_{\mathcal{F}})$. This means that for every $\epsilon > 0$ there is a positive integer $N_\epsilon$ such that

$$\frac{D(x_i(t),x_j(t))}{e_\gamma(t,t_0)} < \epsilon \quad \text{for all } i,j > N_\epsilon, \text{for all } t \in [t_0,t_0+a]_{\mathbb{T}}.$$

Thus, according to part (i),

$$D\big(x_i(t),x_j(t)\big) < \epsilon e_\gamma(t_0+a,t_0) \quad \text{for all } i,j > N_\epsilon, \text{for all } t \in [t_0,t_0+a]_{\mathbb{T}}.$$

Since $(C([t_0,t_0+a]_{\mathbb{T}},\mathbb{R}_{\mathcal{F}}),d_0)$ is a complete metric space (see [21]), there exists $x \in C([t_0,t_0+a]_{\mathbb{T}},\mathbb{R}_{\mathcal{F}})$ such that

$$\lim_{i\to\infty} D\big(x_i(t),x(t)\big) = 0 \quad \text{for all } t \in [t_0,t_0+a]_{\mathbb{T}},$$

and as a result of (i), we have $\lim_{i\to\infty} d_\gamma(x_i(t),x(t)) = 0$. Hence the Cauchy sequence $x_i$ in $C([t_0,t_0+a]_{\mathbb{T}},\mathbb{R}_{\mathcal{F}})$ is convergent and the limit is a fuzzy continuous function on $[t_0,t_0+a]_{\mathbb{T}}$. Thus $(C([t_0,t_0+a]_{\mathbb{T}};\mathbb{R}_{\mathcal{F}}),d_\gamma)$ is a complete metric space. □

Note that we show $\|\cdot\|_\gamma$ has properties similar to the properties of a norm in the usual crisp sense, without being a norm. It is not a norm because $C([a,b]_{\mathbb{T}},\mathbb{R}_{\mathcal{F}})$ is not a vector space (see part (ii) of Theorem 2.2) and, consequently, $C([a,b]_{\mathbb{T}},\mathbb{R}_{\mathcal{F}})$ with $\|\cdot\|_\gamma$ is not a normed space.

**Lemma 4.3** *The mapping* $\|\cdot\|_\gamma : \mathbb{R}_{\mathcal{F}} \to [0,\infty)$ *has the following properties*:

(i) $\|x\|_\gamma = 0$ *if and only if* $x = 0$;
(ii) $\|\lambda \cdot x\|_\gamma = |\lambda|\|x\|_\gamma$ *for all* $x \in C([t_0,t_0+a]_{\mathbb{T}},\mathbb{R}_{\mathcal{F}})$ *and* $\lambda \in \mathbb{R}$;
(iii) $\|x+y\|_\gamma \leq \|x\|_\gamma + \|y\|_\gamma$ *for all* $x,y \in \mathbb{R}_{\mathcal{F}}$.

*Proof*

(i) $\|\cdot\|_\gamma \geq 0$ and it is obvious that $\|x\|_\gamma = 0$ if and only if $x = 0$.

(ii) for $\lambda \in \mathbb{R}$ and $x \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}})$,

$$\begin{aligned} \|\lambda x\|_\gamma &= \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \frac{D(\lambda x(t), \tilde{0})}{e_\gamma(t,t_0)} \\ &= |\lambda| \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \frac{D(x(t), \tilde{0})}{e_\gamma(t,t_0)} \\ &= |\lambda| \|x\|_\gamma, \end{aligned}$$

and

(iii) for $x, y \in C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}})$,

$$\begin{aligned} \|x+y\|_\gamma &= \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \frac{D((x+y)(t), \tilde{0})}{e_\gamma(t,t_0)} \\ &\leq \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \frac{D(x(t), \tilde{0})}{e_\gamma(t,t_0)} + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \frac{D(y(t), \tilde{0})}{e_\gamma(t,t_0)} \\ &= \|x\|_\gamma + \|y\|_\gamma. \end{aligned}$$

□

So far we have proved that $C([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}})$ is a complete metric space with the distance

$$d_\gamma(x,y) = \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \frac{D(x(t), y(t))}{e_\gamma(t,t_0)}.$$

Let $C^1([t_0, t_0+a]_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$ denote the set of continuous functions $x : [t_0, t_0+a]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$ whose derivative $\Delta_H x : [t_0, t_0 + a]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$ exists as a continuous function. For $x, y \in C^1([t_0, t_0 + a]_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$, consider the following distance:

$$d_\gamma^1(x,y) = d_\gamma(x,y) + d_\gamma(\Delta_H x, \Delta_H y).$$

It easily follows from part (i) of Lemma 4.2 that $d_\gamma^1$ is also a metric.

**Lemma 4.4** *The couple* $(C^1([t_0, t_0 + a]_{\mathbb{T}}; \mathbb{R}_{\mathcal{F}}), d_\gamma^1)$ *is a complete metric space.*

*Proof* Let $\{x_n\}_{n=1}^\infty \subset C^1([t_0, t_0 + a]_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$ be a Cauchy sequence in $(C([t_0, t_0 + a]_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}}), d_\gamma^1)$, that is,

$$d_\gamma^1(x_n, x_m) = d_\gamma(x_n, x_m) + d_\gamma(\Delta_H x_n, \Delta_H x_m) \to 0, \quad n, m \to \infty.$$

Then the sequences $\{x_n\}_{n=1}^\infty$ and $\{\Delta_H x\}_{n=1}^\infty$ are Cauchy sequences in $(C([t_0, t_0+a]_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}}), d_\gamma)$, which is complete. Then there exist $x, y \in C([t_0, t_0 + a]_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$ such that $\{x_n\} \to x$ and $\{\Delta_H x_n\} \to y$ as $n \to +\infty$. We have to prove that $x \in C^1([t_0, t_0 + a]_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$ and that $\Delta_H x = y$. In this case,

$$d_\gamma^1(x_n, x) = d_\gamma(x_n, x) + d_\gamma(\Delta_H x_n, \Delta_H x) = d_\gamma(x_n, x) + d_\gamma(\Delta_H x_n, y) \to 0, \quad n \to \infty,$$

which proves that $\{x_n\} \to x$ in $(C^1([t_0, t_0 + a]_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}}), d_\gamma^1)$ is a complete metric space. If we check that $x(t) = x(t_0) + \int_{t_0}^t y(s)\Delta s$, the continuity of $y$ and application of Theorem 7 [9] provide that $x \in C^1([t_0, t_0 + a]_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$ and $\Delta_H x = y$. We will use that $x_n(t) = x_n(t_0) + \int_{t_0}^t \Delta_H x_n(s)\Delta s$. Let $\psi(t) = x(t_0) + \int_{t_0}^t y(s)\Delta s$, then

$$\begin{aligned}
d_\gamma(x,\psi) &= \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{D(x(t), x(t_0) + \int_{t_0}^t y(s)\Delta s)}{e_\gamma(t,t_0)} \right\} \\
&\leq \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{(D(x(t), x_n(t)) + D(x_n(t), x(t_0) + \int_{t_0}^t y(s)\Delta s))}{e_\gamma(t,t_0)} \right\} \\
&= \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{(D(x(t), x_n(t)) + D(x_n(t_0) + \int_{t_0}^t \Delta_H x_n(s)\Delta s, x(t_0) + \int_{t_0}^t y(s)\Delta s))}{e_\gamma(t,t_0)} \right\} \\
&\leq d_\gamma(x,x_n) + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{(D(x_n(t_0), x(t_0)) + D(\int_{t_0}^t \Delta_H x_n(s)\Delta s, \int_{t_0}^t y(s)\Delta s))}{e_\gamma(t,t_0)} \right\} \\
&\leq d_\gamma(x,x_n) + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{(D(x_n(t_0), x(t_0)) + \int_{t_0}^t D(\Delta_H x_n(s), y(s))\Delta s))}{e_\gamma(t,t_0)} \right\} \\
&\leq d_\gamma(x,x_n) + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{(\frac{D(x_n(t_0), x(t_0))}{e_\gamma(t_0,t_0)} + \int_{t_0}^t D(\Delta_H x_n(s), y(s))\Delta s))}{e_\gamma(t,t_0)} \right\} \\
&\leq d_\gamma(x,x_n) + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{d_\gamma(x_n,x)}{e_\gamma(t,t_0)} + \frac{\int_{t_0}^t \frac{D(\Delta_H x_n(s), y(s)) e_\gamma(s,t_0)}{e_\gamma(s,t_0)}\Delta s}{e_\gamma(t,t_0)} \right\} \\
&\leq d_\gamma(x,x_n) + d_\gamma(x_n,x) + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{\int_{t_0}^t \frac{D(\Delta_H x_n(s), y(s)) e_\gamma(s,t_0)}{e_\gamma(s,t_0)}\Delta s}{e_\gamma(t,t_0)} \right\} \\
&\leq d_\gamma(x,x_n) + d_\gamma(x_n,x) + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{\int_{t_0}^t d_\gamma(\Delta_H x_n, y) e_\gamma(s,t_0)\Delta s}{e_\gamma(t,t_0)} \right\} \\
&\leq d_\gamma(x,x_n) + d_\gamma(x_n,x) + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ d_\gamma(\Delta_H x_n, y) \frac{e_\gamma(t,t_0) - 1}{\gamma e_\gamma(t,t_0)} \right\} \\
&= d_\gamma(x,x_n) + d_\gamma(x_n,x) + \frac{d_\gamma(\Delta_H x_n, y)}{\gamma} \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ 1 - \frac{1}{e_\gamma(t,t_0)} \right\} \\
&= d_\gamma(x,x_n) + d_\gamma(x_n,x) + \frac{d_\gamma(\Delta_H x_n, y)}{\gamma} \left\{ 1 - \frac{1}{e_\gamma(t_0+a,t_0)} \right\} \to 0, \quad n \to \infty.
\end{aligned}$$

This proves that $d(x,\psi) = 0$, and therefore $x(t) = \psi(t)$, $t \in [t_0, t_0 + a]_{\mathbb{T}}$, or $x(t) = x(t_0) + \int_{t_0}^t y(s)\,ds$, $t \in [t_0, t_0 + a]_{\mathbb{T}}$, and the complete character of $C^1([t_0, t_0 + a]_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$ is achieved. □

Before starting the main discussion, we give a definition that is necessary to continue.

**Definition 4.5** Let $\mathbb{T}$ be a time scale. A function $f : \mathbb{T} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is called

(i) *rd*-continuous if $g$ defined by $g(t) = f(t, x(t))$ is *rd*-continuous for any continuous function $x : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$;

(ii) Lipschitz continuous with respect to the second variable on a set $S \subset \mathbb{T} \times \mathbb{R}_{\mathcal{F}}$ if there exists a constant $L > 0$ such that

$$D\big(f(t,x_1),f(t,x_2)\big) \leq LD(x_1,x_2) \quad \text{for all } (t,x_1),(t,x_2) \in S.$$

Consider the following fuzzy dynamic equations:

$$\Delta_H x(t) = f\big(t,x(t)\big), \qquad x(t_0) = x_0, \quad t \in [t_0,t_0+a]_{\mathbb{T}}. \tag{11}$$

**Lemma 4.6** *For $t_0 \in \mathbb{T}$, the fuzzy dynamic equation $\Delta_H x(t) = f(t,x(t))$, $x(t_0) = x_0 \in \mathbb{R}_{\mathcal{F}}$, where $f : \mathbb{T} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is rd-continuous, is equivalent to one of the following integral equations*:

$$\begin{cases} x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))\Delta s, & t \in [t_0,t_0+a]_{\mathbb{T}} \\ or \\ x_0 = x(t) + (-1) \cdot \int_{t_0}^{t} f(s,x(s))\Delta s, & t \in [t_0,t_0+a]_{\mathbb{T}} \end{cases} \tag{12}$$

*on the interval $[t_0,t_0+a]_{\mathbb{T}}$, depending on the $\ominus_{gH}$ considered in the definition of delta derivative,* (i)$\Delta_H$ *or* (ii)$\Delta_H$*, respectively.*

*Proof* Let us suppose that $x$ is a solution of the fuzzy dynamic equation $\Delta_H x(t) = f(t,x(t))$, $x(t_0) = x_0 \in \mathbb{R}_{\mathcal{F}}$. Then, by integration, we get

$$\int_{t_0}^{t} \Delta_H x(s)\Delta s = \int_{t_0}^{t} f\big(s,x(s)\big)\Delta s.$$

So,

$$\begin{cases} x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))\Delta s, \\ \text{or} \\ x_0 = x(t) + (-1) \cdot \int_{t_0}^{t} f(s,x(s))\Delta s, \end{cases}$$

where in both cases we have a solution $x$ of the $\Delta_H$-integral equation.

In fact, a solution to the integral equations in (12) is a continuous function satisfying the conditions in (12). Now, if $x$ is a solution to one of $\Delta$-integral equations (12), we can write

$$x(t+h) = x_0 + \int_{t_0}^{t+h} f\big(s,x(s)\big)\Delta s$$

and

$$x\big(\sigma(t)\big) = x_0 + \int_{t_0}^{\sigma(t)} f\big(s,x(s)\big)\Delta s,$$

or

$$x(t+h) = x_0 \ominus (-1) \cdot \int_{t_0}^{t+h} f\big(s,x(s)\big)\Delta s,$$

and

$$x(\sigma(t)) = x_0 \ominus (-1) \cdot \int_{t_0}^{\sigma(t)} f(s, x(s)) \Delta s.$$

Therefore, if $t$ is a right-scattered point $\sigma(t) > t$

$$\frac{x(\sigma(t)) \ominus_{gH} x(t)}{\mu(t)} = \frac{1}{\mu(t)} \int_t^{\sigma(t)} f(s, x(s)) \Delta s.$$

Since $\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t)$, it follows

$$\frac{x(\sigma(t)) \ominus_{gH} x(t)}{\mu(t)} = f(t, x(t)).$$

And if $t$ is a right-dense point $\sigma(t) = t$, we have (in the metric $D$)

$$\lim_{h \to 0^+} \frac{x(t+h) \ominus_{gH} x(t)}{h} = \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} f(s, x(s)) \Delta s,$$

and we observe that

$$\begin{aligned} & D\left[\int_t^{t+h} f(s, x(s)) \Delta s, hf(t, x(t))\right] \\ &\quad = D\left[\int_t^{t+h} f(s, x(s)) \Delta s, \int_t^{t+h} f(t, x(t)) \Delta s\right] \\ &\quad \le \int_t^{t+h} D(f(s, x(s)), f(t, x(t))) \Delta s. \end{aligned}$$

Since $f$ is continuous at $t$ ($t$ is right-dense), it follows that for each $\epsilon > 0$ there exists a neighborhood $U_{\mathbb{T}}$ such that for each $s \in U_{\mathbb{T}}$, $D(f(t, x(t)), f(s, x(s))) < \epsilon$. Hence, by taking limit as $h \to 0^+$, we have

$$\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} f(s, x(s)) \Delta s = f(t, x(t)), \quad \text{in the metric } D,$$

therefore

$$\lim_{h \to 0^+} D\left[\frac{x(t+h) \ominus_{gH} x(t)}{h}, f(t, x(t))\right] = 0.$$

Similarly, the left fuzzy delta derivative of $f$ in $t$ is $f(t, x(t))$. This means that $x$ is a solution to the fuzzy dynamic equation $\Delta_H x(t) = f(t, x(t))$. □

From the proof of Lemma 4.6, it is deduced that from the first expression in (12) we have a (i)$\Delta_H$ differentiable solution and from the second expression in (12) we have a (ii)$\Delta_H$-differentiable solution. Based on Lemma 4.6, every first order fuzzy dynamic equation on time scales can be equivalent by one of two integral equations (12), so we have the following theorem for second order fuzzy dynamic equations.

Consider the following second order fuzzy dynamic equations:

$$\Delta_H^2 x = f(t, x, \Delta_H x), \qquad x(t_0) = k_1, \qquad \Delta_H x(t_0) = k_2, \tag{13}$$

where $k_1, k_2 \in \mathbb{R}_{\mathcal{F}}$.

**Theorem 4.7** *Assume that $f : [t_0, t_0 + a]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$ is rd-continuous. A mapping $x : [t_0, t_0 + a] \to \mathbb{R}_{\mathcal{F}}$ is a solution to initial value problem* (13) *if and only if $x$ and $\Delta_H x$ are continuous and satisfy one of the following integral equations*:

(i) $x(t) = k_2(t - t_0) + \int_{t_0}^{t} (\int_{t_0}^{z} f(s, x(s), \Delta_H x(s))\Delta s)\Delta z + k_1$, *where $\Delta_H x$ and $\Delta_H^2 x$ are* (i)$\Delta_H$*-differentials, or*

(ii) $x(t) = \ominus(-1)(k_2(t - t_0) \ominus (-1) \int_{t_0}^{t} (\int_{t_0}^{z} f(s, x(s), \Delta_H x(s))\Delta s)\Delta z) + k_1$, *where $\Delta_H x$ and $\Delta_H^2 x$ are* (ii)$\Delta_H$*-differentials, or*

(iii) $x(t) = \ominus(-1)(k_2(t - t_0) + \int_{t_0}^{t} (\int_{t_0}^{z} f(s, x(s), \Delta_H x(s))\Delta s)\Delta z) + k_1$, *where $\Delta_H x$ is* (i)$\Delta_H$ *differential and $\Delta_H^2 x$ is* (ii)$\Delta_H$*-differential, or*

(iv) $x(t) = k_2(t - t_0) \ominus (-1) \int_{t_0}^{t} (\int_{t_0}^{z} f(s, x(s), \Delta_H x(s))\Delta s)\Delta z + k_1$, *where $\Delta_H x$ is* (ii)$\Delta_H$ *differential and $\Delta_H^2 x$ is* (ii)$\Delta_H$*-differential.*

*Proof* It can be proved easily using Lemma 4.6. □

**Theorem 4.8** *Let $f : [t_0, t_0 + a]_{\mathbb{T}} \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be continuous, and suppose that there exist $L_1, L_2 > 0$ such that*

$$D\big(f(t, x_1, x_2), f(t, y_1, y_2)\big) \le L_1 D(x_1, y_1) + L_2 D(x_2, y_2)$$

*for all $t \in [t_0, t_0 + a]_{\mathbb{T}}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}_{\mathcal{F}}$. Then the initial value problem* (13) *has a unique solution on $[t_0, t_0 + a]_{\mathbb{T}}$ for each case.*

*Proof* We only prove it for case (ii) of Theorem 4.7, the proofs for other cases are similar. Now, consider the following operator:

$$(Tx)(t) = \ominus(-1)\left(k_2(t - t_0) \ominus (-1) \int_{t_0}^{t} \left(\int_{t_0}^{z} f\big(x, x(s), \Delta_H x\big)\Delta s\right)\Delta z\right) + k_1. \tag{14}$$

Also note that

$$\big(\Delta_H T(x)\big)(t) = k_2 \ominus (-1) \int_{t_0}^{t} f\big(s, x(s), \Delta_H x(s)\big)\Delta s. \tag{15}$$

Now we will show that by choosing a big enough value $\gamma$, $T$ map is a contraction, so Banach's fixed point theorem provides the existence of a unique fixed point for $T$, that is, a unique solution for (13) in case (ii).

$$\begin{aligned} &d_\gamma^1(Tx, Ty) \\ &\quad = d_\gamma(Tx, Ty) + d_\gamma\big(\Delta_H(Tx), \Delta_H(Ty)\big) \\ &\quad = \sup_{t \in [t_0, t_0 + a]_{\mathbb{T}}} \left\{ \frac{1}{e_\gamma(t, t_0)} \right. \end{aligned}$$

$$
\begin{aligned}
&\times D\biggl(k_1 \ominus (-1)\biggl(k_2(t-t_0) \ominus (-1)\int_{t_0}^{t}\biggl(\int_{t_0}^{z} f\bigl(s,x(s),\Delta_H x(s)\bigr)\Delta s\biggr)\Delta z\biggr),\\
&k_1 \ominus (-1)\biggl(k_2(t-t_0) \ominus (-1)\int_{t_0}^{t}\biggl(\int_{t_0}^{z} f\bigl(s,y(s),\Delta_H y(s)\bigr)\Delta s\biggr)\Delta z\biggr)\biggr)\biggr\}\\
&\quad + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}D\biggl(k_2 \ominus (-1)\int_{t_0}^{t} f\bigl(s,x(s),\Delta_H x(s)\bigr)\Delta s,\\
&k_2 \ominus (-1)\int_{t_0}^{t} f\bigl(s,y(s),\Delta_H y(s)\bigr)\Delta s\biggr)\biggr\}\\
&= \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}D\biggl((-1)\biggl(k_2(t-t_0) \ominus (-1)\int_{t_0}^{t}\biggl(\int_{t_0}^{z} f\bigl(s,x(s),\Delta_H x(s)\bigr)\Delta s\biggr)\Delta z\biggr),\\
&(-1)\biggl(k_2(t-t_0) \ominus (-1)\int_{t_0}^{t}\biggl(\int_{t_0}^{z} f\bigl(s,y(s),\Delta_H y(s)\bigr)\Delta s\biggr)\Delta z\biggr)\biggr)\biggr\}\\
&\quad + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}D\biggl(\int_{t_0}^{t} f\bigl(s,x(s),\Delta_H x(s)\bigr)\Delta s, \int_{t_0}^{t} f\bigl(s,y(s),\Delta_H y(s)\bigr)\Delta s\biggr)\biggr\}\\
&= \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}D\biggl(\biggl(\int_{t_0}^{t}\biggl(\int_{t_0}^{z} f\bigl(s,x(s),\Delta_H x(s)\bigr)\Delta s\biggr)\Delta z\biggr),\\
&\biggl(\int_{t_0}^{t}\biggl(\int_{t_0}^{z} f\bigl(s,y(s),\Delta_H y(s)\bigr)\Delta s\biggr)\Delta z\biggr)\biggr)\biggr\}\\
&\quad + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}D\biggl(\int_{t_0}^{t} f\bigl(s,x(s),\Delta_H x(s)\bigr)\Delta s, \int_{t_0}^{t} f\bigl(s,y(s),\Delta_H y(s)\bigr)\Delta s\biggr)\biggr\}\\
&\le \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}\biggl(\int_{t_0}^{t}\biggl(\int_{t_0}^{z} D\bigl(f\bigl(s,x(s),\Delta_H x(s)\bigr), f\bigl(s,y(s),\Delta_H y(s)\bigr)\bigr)\Delta s\biggr)\Delta z\biggr)\biggr\}\\
&\quad + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}\biggl(\int_{t_0}^{t} D\bigl(f\bigl(s,x(s),\Delta_H x(s)\bigr), f\bigl(s,y(s),\Delta_H y(s)\bigr)\bigr)\Delta s\biggr)\biggr\}\\
&\le \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}\biggl(\int_{t_0}^{t}\biggl(\int_{t_0}^{z}\bigl(L_1 D\bigl(x(s),y(s)\bigr) + L_2 D\bigl(\Delta_H x(s),\Delta_H y(s)\bigr)\bigr)\Delta s\biggr)\Delta z\biggr)\biggr\}\\
&\quad + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}\biggl(\int_{t_0}^{t}\bigl(L_1 D\bigl(x(s),y(s)\bigr) + L_2 D\bigl(\Delta_H x(s),\Delta_H y(s)\bigr)\bigr)\Delta s\biggr)\biggr\}\\
&\le \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}\\
&\quad\times\biggl(\int_{t_0}^{t}\biggl(\int_{t_0}^{z}\bigl(L_1 d_\gamma\bigl(x(s),y(s)\bigr) + L_2 d_\gamma(\Delta_H x,\Delta_H y)\bigr)e_\gamma(s,t_0)\Delta s\biggr)\Delta z\biggr)\biggr\}\\
&\quad + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}\biggl(\int_{t_0}^{t}\bigl(L_1 d_\gamma(x,y) + L_2 d_\gamma(\Delta_H x,\Delta_H y)\bigr)e_\gamma(s,t_0)\Delta s\biggr)\biggr\}\\
&= \bigl(L_1 d_\gamma(x,y) + L_2 d_\gamma(\Delta_H x,\Delta_H y)\bigr)\biggl(\sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}\biggl(\int_{t_0}^{t}\biggl(\int_{t_0}^{z} e_\gamma(s,t_0)\Delta s\biggr)\Delta z\biggr)\biggr\}\\
&\quad + \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}\biggl(\int_{t_0}^{t} e_\gamma(s,t_0)\Delta s\biggr)\biggr\}\biggr)\\
&= \bigl(L_1 d_\gamma(x,y) + L_2 d_\gamma(\Delta_H x,\Delta_H y)\bigr)\biggl(\sup_{t\in[t_0,t_0+a]_{\mathbb{T}}}\biggl\{\frac{1}{e_\gamma(t,t_0)}\biggl(\int_{t_0}^{t}\biggl(\frac{e_\gamma(z,t_0)-1}{\gamma}\biggr)\Delta z\biggr)\biggr\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{e_\gamma(t,t_0)-1}{\gamma e_\gamma(t,t_0)} \right\} \Bigg) \\
&= \big(L_1 d_\gamma(x,y) + L_2 d_\gamma(\Delta_H x, \Delta_H y)\big) \Bigg( \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{1}{\gamma e_\gamma(t,t_0)} \left( \frac{e_\gamma(t,t_0)-1}{\gamma} - (t-t_0) \right) \right\} \\
&+ \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{e_\gamma(t,t_0)-1}{\gamma e_\gamma(t,t_0)} \right\} \Bigg) \\
&= \big(L_1 d_\gamma(x,y) + L_2 d_\gamma(\Delta_H x, \Delta_H y)\big) \Bigg( \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{1}{\gamma^2 e_\gamma(t,t_0)} \big( e_\gamma(t,t_0) - 1 - \gamma(t-t_0) \big) \right\} \\
&+ \sup_{t\in[t_0,t_0+a]_{\mathbb{T}}} \left\{ \frac{1}{\gamma} \left( 1 - \frac{1}{e_\gamma(t,t_0)} \right) \right\} \Bigg).
\end{aligned}
$$

Since $(\frac{1}{\gamma^2 e_\gamma(t,t_0)}(e_\gamma(t,t_0)-1-\gamma(t-t_0)))^\Delta = \frac{t-t_0}{e_\gamma(\sigma(t),t_0)} > 0$ and $(\frac{1}{\gamma}(1-\frac{1}{e_\gamma(t,t_0)}))^\Delta = \frac{1}{e_\gamma(\sigma(t),t_0)} > 0$, hence the suprema of two sets above is attained at point $t_0 + a$. Therefore,

$$
\begin{aligned}
&d_\gamma^1(Tx,Ty) \\
&\quad \le \max\{L_1,L_2\} d_\gamma^1(x,y) \Bigg( \frac{1}{\gamma^2 e_\gamma(t_0+a,t_0)} \big( e_\gamma(t_0+a,t_0) - 1 - \gamma(a) \big) \\
&\quad + \frac{1}{\gamma} \left( 1 - \frac{1}{e_\gamma(t_0+a,t_0)} \right) \Bigg),
\end{aligned}
$$

according to

$$
\lim_{\gamma\to\infty} \left( \frac{1}{\gamma^2 e_\gamma(t_0+a,t_0)} \big( e_\gamma(t_0+a,t_0) - 1 - \gamma(a) \big) + \frac{1}{\gamma} \left( 1 - \frac{1}{e_\gamma(t_0+a,t_0)} \right) \right) \to 0, \tag{16}
$$

we can choose $\gamma > 0$ such that

$$
\begin{aligned}
&\max\{L_1,L_2\} \left( \frac{1}{\gamma^2 e_\gamma(t_0+a,t_0)} \big( e_\gamma(t_0+a,t_0) - 1 - \gamma(a) \big) + \frac{1}{\gamma} \left( 1 - \frac{1}{e_\gamma(t_0+a,t_0)} \right) \right) \\
&\quad < 1
\end{aligned}
$$

thus $T$ is a contractive mapping. □

## 5 Conclusion

In this paper, we introduced the second order fuzzy dynamic equations on time scales and defined a new metric. We proved the existence and uniqueness of solutions to first order fuzzy dynamic equations on time scales in [22]. But in this work we proved the existence and uniqueness of solutions to second order fuzzy dynamic equations on time scales. Although we apply this method for fuzzy dynamic equations, we can apply the presented method for crisp ordinary differential equations.

#### Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

#### Authors' contributions

All authors drafted the manuscript, and they read and approved the final version of the manuscript.

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