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Sequential fractional differential equations with nonlocal boundary conditions on an arbitrary interval

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Abstract

We discuss the existence and uniqueness of solutions for a nonlocal three-point boundary value problem of sequential fractional differential equations on an arbitrary interval $[\xi,\zeta], \xi,\zeta \in \mathbb{R}$. Our results rely on standard fixed point theorems and are well illustrated with examples. The case for non-homogeneous three-point boundary conditions is also discussed.

MSC: 34A08; 34B10

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1 Introduction

During the last few decades, non-integer (arbitrary) order calculus has evolved into an interesting and useful area of research in view of the extensive application of its modeling tools in applied and technical sciences. Nowadays, fractional-order differential and integral operators, which are nonlocal in nature, appear in mathematical models of many real world phenomena such as anomalous diffusion, ecological effects, blood flow issues, spreading of disease, control processes, etc. Wide-spread application of fractional calculus has motivated many researchers to develop the theoretical aspects of this branch of mathematical analysis. In particular, there has been shown a great interest in the study of fractional-order boundary value problems (FBVPs). The literature on FBVPs is now much enriched and contains a variety of interesting results involving different kinds of boundary conditions. For application details and theoretical development, we refer the reader to [1–15] and the references cited therein.

In a recent article [13], the author studied a two-point fractional-order boundary value problem (BVP) on an arbitrary interval. Motivated by [13], we investigate a three-point boundary value problem of sequential fractional differential equations given by

$$\left({}^cD^{\beta+1} + \kappa\, {}^cD^{\beta}\right)x(t) = \phi\left(t, x(t)\right), \quad 1 < \beta < 2, \kappa > 0, \xi < t < \zeta, \tag{1.1}$$

$$x(\xi) = 0, \qquad x(\eta) = 0, \qquad x(\zeta) = 0, \qquad -\infty < \xi < \eta < \zeta < \infty, \tag{1.2}$$

where ${}^cD^\beta$ denotes the Caputo fractional derivative of order β, ϕ is a given continuous function and κ, ξ, ζ, η are real constants. New existence and uniqueness results for the problem (1.1)-(1.2) are obtained in Section 2. The non-homogeneous boundary condition case is briefly described in Sections 3. It is imperative to note that the results obtained in this paper are general in the sense that they correspond to any specific interval by fixing $\xi, \zeta \in \mathbb{R}$.

Definition 1.1 ([16]) The Caputo derivative of order α for a function $\varphi : [\xi, \infty) \to \mathbb{R}$ with $\varphi(t) \in C^n[0, \infty)$ is defined by

$$ {}^cD^\alpha \varphi(t) = \frac{1}{\Gamma(n-\alpha)} \int_\xi^t \frac{\varphi^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds = I^{n-\alpha}\varphi^{(n)}(t), \quad t > \xi, n-1 < \alpha < n, $$

where

$$ I^q \chi(t) = \frac{1}{\Gamma(q)} \int_\xi^t \frac{\chi(s)}{(t-s)^{1-q}}\, ds, \quad t > \xi, $$

is called Riemann-Liouville fractional integral of order $q > 0$ ($q = n - \alpha$) for a function $\chi : [0, \infty) \to \mathbb{R}$ if the integral involved is point-wise defined on $[0, \infty)$.

Property 1.2 It has been shown in [16] that

$$ I^{\alpha\, c}D^\alpha \varphi(t) = \varphi(t) - c_0 - c_1(t-\xi) - \cdots - c_{n-1}(t-\xi)^{n-1}, \quad t > \xi, n-1 < \alpha < n, \tag{1.3} $$

where c_i ($i = 1, \dots, n-1$) are arbitrary constants.

The following lemma dealing with the linear variant of the problem (1.1)-(1.2) plays a key role in the forthcoming analysis.

Lemma 1.3 *For any $y \in C[\xi, \zeta]$, the solution of the linear sequential fractional differential equation*

$$ \left({}^cD^{\beta+1} + \kappa\, {}^cD^\beta\right)x(t) = y(t), \quad 1 < \beta < 2, \xi < t < \zeta, \kappa > 0, \tag{1.4} $$

supplemented with the three-point boundary conditions (1.2) *is given by*

$$ \begin{aligned} x(t) &= \int_\xi^t e^{-\kappa(t-s)} \left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\, du \right) ds \\ &\quad + \sigma_1(t) \int_\xi^\eta e^{-\kappa(\eta-s)} \left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\, du \right) ds \\ &\quad + \sigma_2(t) \int_\xi^\zeta e^{-\kappa(\zeta-s)} \left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\, du \right) ds, \end{aligned} \tag{1.5} $$

where

$$ \sigma_1(t) = \frac{\lambda_4 \rho_1(t) - \lambda_3 \rho_2(t)}{\mu}, \qquad \sigma_2(t) = \frac{\lambda_1 \rho_2(t) - \lambda_2 \rho_1(t)}{\mu}, \tag{1.6} $$

$$\rho_1(t) = \kappa\left(1 - e^{-\kappa(t-\xi)}\right), \qquad \rho_2(t) = \kappa(t-\xi) + e^{-\kappa(t-\xi)} - 1, \tag{1.7}$$

$$\mu = \lambda_2\lambda_3 - \lambda_1\lambda_4 \neq 0, \tag{1.8}$$

$$\begin{aligned} &\lambda_1 = \kappa\left(1 - e^{-\kappa(\eta-\xi)}\right), \qquad \lambda_2 = \kappa(\eta-\xi) + e^{-\kappa(\eta-\xi)} - 1, \\ &\lambda_3 = \kappa\left(1 - e^{-\kappa(\zeta-\xi)}\right), \qquad \lambda_4 = \kappa(\zeta-\xi) + e^{-\kappa(\zeta-\xi)} - 1. \end{aligned} \tag{1.9}$$

Proof Applying the operator I^β on (1.4) and using the Property 1.2, we get

$$I^\beta\left[{}^cD^\beta(D+\kappa)x(t)\right] = I^\beta y(t),$$

which, taking into account (1.3), yields

$$(D+\kappa)x(t) = \int_\xi^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)\,ds + c_0 + c_1(t-\xi), \tag{1.10}$$

where c_0, c_1 are arbitrary constants. Rewriting (1.10) as

$$D\left(e^{\kappa t}x(t)\right) = e^{\kappa t}\left(\int_\xi^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)\,ds + c_0 + c_1(t-\xi)\right),$$

and then integrating from ξ to t, we get

$$e^{\kappa t}x(t) - e^{\kappa\xi}x(\xi) = \int_\xi^t e^{\kappa s}\left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\,du + c_0 + c_1(s-\xi)\right) ds.$$

Using $x(\xi) = 0$ and completing the integration, we get

$$\begin{aligned} x(t) &= \int_\xi^t e^{-\kappa(t-s)}\left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\,du\right) ds \\ &\quad + \frac{c_0}{\kappa}\left[1 - e^{-\kappa(t-\xi)}\right] + \frac{c_1}{\kappa^2}\left[\kappa(t-\xi) + e^{-\kappa(t-\xi)} - 1\right]. \end{aligned} \tag{1.11}$$

Making use of the conditions $x(\eta) = 0$ and $x(b) = 0$ in (1.11), we obtain

$$\lambda_1 c_0 + \lambda_2 c_1 = -\kappa^2 \int_\xi^\eta e^{-\kappa(\eta-s)}\left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\,du\right) ds, \tag{1.12}$$

$$\lambda_3 c_0 + \lambda_4 c_1 = -\kappa^2 \int_\xi^\zeta e^{-\kappa(\zeta-s)}\left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\,du\right) ds. \tag{1.13}$$

Solving the system (1.12)-(1.13), we find that

$$\begin{aligned} c_0 = \frac{\kappa^2}{\mu}\Bigg[&\lambda_4 \int_\xi^\eta e^{-\kappa(\eta-s)}\left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\,du\right) ds \\ &- \lambda_2 \int_\xi^\zeta e^{-\kappa(\zeta-s)}\left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\,du\right) ds\Bigg] \end{aligned}$$

and

$$c_1 = \frac{\kappa^2}{\mu}\left[\lambda_1 \int_\xi^\zeta e^{-\kappa(\zeta-s)}\left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\,du\right) ds - \lambda_3 \int_\xi^\eta e^{-\kappa(\eta-s)}\left(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} y(u)\,du\right) ds\right],$$

where we have used (1.9). Inserting the values of c_0, c_1 in (1.11) and using the notations (1.6)-(1.9) completes the solution (1.5). By direct computation, one can establish the converse of this lemma. The proof is completed. □

2 Main results

By Lemma 1.3, the problem (1.1)-(1.2) can be transformed into a fixed point problem as

$$x = \mathcal{G}x, \tag{2.1}$$

where the operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ is defined by

$$(\mathcal{G}x)(t) = \int_\xi^t e^{-\kappa(t-s)}\left(I^\beta \phi\big(s, x(s)\big)\right) ds + \sigma_1(t) \int_\xi^\eta e^{-\kappa(\eta-s)}\left(I^\beta \phi\big(s, x(s)\big)\right) ds + \sigma_2(t) \int_\xi^\zeta e^{-\kappa(\zeta-s)}\left(I^\beta \phi\big(s, x(s)\big)\right) ds,$$

$$I^\beta \phi\big(t, x(t)\big) = \frac{1}{\Gamma(\beta)} \int_\xi^t \frac{\phi(s, x(s))}{(t-s)^{1-\beta}}\, ds,$$

σ_1, σ_2 are defined by (1.6). Here $\mathcal{X} = C([\xi, \zeta], \mathbb{R})$ denotes the Banach space of all continuous functions from $[\xi, \zeta] \to \mathbb{R}$ equipped with the norm $\|x\| = \sup\{|x(t)| : t \in [\xi, \zeta]\}$.

For the sake of computational convenience, we set the notation

$$\widehat{\sigma_1} = \max_{t\in[\xi,\zeta]} \left|\sigma_1(t)\right|, \qquad \widehat{\sigma_2} = \max_{t\in[\xi,\zeta]} \left|\sigma_2(t)\right|, \tag{2.2}$$

$$\delta = \frac{1}{\kappa\Gamma(\beta+1)}\left\{(1+\widehat{\sigma_2})(\zeta-\xi)^\beta\left(1-e^{-\kappa(\zeta-\xi)}\right) + \widehat{\sigma_1}(\eta-\xi)^\beta\left(1-e^{-\kappa(\eta-\xi)}\right)\right\}, \tag{2.3}$$

$$\delta_1 = \delta - \frac{(\zeta-\xi)^\beta(1-e^{-\kappa(\zeta-\xi)})}{\kappa\Gamma(\beta+1)}. \tag{2.4}$$

In the forthcoming work, we need the following assumptions:

(H_1) $|\phi(t,x) - \phi(t,y)| \le \ell|x-y|$, for all $t \in [\xi, \zeta]$, $x, y \in \mathbb{R}$, $\ell > 0$;
(H_2) $|\phi(t,x)| \le \vartheta(t)$, for all $(t,x) \in [\xi, \zeta] \times \mathbb{R}$ and $\vartheta \in \mathcal{C}([\xi, \zeta], \mathbb{R}^+)$.

Now we present our first existence result for the problem (1.1)-(1.2), which relies on Krasnoselskii's fixed point theorem [17].

Theorem 2.1 *Let $\phi : [\xi, \zeta] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the conditions (H_1) and (H_2). Then there exists at least one solution for the problem* (1.1)-(1.2) *on $[\xi, \zeta]$ if*

$$\ell\delta_1 < 1, \tag{2.5}$$

where δ_1 is defined by (2.4).

Proof Setting $\sup_{t\in[\xi,\zeta]}|\vartheta(t)| = \|\vartheta\|$, we fix

$$\begin{aligned}\varrho \geq \frac{\|\vartheta\|}{\kappa\Gamma(\beta+1)}\big\{&(\zeta-\xi)^\beta\big(1-e^{-\kappa(\zeta-\xi)}\big)\\ &+(\zeta-\xi)^\beta\big(1-e^{-\kappa(\zeta-\xi)}\big)\widehat{\sigma}_2+(\eta-\xi)^\beta\big(1-e^{-\kappa(\eta-\xi)}\big)\widehat{\sigma}_1\big\},\end{aligned} \tag{2.6}$$

and we consider $B_\varrho = \{x \in \mathcal{X} : \|x\| \leq \varrho\}$. Introduce the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on B_ϱ as follows:

$$(\mathcal{G}_1x)(t) = \int_\xi^t e^{-\kappa(t-s)}\big(I^\beta\phi\big(s,x(s)\big)\big)\,ds, \tag{2.7}$$

$$\begin{aligned}(\mathcal{G}_2x)(t) = {}&\sigma_1(t)\int_\xi^\eta e^{-\kappa(\eta-s)}\big(I^\beta\phi\big(s,x(s)\big)\big)\,ds\\ &+\sigma_2(t)\int_a^\zeta e^{-\kappa(\zeta-s)}\big(I^\beta\phi\big(s,x(s)\big)\big)\,ds.\end{aligned}$$

Observe that $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$. Now we verify the hypotheses of Krasnoselskii's fixed point theorem in the following steps.

(i) For $x, y \in B_\varrho$, we have

$$\begin{aligned}&\|\mathcal{G}_1x+\mathcal{G}_2y\|\\ &\quad= \sup_{t\in[\xi,\zeta]}\big|(\mathcal{G}_1x)(t)+(\mathcal{G}_2y)(t)\big|\\ &\quad\leq \sup_{t\in[\xi,\zeta]}\bigg\{\int_\xi^t e^{-\kappa(t-s)}\big(I^\beta\big|\phi\big(s,x(s)\big)\big|\big)\,ds+\big|\sigma_1(t)\big|\int_\xi^\eta e^{-\kappa(\eta-s)}\big(I^\beta\big|\phi\big(s,x(s)\big)\big|\big)\,ds\\ &\qquad+\big|\sigma_2(t)\big|\int_\xi^\zeta e^{-\kappa(\zeta-s)}\big(I^\beta\big|\phi\big(s,x(s)\big)\big|\big)\,ds\bigg\}\\ &\quad\leq \|\vartheta\|\sup_{t\in[\xi,\zeta]}\bigg\{\frac{(t-\xi)^\beta}{\Gamma(\beta+1)}\int_\xi^t e^{-\kappa(t-s)}\,ds\\ &\qquad+\big|\sigma_1(t)\big|\frac{(\eta-\xi)^q}{\Gamma(\beta+1)}\int_\xi^\eta e^{-\kappa(\eta-s)}\,ds+\big|\sigma_2(t)\big|\frac{(\zeta-\xi)^\beta}{\Gamma(\beta+1)}\int_\xi^\zeta e^{-\kappa(\zeta-s)}\,ds\bigg\}\\ &\quad\leq \frac{\|\vartheta\|}{\kappa\Gamma(\beta+1)}\big\{(\zeta-\xi)^\beta\big(1-e^{-\kappa(\zeta-\xi)}\big)\\ &\qquad+(\zeta-\xi)^\beta\big(1-e^{-\kappa(\zeta-\xi)}\big)\widehat{\sigma}_2+(\eta-\xi)^\beta\big(1-e^{-\kappa(\eta-\xi)}\big)\widehat{\sigma}_1\big\}\leq\varrho,\end{aligned}$$

where we have used (2.6). Thus $\mathcal{G}_1x + \mathcal{G}_2y \in B_\varrho$.

(ii) Using the assumption (H_1) together with (2.5), it is easy to show that $\mathcal{G}_2$ is a contraction.

(iii) Using the continuity of ϕ, it is easy to show that the operator $\mathcal{G}_1$ is continuous. Further, $\mathcal{G}_1$ is uniformly bounded on B_ϱ as

$$\|\mathcal{G}_1x\| = \sup_{t\in[\xi,\zeta]}\big|(\mathcal{G}_1x)(t)\big| \leq \frac{\|\vartheta\|(\zeta-\xi)^\beta}{\Gamma(\beta+1)}\big(1-e^{-\kappa(\zeta-\xi)}\big).$$

In order to establish that $\mathcal{G}_1$ is compact, we define $\sup_{(t,x)\in[\xi,\zeta]\times B_\varrho}|\phi(t,x)| = \overline{\phi}$. Thus, for $\xi < t_1 < t_2 < \zeta$, we have

$$\begin{aligned}
&\bigl|(\mathcal{G}_1x)(t_2) - (\mathcal{G}_1x)(t_1)\bigr| \\
&= \biggl|\int_\xi^{t_1}\bigl[e^{-\kappa(t_2-s)} - e^{-\kappa(t_1-s)}\bigr]\biggl(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\phi\bigl(u,x(u)\bigr)\,du\biggr)\,ds \\
&\quad + \int_{t_1}^{t_2} e^{-\kappa(t_1-s)}\biggl(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\phi\bigl(u,x(u)\bigr)\,du\biggr)\,ds\biggr| \\
&\leq \frac{\overline{\phi}}{\kappa\Gamma(\beta+1)}\bigl\{2\bigl(1-e^{-\kappa(t_2-t_1)}\bigr) + \bigl|e^{-\kappa(t_2-\xi)} - e^{-\kappa(t_1-\xi)}\bigr|\bigr\} \to 0 \quad \text{as } t_1 \to t_2,
\end{aligned}$$

independent of x. This shows that $\mathcal{G}_1$ is relatively compact on B_ϱ. As all the conditions of the Arzelá-Ascoli theorem are satisfied, so $\mathcal{G}_1$ is compact on B_ϱ. In view of steps (i)-(iii), the conclusion of Krasnoselskii's fixed point theorem applies and hence there exists at least one solution for the problem (1.1)-(1.2) on $[\xi,\zeta]$. □

Remark 2.2 Interchanging the roles of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ in the foregoing result, we can obtain a second result by requiring the condition:

$$\frac{\ell}{\kappa\Gamma(\beta+1)}(\zeta-\xi)^\beta\bigl(1-e^{-\kappa(\zeta-\xi)}\bigr) < 1,$$

instead of (2.5).

In the next result, we establish the uniqueness of solutions for the problem (1.1)-(1.2).

Theorem 2.3 *Let $\phi : [\xi,\zeta]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that the assumption (H_1) holds with $\ell^{-1} > \delta$, where δ is given by* (2.3). *Then the problem* (1.1)-(1.2) *has a unique solution on* $[\xi,\zeta]$.

Proof Let us define $\sup_{t\in[\xi,\zeta]}|\phi(t,0)| = \phi_m$ and select $r \geq \frac{\delta\phi_m}{1-\ell\delta}$ to establish that $\mathcal{G}\mathcal{E}_r \subset \mathcal{E}_r$, where $\mathcal{E}_r = \{x\in\mathcal{X} : \|x\| \leq r\}$ and $\mathcal{G}$ is defined by (2.1). Using the condition (H_1), we have

$$\begin{aligned}
\bigl|\phi(t,x)\bigr| &= \bigl|\phi(t,x) - \phi(t,0) + \phi(t,0)\bigr| \leq \bigl|\phi(t,x) - \phi(t,0)\bigr| + \bigl|\phi(x,0)\bigr| \\
&\leq \ell\|x\| + \phi_m \leq \ell r + \phi_m.
\end{aligned} \tag{2.8}$$

Then, for $x\in\mathcal{E}_r$, we obtain

$$\begin{aligned}
\bigl\|\mathcal{G}(x)\bigr\| &= \sup_{t\in[\xi,\zeta]}\bigl|\mathcal{G}(x)(t)\bigr| \\
&\leq \sup_{t\in[\xi,\zeta]}\biggl\{\int_\xi^t e^{-\kappa(t-s)}\biggl(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\bigl|\phi\bigl(u,x(u)\bigr)\bigr|\,du\biggr)\,ds \\
&\quad + \bigl|\sigma_1(t)\bigr|\int_\xi^\eta e^{-\kappa(\eta-s)}\biggl(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(q)}\bigl|\phi\bigl(u,x(u)\bigr)\bigr|\,du\biggr)\,ds \\
&\quad + \bigl|\sigma_2(t)\bigr|\int_\xi^\zeta e^{-\kappa(\zeta-s)}\biggl(\int_\xi^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\bigl|\phi\bigl(u,x(u)\bigr)\bigr|\,du\biggr)\,ds\biggr\}
\end{aligned}$$

$$\begin{aligned}
&\leq (\ell r+\phi_m)\sup_{t\in[\xi,\zeta]}\left\{\int_\xi^t e^{-\kappa(t-s)}\frac{(s-u)^\beta}{\Gamma(\beta+1)}\,ds\right.\\
&\quad\left.+\left|\sigma_1(t)\right|\int_\xi^\eta e^{-\kappa(\eta-s)}\frac{(s-u)^\beta}{\Gamma(\beta+1)}\,ds+\left|\sigma_2(t)\right|\int_\xi^\zeta e^{-\kappa(\zeta-s)}\frac{(s-u)^\beta}{\Gamma(\beta+1)}\,ds\right\}\\
&\leq \frac{(\ell r+\phi_m)}{\kappa\Gamma(\beta+1)}\big\{(\zeta-\xi)^\beta\big(1-e^{-\kappa(\zeta-\xi)}\big)\\
&\quad+(\zeta-\xi)^\beta\big(1-e^{-\kappa(\zeta-\xi)}\big)\widehat{\sigma}_2+(\eta-\xi)^\beta\big(1-e^{-\kappa(\eta-\xi)}\big)\widehat{\sigma}_1\big\}\\
&=(\ell r+\phi_m)\delta\leq r.
\end{aligned}$$

This shows that $\mathcal{G}x\in\mathcal{E}_r$, $x\in\mathcal{E}_r$. Thus $\mathcal{G}\mathcal{E}_r\subset\mathcal{E}_r$. Now we show that $\mathcal{G}$ is a contraction. For this purpose, let $x,y\in\mathcal{X}$. Then, for each $t\in[\xi,\zeta]$, we have

$$\begin{aligned}
\big\|(\mathcal{G}x)-(\mathcal{G}y)\big\| &\leq \sup_{t\in[\xi,\zeta]}\left\{\int_\xi^t e^{-\kappa(t-s)}\left(\int_\xi^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\big|\phi\big(u,x(u)\big)-\phi\big(u,y(u)\big)\big|\,du\right)ds\right.\\
&\quad+\left|\sigma_1(t)\right|\int_\xi^\eta e^{-\kappa(\eta-s)}\left(\int_\xi^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\big|\phi\big(u,x(u)\big)-\phi\big(u,y(u)\big)\big|\,du\right)ds\\
&\quad\left.+\left|\sigma_2(t)\right|\int_\xi^\zeta e^{-\kappa(\zeta-s)}\left(\int_\xi^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\big|\phi\big(u,x(u)\big)-\phi\big(u,y(u)\big)\big|\,du\right)ds\right\}\\
&\leq \ell\sup_{t\in[\xi,\zeta]}\left\{\int_\xi^t e^{-\kappa(t-s)}\frac{(s-u)^\beta}{\Gamma(\beta+1)}\,ds+\left|\sigma_1(t)\right|\int_\xi^\eta e^{-\kappa(\eta-s)}\frac{(s-u)^\beta}{\Gamma(\beta+1)}\,ds\right.\\
&\quad\left.+\left|\sigma_2(t)\right|\int_\xi^\zeta e^{-\kappa(\zeta-s)}\frac{(s-u)^\beta}{\Gamma(\beta+1)}\,ds\right\}\|x-y\|\\
&\leq \ell\frac{1}{\kappa\Gamma(\beta+1)}\big\{(\zeta-\xi)^\beta\big(1-e^{-\kappa(\zeta-\xi)}\big)\\
&\quad+(\zeta-\xi)^\beta\big(1-e^{-\kappa(\zeta-\xi)}\big)\widehat{\sigma}_2+(\eta-\xi)^\beta\big(1-e^{-\kappa(\eta-\xi)}\big)\widehat{\sigma}_1\big\}\|x-y\|\\
&=\ell\delta\|x-y\|,
\end{aligned}$$

which, in view of the given conditions $\ell^{-1}>\delta$, implies that $\mathcal{G}$ is a contraction. In consequence, it follows by the contraction mapping principle that there exists a unique solution for the problem (1.1)-(1.2) on $[\xi,\zeta]$. □

Example 2.4 Consider the sequential fractional differential equation

$$\big({}^cD^{\frac{11}{4}}+{}^cD^{\frac{7}{4}}\big)x(t)=\left(\frac{|x|}{1+|x|}+\tan^{-1}x(t)\right)\frac{A}{2\sqrt{t^2+15}}+\sin t,\quad 1<t<2, \tag{2.9}$$

supplemented with the boundary conditions:

$$x(1)=0,\qquad x(3/2)=0,\qquad x(2)=0. \tag{2.10}$$

Here, $\beta=7/4$, $\kappa=1$, $\eta=3/2$, $\xi=1$, $\zeta=2$, A is a positive constant to be determined later and

$$\phi(t,x)=\left(\frac{|x|}{1+|x|}+\tan^{-1}x\right)\frac{A}{2\sqrt{t^2+15}}+\sin t.$$

Clearly $|\phi(t,x) - \phi(t,y)| \leq (A/4)|x - y|$ with $\ell = A/4$. Using the given values, it is found that $\delta_1 = 0.09823$ and $\delta = 0.78602$ (δ and δ_1 are, respectively, given by (2.3) and (2.4)). It is easy to check that $|f(t,x)| \leq 1 + A(2+\pi)/(4\sqrt{t^2+15}) = \vartheta(t)$ and $\ell\delta_1 < 1$ when $A < 43.32943$. As all the conditions of Theorem 2.1 hold true, therefore the conclusion of Theorem 2.1 applies to the problem (2.9)-(2.10) on $[1,2]$. On the other hand, $\ell\delta < 1$ whenever $A < 5.0889$. Thus there exists a unique solution for problem (2.9)-(2.10) on $[1,2]$ by Theorem 2.3.

Remark 2.5 We can formulate several existence results for the problem (1.1)-(1.2) by assuming different conditions on the nonlinear function involved in the problem by applying different tools of the fixed point theory. For instance, there exists at least one solution for the problem (1.1)-(1.2) on $[\xi,\zeta]$ if $|\phi(t,x)| \leq \widehat{p}(t)\widehat{\psi}(\|x\|)$ for each $(t,x) \in [\xi,\zeta] \times \mathbb{R}$ and that $M[\widehat{\psi}(M)\|\widehat{p}\|\delta]^{-1} > 1$, where $\widehat{\psi} : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function, $\widehat{p} \in C([\xi,\zeta],\mathbb{R}^+)$, M is a positive constant and δ is defined by (2.3). This result can be established by applying nonlinear alternative for single valued maps [18]. We can also obtain an existence result by applying Leray-Schauder's degree theory by assuming $\phi : [\xi,\zeta] \times \mathbb{R} \to \mathbb{R}$ to be a continuous function such that $|\phi(t,x)| \leq \varpi|x| + M_1$ for all $(t,x) \in [\xi,\zeta] \times \mathbb{R}, 0 \leq \varpi < \delta^{-1}, M_1 > 0$.

3 Non-homogeneous boundary conditions

In this section we consider a three-point boundary value problem of sequential fractional differential equations with non-homogeneous boundary conditions given by

$$\left({}^cD^{\beta+1} + \kappa\,{}^cD^{\beta}\right)x(t) = \phi\big(t,x(t)\big), \quad 1 < \beta < 2, \kappa > 0, \xi < t < \zeta, \tag{3.1}$$

$$x(\xi) = \omega_1, \qquad x(\eta) = \omega_2, \qquad x(\zeta) = \omega_3, \qquad -\infty < \xi < \eta < \zeta < \infty, \tag{3.2}$$

where $\omega_i, i = 1,2,3$ are real constants.

As before we formulate the following lemma for the linear variant of the problem (3.1)-(3.2).

Lemma 3.1 *For any $y \in C[\xi,\zeta]$, the solution of the linear sequential fractional differential equation $({}^cD^{\beta+1} + \kappa\,{}^cD^{\beta})x(t) = y(t), 1 < \beta < 2, \xi < t < \zeta, \kappa > 0$ supplemented with* (3.2) *is given by*

$$\begin{aligned} x(t) = {} & \int_\xi^t e^{-\kappa(t-s)}\big(I^\beta y(s)\big)\,ds + \omega_1 e^{-\kappa(t-\xi)} \\ & + \sigma_1(t)\left\{\int_\xi^\eta e^{-\kappa(\eta-s)}\big(I^\beta y(s)\big)\,ds + \big(\omega_1 e^{-\kappa(\eta-\xi)} - \omega_2\big)\right\} \\ & + \sigma_2(t)\left\{\int_\xi^\zeta e^{-\kappa(\zeta-s)}\big(I^\beta y(s)\big)\,ds + \big(\omega_1 e^{-\kappa(\zeta-\xi)} - \omega_3\big)\right\}, \end{aligned}$$

where σ_1, σ_2 are defined by (1.6).

In view of Lemma 3.1, we introduce an operator $\widehat{\mathcal{G}}: \mathcal{X} \to \mathcal{X}$ associated with the problem (3.1)-(3.2) as follows:

$$\begin{aligned}
(\widehat{\mathcal{G}}x)(t) = {} & \int_{\xi}^{t} e^{-\kappa(t-s)} \big(I^{\beta}\phi\big(s,x(s)\big)\big)\, ds + \omega_1 e^{-\kappa(t-\xi)} \\
& + \sigma_1(t)\left\{\int_{\xi}^{\eta} e^{-\kappa(\eta-s)} \big(I^{\beta}\phi\big(s,x(s)\big)\big)\, ds + \big(\omega_1 e^{-\kappa(\eta-\xi)} - \omega_2\big)\right\} \\
& + \sigma_2(t)\left\{\int_{\xi}^{\zeta} e^{-\kappa(\zeta-s)} \big(I^{\beta}\phi\big(s,x(s)\big)\big)\, ds + \big(\omega_1 e^{-\kappa(\zeta-\xi)} - \omega_3\big)\right\}.
\end{aligned}$$

Also we have

$$\begin{aligned}
\widehat{\delta} = {} & \frac{1}{\kappa\Gamma(\beta+1)}\big\{(1+\widehat{\sigma}_2)\big((\zeta-\xi)^{\beta}\big(1-e^{-\kappa(\zeta-\xi)}\big) + \omega_1 e^{-\kappa(\zeta-\xi)}\big) \\
& + \widehat{\sigma}_1\big((\eta-\xi)^{\beta}\big(1-e^{-\kappa(\eta-\xi)}\big) + \omega_1 e^{-\kappa(\eta-\xi)}\big) - \widehat{\sigma}_1\omega_2 - \widehat{\sigma}_2\omega_3\big\},
\end{aligned}$$

where $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ define by (2.2).

As in Section 2, we can obtain the existence results for the problem (3.1)-(3.2) with the aid of the operator $\widehat{\mathcal{G}}: \mathcal{X} \to \mathcal{X}$ and the constant $\widehat{\delta}$ defined above.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed to each part of this work equally and read and approved the final version of the manuscript.

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