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Robust stabilization of hybrid uncertain stochastic systems with time-varying delay by discrete-time feedback control

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Abstract

This paper is concerned with the robust stabilization problem for a class of continuous-time hybrid uncertain stochastic systems with time-varying delay. Our attention is focused on the design of a feedback controller based on discrete-time state observations such that, for all admissible uncertainties, the closed-loop system is robustly exponentially stable in the mean square, independent of the time delay. By employing the Razumikhin technique, sufficient conditions are firstly established to guarantee the existence of the desired robust controller; they are given in terms of linear matrix inequalities (LMIs). It should be pointed out that in this paper the time-varying delay is just required to be a bounded function rather than a bounded differentiable one as in most existing results. The developed theory is illustrated by a numerical example.

Keywords: hybrid uncertain stochastic systems; robust stabilization; time-varying delay; discrete-time feedback control; Razumikhin technique

1 Introduction

As an important class of hybrid systems, hybrid stochastic differential equations (SDEs) (also known as SDEs with Markovian switching) have been widely employed to model many practical systems that have variable structures subject to random abrupt changes, which may result from abrupt phenomena such as random failures and repairs of components, sudden environment changes, etc. One of the important issues in the study of hybrid SDEs is the automatic control, with subsequent emphasis being placed on the analysis of stability. A great number of significant results on this subject have been reported in the literature; see, for instance, [1–11] and the references therein. In particular, we refer the reader to the book [8].

It is well known that parameter uncertainties exist inevitably in various dynamic systems due to modeling and measurement errors etc. And also, many practical systems, such as aircraft systems, chemical process systems and electrical networks systems, frequently encounter time delays. These often have an unstable effect and lead to undesirable dynamic behaviors of control systems. Therefore, the problem of stability analysis for uncertain time-delay systems has developed into a hot topic, and a huge number of papers have appeared. For example, Wang et al. [12] presented several sufficient conditions, delay-independent or delay-dependent, that guarantee the robust stability of uncertain time-delay systems subjected to parametric perturbations. In [13], the problem of robust stability of uncertain systems with interval time-varying delay was considered by constructing a Lyapunov functional with uncorrelated augmented matrix items and a tighter bounding technology. As for the stochastic case, Mao [4] investigated the exponential stability for a class of linear stochastic delay interval systems with Markovian switching. Improved delay-dependent stability criteria for uncertain stochastic dynamic systems with time-varying delays were proposed in [14]. We further refer the reader to [6, 10, 15–21] and the references therein. It is noted that most of the existing results on the stability of uncertain (hybrid stochastic) delay systems require the time delay to be a constant or a differentiable function whose derivative is bounded by a positive constant less than 1, and this is very restrictive.

When a system is unstable, we usually need to design controllers to stabilize the original system. A class of controllers commonly used are feedback controllers with or without delays. Wang et al. in [5] designed a state feedback controller to stabilize bilinear uncertain time-delay stochastic systems with Markovian jumping parameters in the mean-square sense. The almost surely exponential stabilization problem of hybrid stochastic differential equations by stochastic feedback controllers was investigated in [22]. A robust delayed-state-feedback controller that exponentially stabilized uncertain stochastic delay systems was proposed in [23]. It is observed that most of the pioneering works on the stabilization problem of stochastic systems often employed a regular feedback control which requires continuous observations of the state. This is expensive and sometimes not possible as the observations are often of discrete time in practice. In 2013, Mao [24] initiated the study of the mean-square exponential stabilization of continuous-time hybrid stochastic differential equations by feedback controls based on discrete-time state observations, which cost less and are more realistic. Shortly after, Mao et al. in [25] provided us with a better bound on the duration between two consecutive state observations, while You et al. [26] removed the global Lipschitz assumption on coefficients and further investigated the asymptotic stabilization of nonlinear hybrid stochastic systems. Furthermore, taking the parameter uncertainties into account, You et al. [27] discussed the robust discrete-state-feedback stabilization of hybrid uncertain stochastic systems, and Li et al. [28], going a step further, studied the robust stabilization of uncertain stochastic systems with Markovian switching by feedback control based on discrete-time state and mode observations. However, to the best knowledge of the authors, little work has been done for robust stabilization of hybrid uncertain stochastic systems with time-varying delay by discrete-time feedback control.

Motivated by the above situation, we will investigate the robust discrete-state-feedback stabilization for a class of hybrid uncertain stochastic systems with time-varying delay. It should be pointed out that this work is challenging because, in order to overcome the difficulties arisen from the delay term $x(t-\delta(t))$, we must find an efficient method that is different from those in [24–27]. If we use a similar method like that in [25, 27], we will have to estimate the differences between $x(t)$ and $x(\eta(t))$, $x(t)$ and $x(t-\delta(t))$ as well as $x(\eta(t))$ and $x(t-\delta(t))$, which is very complicated, and the estimation bounds are dependent on the delay term such that it becomes very difficult to achieve our goal. In case the Lyapunov functional method is adopted, firstly it is very difficult to construct an appropriate Lyapunov functional; on the other hand, the time-varying delay is generally required to be a bounded differentiable function, which is restrictive. Fortunately, we come up with the Razumikhin technique, by which not only the robust stabilization problem for hybrid uncertain stochastic systems with time-varying delay will be settled, but also the restriction that the time-varying delay should be differentiable as in most existing results will be removed. The rest of this article is arranged as follows. Section 2 recalls some notations, related definition and some useful lemmas and states the subject to be studied. The main results are presented in Section 3. Section 4 covers a numerical example to demonstrate the main results. Finally, the article is concluded in Section 5.

2 Preliminaries and problem statement

Throughout this paper, we use the following notations. For $x \in R^n$, $|x|$ denotes its Euclidean norm. If A is a vector or matrix, its transpose is denoted by A^T. For a matrix A, we let $|A| = \sqrt{\operatorname{trace}(A^TA)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. If A is a symmetric matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalues, respectively. For two symmetric matrices A and B, $A > (<, \geq, \leq) B$ means that $A - B$ is positive definite (negative definite, positive semidefinite, negative semidefinite). The integer part of a real number x will be denoted by $[x]$. If both a, b are real numbers, then $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right-continuous with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Let $w(t) = (w_1(t), \ldots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. $L^2(\Omega; \mathbb{R}^n)$ stands for the family of all $\mathbb{R}^n$-valued random variables X such that $\mathbb{E}|X|^2 < \infty$. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to R^n with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ the family of all bounded, $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables. For $p > 0$ and $t \geq 0$, $L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ stands for the family of all $\mathcal{F}_t$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\sup_{-\tau\leq\theta\leq 0} E|\phi(\theta)|^p < \infty$. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that almost all sample paths of $r(t)$ are constant except for a finite number of simple jumps in any finite subinterval of $R_+ := [0, \infty)$. We stress that almost all sample paths of $r(t)$ are right-continuous.

Let us consider the following controlled hybrid uncertain stochastic system with time-varying delay:

$$\begin{aligned} dx(t) = {} & \bigl\{\bigl[A\bigl(r(t)\bigr) + \Delta A\bigl(t, r(t)\bigr)\bigr]x(t) + \bigl[A_d\bigl(r(t)\bigr) + \Delta A_d\bigl(t, r(t)\bigr)\bigr]x\bigl(t - \delta(t)\bigr) \\ & + C\bigl(r(t)\bigr)u\bigl(x\bigl(\eta(t)\bigr), r(t)\bigr)\bigr\}\, dt \end{aligned}$$

$$\begin{aligned}&+\sum_{k=1}^{m}\bigl\{\bigl[B^k\bigl(r(t)\bigr)+\Delta B^k\bigl(t,r(t)\bigr)\bigr]x(t)\\&+\bigl[B_d^k\bigl(r(t)\bigr)+\Delta B_d^k\bigl(t,r(t)\bigr)\bigr]x\bigl(t-\delta(t)\bigr)\bigr\}\,dw_k(t)\end{aligned}\tag{2.1}$$

on $t \geq 0$, with initial data

$$x_0=\xi\in C^b_{\mathcal{F}_0}\bigl([-h,0];\mathbb{R}^n\bigr),\qquad r(0)=r_0\in S,\tag{2.2}$$

where the time delay $\delta(t)$ is a Borel-measurable function with $0 \leq \delta(t) \leq h$ for all $t \geq 0$ and for any $i \in S$, $A(i) = A_i$, $A_d(i) = A_{di}$, $C(i) = C_i$, $B^k(i) = B^k_i$, $B^k_d(i) = B^k_{di}$ are given constant matrices, while uncertainties $\Delta A(t,i)$, $\Delta A_d(t,i)$, $\Delta B^k(t,i)$, $\Delta B^k_d(t,i)$ are assumed to be norm-bounded, i.e.,

$$\begin{aligned}&\Delta A(t,i)=L_AF_A(t)E_i,\qquad \Delta A_d(t,i)=L_AF_A(t)E_{di},\\&\Delta B^k(t,i)=L_BF_B(t)N_i^k,\qquad \Delta B_d^k(t,i)=L_BF_B(t)N_{di}^k\end{aligned}\tag{2.3}$$

with known constant matrices $L_A, E_i, E_{di}, L_B, N^k_i, N^k_{di}$ and unknown matrix functions $F_A(t)$ and $F_B(t)$ having Lebesgue-measurable elements and satisfying

$$F_A^T(t)F_A(t)\leq I,\qquad F_B^T(t)F_B(t)\leq I,\quad \forall t\geq 0.\tag{2.4}$$

Any uncertainties $\Delta A(t,i), \Delta A_d(t,i), \Delta B^k(t,i)$ and $\Delta B^k_d(t,i)$ satisfying (2.3) and (2.4) are said to be admissible. We choose the controller of the form

$$u\bigl(x\bigl(\eta(t)\bigr),r(t)\bigr)=K\bigl(r(t)\bigr)x\bigl(\eta(t)\bigr)$$

with $\eta(t) = [t/\tau]\tau$ for $t \geq 0$. Then the controlled system (2.1) becomes

$$\begin{aligned}dx(t)={}&\bigl\{\bigl[A\bigl(r(t)\bigr)+\Delta A\bigl(t,r(t)\bigr)\bigr]x(t)+\bigl[A_d\bigl(r(t)\bigr)+\Delta A_d\bigl(t,r(t)\bigr)\bigr]x\bigl(t-\delta(t)\bigr)\\&+C\bigl(r(t)\bigr)K\bigl(r(t)\bigr)x\bigl(\eta(t)\bigr)\bigr\}\,dt\\&+\sum_{k=1}^{m}\bigl\{\bigl[B^k\bigl(r(t)\bigr)+\Delta B^k\bigl(t,r(t)\bigr)\bigr]x(t)\\&+\bigl[B_d^k\bigl(r(t)\bigr)+\Delta B_d^k\bigl(t,r(t)\bigr)\bigr]x\bigl(t-\delta(t)\bigr)\bigr\}\,dw_k(t).\end{aligned}\tag{2.5}$$

System (2.5) is in fact a hybrid uncertain stochastic system with mixed bounded variable delays as follows:

$$\begin{aligned}dx(t)={}&\bigl\{\bigl[A\bigl(r(t)\bigr)+\Delta A\bigl(t,r(t)\bigr)\bigr]x(t)+\bigl[A_d\bigl(r(t)\bigr)+\Delta A_d\bigl(t,r(t)\bigr)\bigr]x\bigl(t-\delta(t)\bigr)\\&+C\bigl(r(t)\bigr)K\bigl(r(t)\bigr)x\bigl(t-\zeta(t)\bigr)\bigr\}\,dt\\&+\sum_{k=1}^{m}\bigl\{\bigl[B^k\bigl(r(t)\bigr)+\Delta B^k\bigl(t,r(t)\bigr)\bigr]x(t)\\&+\bigl[B_d^k\bigl(r(t)\bigr)+\Delta B_d^k\bigl(t,r(t)\bigr)\bigr]x\bigl(t-\delta(t)\bigr)\bigr\}\,dw_k(t)\end{aligned}\tag{2.6}$$

if we define the bounded variable delay $\zeta : [0,\infty) \to [0,\tau]$ by

$$\zeta(t) = t - \nu\tau \quad \text{for } \nu\tau \le t < (\nu+1)\tau, \tag{2.7}$$

where $\nu = 0,1,2,\ldots$. It is easy to know that given any initial data $r_0 \in S$ and $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\gamma,0];\mathbb{R}^n)$, where $\gamma = h \vee \tau$, system (2.6) has a unique continuous solution $x(t)$ such that $\mathbb{E}|x(t)|^2 < \infty$ for all $t \ge -\gamma$ (*cf.* [8]). Moreover, to achieve the mean-square exponential stability of system (2.5), we just need to prove that system (2.6) is exponentially stable in the mean-square sense due to the arbitrariness of integer ν. So our study will mainly focus on system (2.6) in the rest of this paper.

At the end of this section, let us introduce the basic definition, two useful lemmas (*cf.* [27]) and the Razumikhin-type theorem (*cf.* [8]) which are useful for further discussion.

Definition 2.1 The controlled system (2.6) with initial data $r_0 \in S$, $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\gamma,0];\mathbb{R}^n)$ is said to be robustly exponentially stable in the mean square if there is a positive constant $\lambda > 0$ such that, for any admissible uncertainties, the solution $x(t)$ satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log\bigl(\mathbb{E}\bigl|x(t)\bigr|^2\bigr) \le -\lambda.$$

Lemma 2.2 *For any vectors $u \in \mathbb{R}^q$, $v \in \mathbb{R}^l$ and a matrix $M \in \mathbb{R}^{q\times l}$, the inequality*

$$2u^T M v \le r u^T M G M^T u + \frac{1}{r} v^T G^{-1} v$$

holds for any symmetric positive definite matrix $G \in \mathbb{R}^{l\times l}$ and number $r > 0$.

Lemma 2.3 *Let A, B, D, F, W be matrices with suitable dimensions. If $W > 0$, $F^T F \le I$, then for any number $\varepsilon > 0$ such that $W^{-1} - \varepsilon DD^T > 0$, it holds that*

$$(A + DFB)^T W (A + DFB) \le A^T \bigl(W^{-1} - \varepsilon DD^T\bigr)^{-1} A + \varepsilon^{-1} B^T B.$$

Theorem 2.4 *Let λ, p, c_1, c_2 be all positive numbers and $q > 1$. Assume that there exists a function $V \in C^{2,1}(\mathbb{R}^n \times [-\gamma,\infty) \times S; \mathbb{R}_+)$ such that*

$$c_1|x|^p \le V(x,t,i) \le c_2|x|^p \quad \text{for all } (x,t,i) \in \mathbb{R}^n \times [-\gamma,\infty) \times S, \tag{2.8}$$

and also for all $t \ge 0$,

$$\mathbb{E}\Bigl[\max_{1\le i\le N} \mathcal{L}V(\phi,t,i)\Bigr] \le -\lambda \mathbb{E}\Bigl[\max_{1\le i\le N} V\bigl(\phi(0),t,i\bigr)\Bigr] \tag{2.9}$$

provided $\phi = \{\phi(\theta) : -\gamma \le \theta \le 0\} \in L^p_{\mathcal{F}_t}([-\gamma,0];\mathbb{R}^n)$ satisfying

$$\mathbb{E}\Bigl[\min_{1\le i\le N} V\bigl(\phi(\theta),t+\theta,i\bigr)\Bigr] < q\mathbb{E}\Bigl[\max_{1\le i\le N} V\bigl(\phi(0),t,i\bigr)\Bigr] \tag{2.10}$$

for all $-\gamma \leq \theta \leq 0$. *Then, for all* $\xi \in C^b_{\mathcal{F}_0}([-\gamma, 0]; \mathbb{R}^n)$, *the solution of a stochastic functional differential equation has the property that*

$$\limsup_{t\to\infty} \frac{1}{t} \log\bigl(\mathbb{E}\bigl|x(t;\xi)\bigr|^p\bigr) \leq -\rho,$$

where $\rho = \min\{\lambda, \log(q)/\gamma\}$. *In other words, the trivial solution of a stochastic functional differential system is pth moment exponentially stable and the pth moment Lyapunov exponent is not greater than* $-\rho$.

3 Main results

Let us rewrite system (2.6) as

$$\begin{aligned} dx(t) = {}& \bigl\{\bigl[A\bigl(r(t)\bigr) + \Delta A\bigl(t, r(t)\bigr) + C\bigl(r(t)\bigr)K\bigl(r(t)\bigr)\bigr]x(t) \\ & + \bigl[A_d\bigl(r(t)\bigr) + \Delta A_d\bigl(t, r(t)\bigr)\bigr]x\bigl(t-\delta(t)\bigr) \\ & - C\bigl(r(t)\bigr)K\bigl(r(t)\bigr)\bigl(x(t) - x\bigl(t-\zeta(t)\bigr)\bigr)\bigr\}\,dt \\ & + \sum_{k=1}^{m}\bigl\{\bigl[B^k\bigl(r(t)\bigr) + \Delta B^k\bigl(t, r(t)\bigr)\bigr]x(t) \\ & + \bigl[B^k_d\bigl(r(t)\bigr) + \Delta B^k_d\bigl(t, r(t)\bigr)\bigr]x\bigl(t-\delta(t)\bigr)\bigr\}\,dw_k(t) \end{aligned} \tag{3.1}$$

on $t \geq \gamma$ with

$$\begin{aligned} x(t) - x\bigl(t-\zeta(t)\bigr) = {}& \int_{t-\zeta(t)}^{t} \bigl\{\bigl[A\bigl(r(s)\bigr) + \Delta A\bigl(s, r(s)\bigr)\bigr]x(s) \\ & + \bigl[A_d\bigl(r(s)\bigr) + \Delta A_d\bigl(s, r(s)\bigr)\bigr]x\bigl(s-\delta(s)\bigr) \\ & + C\bigl(r(s)\bigr)K\bigl(r(s)\bigr)x\bigl(s-\zeta(s)\bigr)\bigr\}\,ds \\ & + \sum_{k=1}^{m}\int_{t-\zeta(t)}^{t} \bigl\{\bigl[B^k\bigl(r(s)\bigr) + \Delta B^k\bigl(s, r(s)\bigr)\bigr]x(s) \\ & + \bigl[B^k_d\bigl(r(s)\bigr) + \Delta B^k_d\bigl(s, r(s)\bigr)\bigr]x\bigl(s-\delta(s)\bigr)\bigr\}\,dw_k(s). \end{aligned}$$

For each $t \geq \gamma$, define an operator $\Phi(t,\cdot) : L^2_{\mathcal{F}_t}([-2\gamma, 0]; \mathbb{R}^n) \to L^2(\Omega; \mathbb{R}^n)$ by

$$\begin{aligned} \Phi(t,\varphi) = {}& \int_{t-\zeta(t)}^{t} \bigl\{\bigl[A\bigl(r(s)\bigr) + \Delta A\bigl(s, r(s)\bigr)\bigr]\varphi(s-t) \\ & + \bigl[A_d\bigl(r(s)\bigr) + \Delta A_d\bigl(s, r(s)\bigr)\bigr]\varphi\bigl(s-\delta(s)-t\bigr) \\ & + C\bigl(r(s)\bigr)K\bigl(r(s)\bigr)\varphi\bigl(s-\zeta(s)-t\bigr)\bigr\}\,ds \\ & + \sum_{k=1}^{m}\int_{t-\zeta(t)}^{t} \bigl\{\bigl[B^k\bigl(r(s)\bigr) + \Delta B^k\bigl(s, r(s)\bigr)\bigr]\varphi(s-t) \\ & + \bigl[B^k_d\bigl(r(s)\bigr) + \Delta B^k_d\bigl(s, r(s)\bigr)\bigr]\varphi\bigl(s-\delta(s)-t\bigr)\bigr\}\,dw_k(s). \end{aligned} \tag{3.2}$$

Moreover, let $x_t = \{x(t+\theta) : -2\gamma \le \theta \le 0\}$ on $t \ge \gamma$, which is regarded as a $C([-2\gamma,0];\mathbb{R}^n)$-valued stochastic process. Then we have

$$x(t) - x\big(t-\zeta(t)\big) = \Phi(t,x_t),$$

and hence system (2.6) can be further written as

$$\begin{aligned} dx(t) = {} & \big\{\big[A\big(r(t)\big) + \Delta A\big(t,r(t)\big) + C\big(r(t)\big)K\big(r(t)\big)\big]x(t) \\ & + \big[A_d\big(r(t)\big) + \Delta A_d\big(t,r(t)\big)\big]x\big(t-\delta(t)\big) - C\big(r(t)\big)K\big(r(t)\big)\Phi(t,x_t)\big\}\,dt \\ & + \sum_{k=1}^{m}\big\{\big[B^k\big(r(t)\big) + \Delta B^k\big(t,r(t)\big)\big]x(t) \\ & + \big[B_d^k\big(r(t)\big) + \Delta B_d^k\big(t,r(t)\big)\big]x\big(t-\delta(t)\big)\big\}\,dw_k(t) \end{aligned} \tag{3.3}$$

on $t \ge \gamma$ with initial data $x(t) = \xi$ for $t \in [-\gamma,0]$ and $x(t) = x(t,\xi)$ for $t \in [0,\gamma]$.

The following lemma gives an estimation for the operator Φ, which is useful for the proof of our main results.

Lemma 3.1 *Set*

$$\begin{aligned} & M_A = 2\max_{i\in S}\big(\|A_i\|^2 + \|L_A\|^2\cdot\|E_i\|^2\big), \qquad M_{A_d} = 2\max_{i\in S}\big(\|A_{di}\|^2 + \|L_A\|^2\cdot\|E_{di}\|^2\big), \\ & M_{CK} = \max_{i\in S}\|C_iK_i\|^2, \qquad M_B = 2\max_{i\in S}\sum_{k=1}^{m}\big(\big\|B_i^k\big\|^2 + \|L_B\|^2\cdot\big\|N_i^k\big\|^2\big), \\ & M_{B_d} = 2\max_{i\in S}\sum_{k=1}^{m}\big(\big\|B_{di}^k\big\|^2 + \|L_B\|^2\cdot\big\|N_{di}^k\big\|^2\big), \end{aligned}$$

and define

$$K_\tau = 6\tau^2(M_A + M_{A_d} + M_{CK}) + 4\tau(M_B + M_{B_d}).$$

Then the operator Φ *defined by* (3.2) *has the property that*

$$\mathbb{E}\big|\Phi(t,\varphi)\big|^2 \le K_\tau \sup_{-2\gamma\le\theta\le0}\mathbb{E}\big|\varphi(\theta)\big|^2$$

for all $t \ge \gamma$ *and* $\varphi \in L^2_{\mathcal{F}_t}([-2\gamma,0];\mathbb{R}^n)$.

Proof By *Hölder's* inequality and Doob's martingale inequality, we can derive from (3.2) that

$$\begin{aligned} \mathbb{E}\big|\Phi(t,\varphi)\big|^2 \le 2\tau\mathbb{E}\int_{t-\zeta(t)}^{t} & \big|\big[A\big(r(s)\big) + \Delta A\big(s,r(s)\big)\big]\varphi(s-t) \\ & + \big[A_d\big(r(s)\big) + \Delta A_d\big(s,r(s)\big)\big]\varphi\big(s-\delta(s)-t\big) \\ & + C\big(r(s)\big)K\big(r(s)\big)\varphi\big(s-\zeta(s)-t\big)\big|^2\,ds \end{aligned}$$

$$
\begin{aligned}
&+4\mathbb{E}\left|\sum_{k=1}^{m}\int_{t-\zeta(t)}^{t}\left[B^k\big(r(s)\big)+\Delta B^k\big(s,r(s)\big)\right]\varphi(s-t)\,dw_k(s)\right|^2\\
&+4\mathbb{E}\left|\sum_{k=1}^{m}\int_{t-\zeta(t)}^{t}\left[B_d^k\big(r(s)\big)+\Delta B_d^k\big(s,r(s)\big)\right]\varphi\big(s-\delta(s)-t\big)\,dw_k(s)\right|^2\\
\le{}& 6\tau\mathbb{E}\int_{t-\tau}^{t}\Big(M_A\big|\varphi(s-t)\big|^2+M_{A_d}\big|\varphi\big(s-\delta(s)-t\big)\big|^2\\
&+M_{CK}\big|\varphi\big(s-\zeta(s)-t\big)\big|^2\Big)\,ds\\
&+4\sum_{k=1}^{m}\int_{t-\tau}^{t}\mathbb{E}\Big(\big\|B^k\big(r(s)\big)+\Delta B^k\big(s,r(s)\big)\big\|^2\big|\varphi(s-t)\big|^2\Big)\,ds\\
&+4\sum_{k=1}^{m}\int_{t-\tau}^{t}\mathbb{E}\Big(\big\|B_d^k\big(r(s)\big)+\Delta B_d^k\big(s,r(s)\big)\big\|^2\big|\varphi\big(s-\delta(s)-t\big)\big|^2\Big)\,ds\\
\le{}& K_\tau\sup_{-2\gamma\le\theta\le 0}\mathbb{E}\big|\varphi(\theta)\big|^2
\end{aligned}
$$

as required. □

Sufficient conditions for robust exponential stability of hybrid uncertain stochastic delay system (3.3) are proposed as follows.

Theorem 3.2 *Assume that there exist positive definite matrices Q_i and positive numbers ρ_{1i}, ρ_{2i}, ε_{1i}, ε_{2i} ($i\in S$) such that, for any $i\in S$,*

$$Q_i^{-1}-\varepsilon_{1i}L_BL_B^T>0,\qquad Q_i^{-1}-\varepsilon_{2i}L_BL_B^T>0 \tag{3.4}$$

and

$$
\begin{aligned}
\bar{Q}_i :={}& Q_i(A_i+C_iK_i)+(A_i+C_iK_i)^TQ_i+(\rho_{1i}+\rho_{2i})Q_iL_AL_A^TQ_i+\rho_{1i}^{-1}E_i^TE_i\\
&+\sum_{k=1}^{m}\big(B_i^k\big)^T\big(Q_i^{-1}-\varepsilon_{1i}L_BL_B^T\big)^{-1}B_i^k+\varepsilon_{1i}^{-1}\sum_{k=1}^{m}\big(N_i^k\big)^TN_i^k+\sum_{j=1}^{N}\gamma_{ij}Q_j
\end{aligned}
\tag{3.5}
$$

are all negative-definite matrices. Set

$$
\begin{aligned}
&\beta=\max_{i\in S}\lambda_{\max}\left(\rho_{2i}^{-1}E_{di}^TE_{di}+\varepsilon_{2i}^{-1}\sum_{k=1}^{m}\big(N_{di}^k\big)^TN_{di}^k+\sum_{k=1}^{m}\big(B_{di}^k\big)^T\big(Q_i^{-1}-\varepsilon_{2i}L_BL_B^T\big)^{-1}B_{di}^k\right),\\
&M_{QA_d}=\max_{i\in S}\|Q_iA_{di}\|^2,\qquad M_{QCK}=\max_{i\in S}\|Q_iC_iK_i\|^2,\\
&\lambda_M=\max_{i\in S}\lambda_{\max}(Q_i),\qquad \lambda_m=\min_{i\in S}\lambda_{\min}(Q_i),\qquad \lambda=\max_{i\in S}\lambda_{\max}(\bar{Q}_i)
\end{aligned}
$$

(and of course $\lambda<0$). If τ is sufficiently small for

$$
\begin{aligned}
&\lambda+2\sqrt{\frac{\lambda_MM_{QA_d}}{\lambda_m}}+2\sqrt{\frac{\lambda_MM_{QCK}K_\tau}{\lambda_m}}\\
&\quad+\lambda_M(M_B+M_{B_d})\sqrt{\frac{\lambda_M}{\lambda_m}}+\frac{\beta\lambda_M}{\lambda_m}<0,
\end{aligned}
\tag{3.6}
$$

then the solution of (3.3) *satisfies*

$$\limsup_{t\to\infty}\frac{1}{t}\log\bigl(\mathbb{E}\bigl|x(t;\xi)\bigr|^2\bigr)\le-\frac{\log(q)}{\gamma},$$

where $q>1$ *is the unique root to the following equation*:

$$\lambda+2\sqrt{\frac{q\lambda_M M_{QA_d}}{\lambda_m}}+2\sqrt{\frac{q\lambda_M M_{QCK}K_\tau}{\lambda_m}}+\lambda_M(M_B+M_{B_d})\sqrt{\frac{q\lambda_M}{\lambda_m}}+\frac{q\beta\lambda_M}{\lambda_m}=-\lambda_M\frac{\log(q)}{\gamma}. \tag{3.7}$$

Proof The proof is an application of Razumikhin-type Theorem 2.4 with $p=2$. Define $V\in C^{2,1}(\mathbb{R}^n\times[-\gamma,\infty)\times S;\mathbb{R}_+)$ by

$$V(x,t,i)=x^TQ_ix.$$

Obviously,

$$\lambda_m|x|^2\le V(x,t,i)\le\lambda_M|x|^2,$$

that is, (2.8) is satisfied with $p=2$. In what follows, we need to show that

$$\mathbb{E}\Bigl[\max_{i\in S}\mathcal{L}V(\varphi,t,i)\Bigr]\le-\frac{\log(q)}{\gamma}\mathbb{E}\Bigl[\max_{i\in S}V\bigl(\varphi(0),t,i\bigr)\Bigr] \tag{3.8}$$

for all $t\ge0$ and those $\varphi=\{\varphi(\theta):-2\gamma\le\theta\le0\}\in L^2_{\mathcal{F}_t}([-2\gamma,0];\mathbb{R}^n)$ satisfying

$$\mathbb{E}\Bigl[\min_{i\in S}V\bigl(\varphi(\theta),t+\theta,i\bigr)\Bigr]<q\mathbb{E}\Bigl[\max_{i\in S}V\bigl(\varphi(0),t,i\bigr)\Bigr],\quad\forall\theta\in[-2\gamma,0]. \tag{3.9}$$

For this purpose, we compute $\mathcal{L}V(\varphi,t,i)$ as follows:

$$\begin{aligned}\mathcal{L}V(\varphi,t,i)={}&2\varphi^T(0)Q_i\bigl(\bigl[A_i+\Delta A(t,i)+C_iK_i\bigr]\varphi(0)\\&+\bigl[A_{di}+\Delta A_d(t,i)\bigr]\varphi\bigl(-\delta(t)\bigr)-C_iK_i\Phi(t,\varphi)\bigr)\\&+\sum_{j=1}^N\gamma_{ij}\varphi^T(0)Q_j\varphi(0)\\&+\sum_{k=1}^m\bigl(\bigl[B_i^k+\Delta B^k(t,i)\bigr]\varphi(0)+\bigl[B_{di}^k+\Delta B_d^k(t,i)\bigr]\varphi\bigl(-\delta(t)\bigr)\bigr)^T\\&\times Q_i\bigl(\bigl[B_i^k+\Delta B^k(t,i)\bigr]\varphi(0)+\bigl[B_{di}^k+\Delta B_d^k(t,i)\bigr]\varphi\bigl(-\delta(t)\bigr)\bigr).\end{aligned} \tag{3.10}$$

For the first term, we can, by (2.3) and Lemma 2.2, derive that

$$\begin{aligned}&2\varphi^T(0)Q_i\bigl(\bigl[A_i+\Delta A(t,i)+C_iK_i\bigr]\varphi(0)\\&\quad+\bigl[A_{di}+\Delta A_d(t,i)\bigr]\varphi\bigl(-\delta(t)\bigr)-C_iK_i\Phi(t,\varphi)\bigr)\end{aligned}$$

$$
\begin{aligned}
&\le 2\varphi^T(0)Q_iA_i\varphi(0) + \rho_{1i}\varphi^T(0)\big(Q_iL_AL_A^TQ_i\big)\varphi(0) + \rho_{1i}^{-1}\varphi^T(0)\big(E_i^TE_i\big)\varphi(0) \\
&\quad + 2\varphi^T(0)Q_iC_iK_i\varphi(0) + 2\varphi^T(0)Q_iA_{di}\varphi\big(-\delta(t)\big) + \rho_{2i}\varphi^T(0)\big(Q_iL_AL_A^TQ_i\big)\varphi(0) \\
&\quad + \rho_{2i}^{-1}\varphi^T\big(-\delta(t)\big)\big(E_{di}^TE_{di}\big)\varphi\big(-\delta(t)\big) - 2\varphi^T(0)Q_iC_iK_i\Phi(t,\varphi) \\
&\le \varphi^T(0)\big(Q_iA_i + A_i^TQ_i + \rho_{1i}Q_iL_AL_A^TQ_i + \rho_{1i}^{-1}E_i^TE_i + Q_iC_iK_i + K_i^TC_i^TQ_i \\
&\quad + \rho_{2i}Q_iL_AL_A^TQ_i\big)\varphi(0) + \alpha_1\big|\varphi(0)\big|^2 + \frac{M_{QA_d}}{\alpha_1}\big|\varphi\big(-\delta(t)\big)\big|^2 \\
&\quad + \alpha_2\big|\varphi(0)\big|^2 + \frac{M_{QCK}}{\alpha_2}\big|\Phi(t,\varphi)\big|^2 + \varphi^T\big(-\delta(t)\big)\big(\rho_{2i}^{-1}E_{di}^TE_{di}\big)\varphi\big(-\delta(t)\big).
\end{aligned}
\tag{3.11}
$$

By (2.3), (2.4) and Lemma 2.3, the last term can be treated as

$$
\begin{aligned}
&\sum_{k=1}^{m}\Big(\varphi^T(0)\big[B_i^k + L_BF_B(t)N_i^k\big]^TQ_i\big[B_i^k + L_BF_B(t)N_i^k\big]\varphi(0) \\
&\quad + 2\varphi^T(0)\big[B_i^k + L_BF_B(t)N_i^k\big]^TQ_i\big[B_{di}^k + L_BF_B(t)N_{di}^k\big]\varphi\big(-\delta(t)\big) \\
&\quad + \varphi^T\big(-\delta(t)\big)\big[B_{di}^k + L_BF_B(t)N_{di}^k\big]^TQ_i\big[B_{di}^k + L_BF_B(t)N_{di}^k\big]\varphi\big(-\delta(t)\big)\Big) \\
&\le \sum_{k=1}^{m}\Big(\varphi^T(0)\big[\big(B_i^k\big)^T\big(Q_i^{-1} - \varepsilon_{1i}L_BL_B^T\big)^{-1}B_i^k + \varepsilon_{1i}^{-1}\big(N_i^k\big)^TN_i^k\big]\varphi(0)\Big) \\
&\quad + \alpha_3\lambda_MM_B\big|\varphi(0)\big|^2 + \frac{\lambda_MM_{B_d}}{\alpha_3}\big|\varphi\big(-\delta(t)\big)\big|^2 \\
&\quad + \sum_{k=1}^{m}\Big(\varphi^T\big(-\delta(t)\big)\big[\big(B_{di}^k\big)^T\big(Q_i^{-1} - \varepsilon_{2i}L_BL_B^T\big)^{-1}B_{di}^k + \varepsilon_{2i}^{-1}\big(N_{di}^k\big)^TN_{di}^k\big]\varphi\big(-\delta(t)\big)\Big).
\end{aligned}
\tag{3.12}
$$

Substituting (3.11), (3.12) together into (3.10), we have

$$
\begin{aligned}
&\mathcal{L}V(\varphi,t,i) \\
&\quad \le \varphi^T(0)\Bigg[Q_i(A_i + C_iK_i) + (A_i + C_iK_i)^TQ_i + (\rho_{1i} + \rho_{2i})Q_iL_AL_A^TQ_i \\
&\qquad + \rho_{1i}^{-1}E_i^TE_i + \sum_{k=1}^{m}\big(B_i^k\big)^T\big(Q_i^{-1} - \varepsilon_{1i}L_BL_B^T\big)^{-1}B_i^k \\
&\qquad + \varepsilon_{1i}^{-1}\sum_{k=1}^{m}\big(N_i^k\big)^TN_i^k + \sum_{j=1}^{N}\gamma_{ij}Q_j\Bigg]\varphi(0) \\
&\qquad + (\alpha_1 + \alpha_2 + \alpha_3\lambda_MM_B)\big|\varphi(0)\big|^2 \\
&\qquad + \bigg(\frac{M_{QA_d}}{\alpha_1} + \frac{\lambda_MM_{B_d}}{\alpha_3}\bigg)\big|\varphi\big(-\delta(t)\big)\big|^2 + \frac{M_{QCK}}{\alpha_2}\big|\Phi(t,\varphi)\big|^2 \\
&\qquad + \varphi^T\big(-\delta(t)\big)\Bigg[\rho_{2i}^{-1}E_{di}^TE_{di} + \sum_{k=1}^{m}\big(B_{di}^k\big)^T\big(Q_i^{-1} - \varepsilon_{2i}L_BL_B^T\big)^{-1}B_{di}^k \\
&\qquad + \varepsilon_{2i}^{-1}\sum_{k=1}^{m}\big(N_{di}^k\big)^TN_{di}^k\Bigg]\varphi\big(-\delta(t)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\le (\lambda + \alpha_1 + \alpha_2 + \alpha_3\lambda_M M_B)\big|\varphi(0)\big|^2 \\
&\quad + \left(\frac{M_{QA_d}}{\alpha_1} + \frac{\lambda_M M_{B_d}}{\alpha_3} + \beta\right)\big|\varphi\big(-\delta(t)\big)\big|^2 + \frac{M_{QCK}}{\alpha_2}\big|\Phi(t,\varphi)\big|^2.
\end{aligned} \tag{3.13}
$$

It follows from (3.9) that

$$
\mathbb{E}\big|\varphi(\theta)\big|^2 < \frac{q\lambda_M}{\lambda_m}\mathbb{E}\big|\varphi(0)\big|^2, \quad \forall\theta \in [-2\gamma, 0]. \tag{3.14}
$$

Setting $\alpha_1 = \sqrt{\frac{q\lambda_M M_{QA_d}}{\lambda_m}}$, $\alpha_2 = \sqrt{\frac{q\lambda_M M_{QCK} K_\tau}{\lambda_m}}$, $\alpha_3 = \sqrt{\frac{q\lambda_M}{\lambda_m}}$, applying Lemma 3.1 and combining (3.14) with (3.13) yield

$$
\begin{aligned}
&\mathbb{E}\Big[\max_{i\in S}\mathcal{L}V(\varphi,t,i)\Big] \\
&\quad\le (\lambda + \alpha_1 + \alpha_2 + \alpha_3\lambda_M M_B)\mathbb{E}\big|\varphi(0)\big|^2 \\
&\qquad + \left(\frac{M_{QA_d}}{\alpha_1} + \frac{\lambda_M M_{B_d}}{\alpha_3} + \beta\right)\mathbb{E}\big|\varphi\big(-\delta(t)\big)\big|^2 \\
&\qquad + \frac{M_{QCK}}{\alpha_2}K_\tau \sup_{-2\gamma\le\theta\le 0}\mathbb{E}\big|\varphi(\theta)\big|^2 \\
&\quad< (\lambda + \alpha_1 + \alpha_2 + \alpha_3\lambda_M M_B)\mathbb{E}\big|\varphi(0)\big|^2 \\
&\qquad + \frac{q\lambda_M}{\lambda_m}\left(\frac{M_{QA_d}}{\alpha_1} + \frac{\lambda_M M_{B_d}}{\alpha_3} + \beta\right)\mathbb{E}\big|\varphi(0)\big|^2 \\
&\qquad + \frac{q\lambda_M}{\lambda_m}\frac{M_{QCK}}{\alpha_2}K_\tau\mathbb{E}\big|\varphi(0)\big|^2 \\
&\quad= \left(\lambda + 2\alpha_1 + 2\alpha_2 + \alpha_3\lambda_M(M_B + M_{B_d}) + \frac{q\lambda_M\beta}{\lambda_m}\right)\mathbb{E}\big|\varphi(0)\big|^2.
\end{aligned}
$$

From (3.7), we find that $\lambda + 2\alpha_1 + 2\alpha_2 + \alpha_3\lambda_M(M_B + M_{B_d}) + \frac{q\lambda_M\beta}{\lambda_m} < 0$. Thus

$$
\begin{aligned}
&\mathbb{E}\Big[\max_{i\in S}\mathcal{L}V(\varphi,t,i)\Big] \\
&\quad\le \frac{1}{\lambda_M}\left(\lambda + 2\alpha_1 + 2\alpha_2 + \alpha_3\lambda_M(M_B + M_{B_d}) + \frac{q\lambda_M\beta}{\lambda_m}\right)\mathbb{E}\Big[\max_{i\in S}V\big(\varphi(0),t,i\big)\Big] \\
&\quad= -\frac{\log(q)}{\gamma}\mathbb{E}\Big[\max_{i\in S}V\big(\varphi(0),t,i\big)\Big],
\end{aligned}
$$

which is the required inequality (3.8). The proof is therefore complete. □

The following theorem provides the LMI method for designing the feedback controller based on discrete-time state observations such that the closed-loop system (3.3) is robustly exponentially stable in the mean square.

Theorem 3.3 *If there exist matrices $Y_i, P_i = P_i^T > 0$ and positive numbers ρ_{1i}, ρ_{2i}, ε_{1i}, ε_{2i} ($i \in S$) such that, for any $i \in S$,*

$$
P_i - \varepsilon_{1i}L_B L_B^T > 0, \qquad P_i - \varepsilon_{2i}L_B L_B^T > 0 \tag{3.15}
$$

and

$$\Pi_i = \begin{pmatrix} \Pi_{11i} & P_iE_i^T & \Pi_{31i}^T & \Pi_{41i}^T & \Pi_{51i}^T \\ E_iP_i & -\rho_{1i}I & 0 & 0 & 0 \\ \Pi_{31i} & 0 & \Pi_{33i} & 0 & 0 \\ \Pi_{41i} & 0 & 0 & \Pi_{44i} & 0 \\ \Pi_{51i} & 0 & 0 & 0 & \Pi_{55i} \end{pmatrix} < 0, \tag{3.16}$$

where

$$\begin{aligned}
&\Pi_{11i} = A_iP_i + C_iY_i + P_iA_i^T + Y_i^TC_i^T + (\rho_{1i} + \rho_{2i})L_AL_A^T + \gamma_{ii}P_i,\\
&\Pi_{31i} = (\sqrt{\gamma_{i1}}P_i, \dots, \sqrt{\gamma_{i,i-1}}P_i, \sqrt{\gamma_{i,i+1}}P_i, \dots, \sqrt{\gamma_{iN}}P_i)^T,\\
&\Pi_{33i} = \operatorname{diag}(-P_1, \dots, -P_{i-1}, -P_{i+1}, \dots, -P_N),\\
&\Pi_{41i} = \bigl(P_i\bigl(B_i^1\bigr)^T, P_i\bigl(B_i^2\bigr)^T, \dots, P_i\bigl(B_i^m\bigr)^T\bigr)^T,\\
&\Pi_{44i} = \operatorname{diag}\bigl(\varepsilon_{1i}L_BL_B^T - P_i, \varepsilon_{1i}L_BL_B^T - P_i, \dots, \varepsilon_{1i}L_BL_B^T - P_i\bigr),\\
&\Pi_{51i} = \bigl(P_i\bigl(N_i^1\bigr)^T, P_i\bigl(N_i^2\bigr)^T, \dots, P_i\bigl(N_i^m\bigr)^T\bigr)^T,\\
&\Pi_{55i} = \operatorname{diag}(-\varepsilon_{1i}I, -\varepsilon_{1i}I, \dots, -\varepsilon_{1i}I).
\end{aligned}$$

Then, by setting $Q_i = P_i^{-1}$ *and* $K_i = Y_iP_i^{-1}$, *the controlled system* (3.3) *will be exponentially stable in the mean square if* $\tau > 0$ *is small enough such that* (3.6) *holds.*

Proof We first observe that by the well-known Schur complements (*cf.* [8]), LMIs (3.16) are equivalent to the following matrix inequalities:

$$\begin{aligned}
&A_iP_i + C_iY_i + P_iA_i^T + Y_i^TC_i^T + (\rho_{1i} + \rho_{2i})L_AL_A^T\\
&\quad + \gamma_{ii}P_i + \rho_{1i}^{-1}P_iE_i^TE_iP_i + \sum_{j\neq i}^{N} \gamma_{ij}P_iP_j^{-1}P_i\\
&\quad + \sum_{k=1}^{m} P_i\bigl(B_i^k\bigr)^T\bigl(P_i - \varepsilon_{1i}L_BL_B^T\bigr)^{-1}B_i^kP_i\\
&\quad + \varepsilon_{1i}^{-1}\sum_{k=1}^{m} P_i\bigl(N_i^k\bigr)^TN_i^kP_i < 0.
\end{aligned} \tag{3.17}$$

Recalling that $K_i = Y_iP_i^{-1}$ and $P_i = P_i^T$, we have

$$\begin{aligned}
&A_iP_i + C_iK_iP_i + P_iA_i^T + P_iK_i^TC_i^T + (\rho_{1i} + \rho_{2i})L_AL_A^T\\
&\quad + \rho_{1i}^{-1}P_iE_i^TE_iP_i + \sum_{j=1}^{N} \gamma_{ij}P_iP_j^{-1}P_i\\
&\quad + \sum_{k=1}^{m} P_i\bigl(B_i^k\bigr)^T\bigl(P_i - \varepsilon_{1i}L_BL_B^T\bigr)^{-1}B_i^kP_i\\
&\quad + \varepsilon_{1i}^{-1}\sum_{k=1}^{m} P_i\bigl(N_i^k\bigr)^TN_i^kP_i < 0.
\end{aligned} \tag{3.18}$$

Multiplying P_i^{-1} from left and then from right, and noting $Q_i = P_i^{-1}$, we see that the matrix inequalities (3.18) are equivalent to the following matrix inequalities:

$$\begin{aligned} &Q_iA_i + Q_iC_iK_i + A_i^TQ_i + K_i^TC_i^TQ_i + (\rho_{1i} + \rho_{2i})Q_iL_AL_A^TQ_i + \rho_{1i}^{-1}E_i^TE_i \\ &\quad + \sum_{j=1}^N \gamma_{ij}Q_j + \sum_{k=1}^m \bigl(B_i^k\bigr)^T\bigl(Q_i^{-1} - \varepsilon_{1i}L_BL_B^T\bigr)^{-1}B_i^k + \varepsilon_{1i}^{-1}\sum_{k=1}^m \bigl(N_i^k\bigr)^TN_i^k < 0, \end{aligned} \tag{3.19}$$

which means the matrices in (3.5) are all negative-definite. So the required assertion follows directly from Theorem 3.2. □

From the above theorem we can see that, to design the robust controller, we should first find solutions for (3.15) and (3.16) and then obtain small τ from condition (3.6) after calculating all related quantities.

Remark The discrete-time state feedback control in this paper is similar to sample-data control. In addition to state feedback control, sample-data control can also be applied to output feedback control and has been extensively used in the area of automatic control for deterministic differential systems (see, e.g., [29, 30]). In 2013, Mao [24] initiated the study on the mean-square exponential stabilization of continuous-time hybrid stochastic differential equations by feedback controls based on discrete-time state observations. By a new technique, we extend the theory to hybrid uncertain stochastic systems with time-varying delay. Further work will be done to develop the theory to solve the discrete-time state feedback stabilization problem for more general systems and obtain better bounds on the duration τ between two consecutive state observations.

4 An example

Consider a two-dimensional controlled hybrid uncertain stochastic differential equation with time-varying delay

$$\begin{aligned} dx(t) = {}&\bigl\{\bigl[A\bigl(r(t)\bigr) + \Delta A\bigl(t,r(t)\bigr)\bigr]x(t) + \bigl[A_d\bigl(r(t)\bigr) + \Delta A_d\bigl(t,r(t)\bigr)\bigr]x\bigl(t-\delta(t)\bigr) \\ &+ C\bigl(r(t)\bigr)K\bigl(r(t)\bigr)x\bigl(\eta(t)\bigr)\bigr\}\,dt \\ &+ \bigl\{\bigl[B\bigl(r(t)\bigr) + \Delta B\bigl(t,r(t)\bigr)\bigr]x(t) \\ &+ \bigl[B_d\bigl(r(t)\bigr) + \Delta B_d\bigl(t,r(t)\bigr)\bigr]x\bigl(t-\delta(t)\bigr)\bigr\}\,dw(t). \end{aligned} \tag{4.1}$$

Here $w(t)$ is a scalar Brownian motion and $r(t)$ is a Markov chain on the state space $S = \{1,2\}$ with the generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$\Delta A(t,r(t))$, $\Delta A_d(t,r(t))$, $\Delta B(t,r(t))$ and $\Delta B_d(t,r(t))$ are defined as in (2.3) and (2.4), and all the coefficients are given by

$$A_1 = \begin{pmatrix} -2.1 & 1 \\ 0.1 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1.9 & -0.1 \\ 1 & 0.9 \end{pmatrix}, \qquad A_{d1} = \begin{pmatrix} -0.04 & 0.01 \\ 0.03 & -0.02 \end{pmatrix},$$

$$A_{d2} = \begin{pmatrix} 0.05 & 0.02 \\ -0.05 & -0.01 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0.12 & 0.1 \\ 0 & -0.2 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.13 \end{pmatrix},$$
$$B_{d1} = \begin{pmatrix} 0.1 & 0 \\ 0.1 & -0.2 \end{pmatrix}, \qquad B_{d2} = \begin{pmatrix} 0.13 & 0 \\ 0 & 0.12 \end{pmatrix}, \qquad L_A = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},$$
$$L_B = \begin{pmatrix} 0.12 & 0 \\ 0 & 0.12 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.13 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0.24 & 0 \\ 0 & -0.31 \end{pmatrix},$$
$$E_{d1} = \begin{pmatrix} -0.02 & 0.01 \\ 0.02 & 0.01 \end{pmatrix}, \qquad E_{d2} = \begin{pmatrix} 0.03 & 0 \\ 0 & -0.15 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix},$$
$$N_2 = \begin{pmatrix} 0.13 & 0 \\ 0 & 0.11 \end{pmatrix}, \qquad N_{d1} = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.08 \end{pmatrix}, \qquad N_{d2} = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.11 \end{pmatrix},$$
$$C_1 = \begin{pmatrix} -8 & 0.1 \\ 0.05 & -10 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} -4 & 0 \\ 0 & -5 \end{pmatrix}.$$

By solving LMIs (3.15) and (3.16), we find the feasible solution

$$P_1 = \begin{pmatrix} 9.0232 & 0.0593 \\ 0.0593 & 9.5935 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 9.5100 & -0.0642 \\ -0.0642 & 9.0621 \end{pmatrix},$$
$$Y_1 = \begin{pmatrix} 8.0006 & -8.4760 \\ 7.9723 & 10.0764 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 17.9618 & -8.6680 \\ 8.6680 & 19.5889 \end{pmatrix},$$

and

$$\rho_{11} = 16.3747, \qquad \rho_{12} = 15.9963, \qquad \rho_{21} = 16.0063, \qquad \rho_{22} = 15.9923,$$
$$\varepsilon_{11} = 15.9319, \qquad \varepsilon_{12} = 15.9317, \qquad \varepsilon_{21} = 15.9983, \qquad \varepsilon_{22} = 15.9983.$$

By Theorem 3.3, we can obtain

$$K_1 = \begin{pmatrix} 0.8925 & -0.8890 \\ 0.8767 & 1.0449 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 1.8824 & -0.9432 \\ 0.9261 & 2.1682 \end{pmatrix}.$$

A further calculation shows that

$$\lambda_M = 0.1109, \qquad \lambda_m = 0.1042, \qquad \lambda = -1.7761,$$
$$M_A = 13.26, \qquad M_{A_d} = 0.01, \qquad M_B = 0.11, \qquad M_{B_d} = 0.105,$$
$$M_{QA_d} = 3.4453 \times 10^{-5}, \qquad M_{QCK} = 2.0674, \qquad M_{CK} = 188.5213, \qquad \beta = 0.0059.$$

It is easy to show that (3.6) holds whenever $\tau < 0.0165$. So, according to Theorem 3.3, if we set K_i $(i = 1, 2)$ as above and make sure $\tau < 0.0165$, then the controlled system (4.1) is mean-square exponentially stable, independent of h - the upper bound for the time-varying delay $\delta(t)$. The numerical simulation (Figures 1 and 2) supports this result clearly. However, to obtain the upper bound for the second Lyapunov exponent, we set $\gamma = h \vee \tau = 0.1$, then

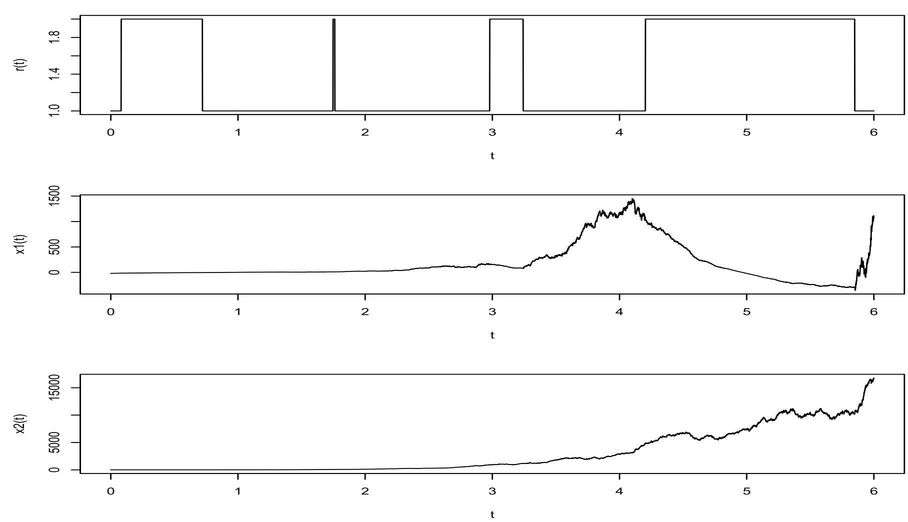

Figure 1 Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the original system of system (4.1) using the Euler-Maruyama method with step size 10^{-6}, delay upper bound $h = 0.1$ and initial values $r(0) = 1$, $x_{10} = -20$ and $x_{20} = 10$.

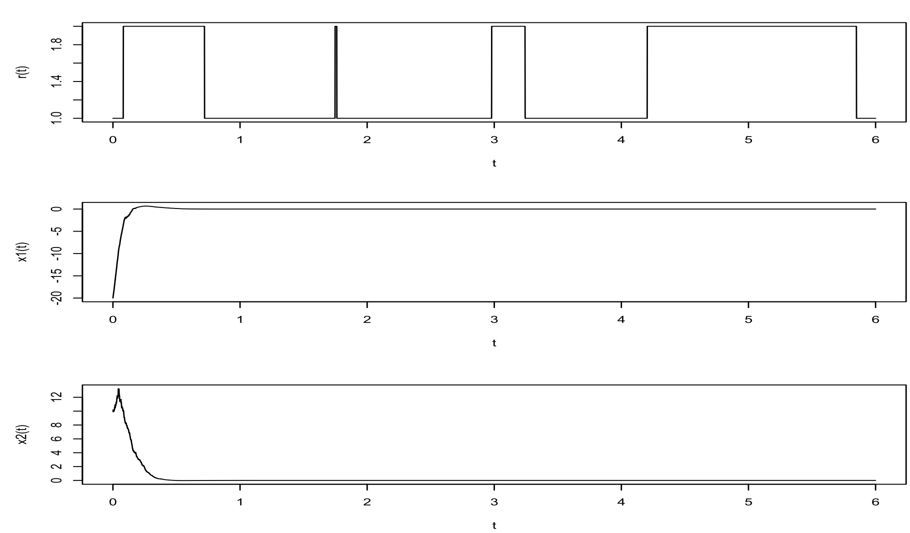

Figure 2 Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the controlled system (4.1) with $\tau = 0.016$ using the Euler-Maruyama method with step size 10^{-6}, delay upper bound $h = 0.1$ and initial values $r(0) = 1$, $x_{10} = -20$ and $x_{20} = 10$.

Eq. (3.7) becomes

$$1.1025\sqrt{q} + 0.0063q - 1.7761 = -1.109\log(q),$$

which has a unique root $q = 1.4724$ on $(1, \infty)$. Hence the solution of (4.1) has the property

$$\limsup_{t\to\infty} \frac{1}{t}\log\bigl(\mathbb{E}\bigl|x(t;\xi)\bigr|^2\bigr) \le -3.8689.$$

5 Conclusion

In this paper, we have showed that an unstable hybrid stochastic system with time-varying delay and norm-bounded uncertainties can be stabilized by the linear feedback control based on discrete-time state observations. By employing the Razumikhin method, the mean-square exponential stability criterion has been established, just requiring the time-varying delay be a bounded variable rather than a bounded differentiable function. The method for designing the robust controller has also been developed.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions to each part of this paper. Both the authors have read and approved the final manuscript.

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