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Iterative technique for coupled integral boundary value problem of non-integer order differential equations

Yongjin Li[1,2*], Kamal Shah[1,2] and Rahmat Ali Khan[1,2]

*Correspondence: stslyj@mail.sysu.edu.cn
[1]Department of Mathematics, Sun Yat-Sen University, Guangzhou, P.R. China
[2]Department of Mathematics, University of Malakand, Dir (Lower), KPK, Pakistan

Abstract

This article is concerned to the investigation of extremal solutions for a system of fractional order differential equations with coupled integral boundary value problem. In initial stage, we establish a comparison result and then using the iterative technique of monotone type together with the procedure of extremal solutions, we develop sufficient conditions to obtain the solutions for the considered fractional differential system. Moreover, the investigated results are also justified by providing suitable examples.

MSC: 34A08; 35A11; 34G10; 34G20

Keywords: coupled system; coupled integral boundary conditions; fractional derivative; monotone iterative technique; extremal system of solutions

1 Introduction

The aims and objectives of this manuscript is to establish conditions to obtain the solutions for the system of arbitrary order differential equations (FDEs) with coupled integral boundary conditions given by

$$\begin{cases} -{}^{c}\mathcal{D}^{\alpha} w(t) = \Theta(t, w(t), z(t)); & t \in (0,1); 1 < \alpha \leq 2, \\ -{}^{c}\mathcal{D}^{\beta} z(t) = \Phi(t, w(t), z(t)); & t \in (0,1); 1 < \beta \leq 2, \\ w(0) = z(0) = 0, \quad w(1) = \int_0^1 z(t)\phi(t)\,dt, & z(1) = \int_0^1 w(t)\varphi(t)\,dt. \end{cases} \tag{1}$$

The functions $\phi, \varphi \in L^1[0,1]$ are non-negative and nondecreasing on $[0,1]$, while $\Theta, \Phi : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are nonlinear continuous functions. ${}^{c}\mathcal{D}$ stands for the Caputo fractional order derivative.

Differential equations of arbitrary order and their systems are the valuable tools for describing many physical, biological, psychological phenomena more accurately as compared to classical differential equations. Moreover, many problems related to engineering and applied science can also be described accurately by using fractional differential equations. Besides, applications of fractional differential equations (FDEs) are also found in the field of computer networking, electro chemistry, viscoelasticity, control theory, aerodynamics, electrodynamics of complex medium, polymer rheology and image and signal processing phenomenon *etc.* (see [1–4]). Therefore, considerable attention was paid to study the field devoted to differential equations of fractional order. In last few decades, boundary value problems of differential equations of fractional order were greatly studied by many researchers for the existence of solutions (see [5–12] and the references therein). This is due to the fact that boundary value problems have significant applications in applied sciences. The aforesaid area is well explored and plenty of research work is available on it. Another important class of differential equations is in the field of system of differential equations with coupled boundary conditions. It has been found that differential system with coupled boundary conditions mostly occur in investigation concerned with mathematical physics, mathematical biology, biochemical system, biomedical engineering and so on (see [13, 14]). Therefore, this field very recently attracted the attention of researchers towards itself.

The monotone iterative technique coupled with the method of upper and lower solutions is a powerful scheme applied to approximate solutions to differential equations of arbitrary order as well as classical order and their systems. The aforesaid techniques were used in some articles to develop conditions for existence of iterative solutions for ordinary and fractional order differential equations (FDES) (see [15–21]). There is no need of special restrictions for the utility and importance of the technique. In the mentioned technique, upper and lower solutions are used as initial iterations and monotonic sequences are developed from the corresponding linear differential equations/system which converge monotonically to their corresponding extremal solutions. By using the aforesaid technique to establish the necessary and sufficient conditions for the existence of iterative solutions to a system of coupled boundary conditions, one needs proper differential inequalities as comparison results. Monotone iterative techniques to develop conditions for extremal solutions to coupled boundary value problems are very rarely studied and very few articles are devoted to this. For instance, Asif and Khan [22] studied the coupled system with four point coupled boundary conditions for the positive solutions given by

$$\begin{cases} -w''(t) = \Theta(t, w(t), z(t)), & -z''(t) = \Phi(t, w(t), z(t)); \quad t \in (0,1), \\ w(0) = 0, \quad w(1) = \alpha z(\xi), & z(0) = 0, \quad z(1) = \beta w(\eta), \end{cases} \tag{2}$$

where $\xi, \eta \in (0,1)$, $0 < \alpha\beta\xi\eta < 1$, $\Theta, \Phi : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and the system become singular at $t = 0$ and $t = 1$. The above system (2), was extended to fractional order under the same coupled boundary conditions by Cui and Zou [23], as given by

$$\begin{cases} -{}^{c}\mathcal{D}^{\alpha} w(t) = \Theta(t, w(t), z(t)), & -{}^{c}\mathcal{D}^{\beta} z(t) = \Phi(t, w(t), z(t)); \quad t \in (0,1), \\ w(0) = 0, \quad w(1) = \alpha z(\xi), & z(0) = 0, \quad z(1) = \beta w(\eta), \end{cases} \tag{3}$$

where $\xi, \eta \in (0,1)$, $0 < \alpha\beta\xi\eta < 1$, $\Theta, \Phi : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and the system become singular at $t = 0$ and $t = 1$. Recently, Shah *et al.* [24], studied the following coupled system with coupled m-point boundary condition for upper and lower solutions:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha} w(t) + \Theta(t, w(t), z(t)) = 0, & {}^{c}\mathcal{D}^{\beta} z(t) + \Phi(t, w(t), z(t)) = 0; \quad t \in (0,1), \\ w(0) = z(0) = 0, \quad w(1) = \sum_{i=1}^{m-2} \delta_i z(\eta_i), & z(1) = \sum_{i=1}^{m-2} \lambda_i w(\xi_i), \end{cases} \tag{4}$$

where $1 < \alpha, \beta \leq 2$, η_i, ξ_i $(i = 1, 2, \ldots, m-2) \in (0, 1)$, $\sum_{i=1}^{m-2} \delta_i \eta_i < 1$, $\sum_{i=1}^{m-2} \gamma_i \xi_i < 1$ and $\Theta, \Phi : (0, 1) \times \mathbb{R}^2 \to \mathbb{R}^2$ are given continuous functions and ${}^c\mathcal{D}$ is stand for the Caputo's fractional order derivative of order α, β, respectively. Similarly Cui and Zou [25], studied the following coupled system with coupled integral boundary conditions:

$$\begin{cases} w''(t) + \Theta(t, w(t), z(t)) = 0, \qquad z''(t) + \Phi(t, w(t), z(t)) = 0; \quad t \in (0, 1), \\ w(0) = 0, \qquad w(1) = \int_0^1 z(t)\, dA(t), \qquad z(0) = 0, \qquad z(1) = \int_0^1 w(t)\, dB(t), \end{cases} \tag{5}$$

where A, B are right continuous on $[0, 1)$ and left continuous at $t = 1$ and nondecreasing on $[0, 1]$, $A(0) = B(0) = 0$, $\int_0^1 \mu(s)\, dA(s)$, $\int_0^1 \mu(s)\, dB(s)$ denote Riemann-Stieljes integral of μ with respect to A, B, respectively. Cui and Sun [26], studied a boundary value problem with coupled integral boundary conditions and developed some useful results.

Motivated by the above work, we establish fractional differential inequalities as a comparison result to study the coupled system (1). By using the monotone iterative technique coupled with the method of upper and lower solutions, we develop conditions for extremal solutions for the system (1). We also derive the corresponding convergent monotone sequences for the lower and upper solutions. Finally, we also developed conditions for uniqueness of the positive solution for the considered coupled system with coupled integral boundary conditions. Further, we provide examples to justify the main results.

2 Preliminaries

Here, we recall some fundamental notions and results of the fractional calculus and functional analysis which are found in [27–29].

Definition 2.1 Let $\alpha > 0$ and $w : [a, +\infty) \to \mathbb{R}$. Then the Riemann-Liouville arbitrary order integral of $h(t)$ is given by

$$\mathcal{I}_{a+}^{\alpha} w(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} w(s)\, ds,$$

where $\alpha \in \mathbb{R}_+$ and 'Γ' is a Gamma function provided that the integral at the right side is pointwise defined on $(0, \infty)$.

Definition 2.2 The fractional order derivative in the Caputo sense of a function w on the interval $[a, b]$ is given by

$${}^c\mathcal{D}_{a+}^{\alpha} w(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n - \alpha - 1} w^{(n)}(s)\, ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ represents the integer part of α, provided that the integral on the right side is point wise defined on $(0, \infty)$.

Lemma 2.1 *The unique solution of fractional differential equation* ${}^c\mathcal{D}^{\alpha} w(t) = 0$, *for* $w \in C(0, 1) \cap L(0, 1)$ *is provided by*

$$\mathcal{I}^{\alpha}\left[{}^c\mathcal{D}^{\alpha} w(t)\right] = w(t) + \sum_{k=0}^{n-1} C_k t^k,$$

for arbitrary $C_k \in \mathbb{R}$, $k = 0, 1, 2, \ldots, n - 1$.

Let

$$\lambda_1 = \int_0^1 t\phi(t)\,dt, \qquad \lambda_2 = \int_0^1 t\varphi(t)\,dt, \qquad \lambda = 1 - \lambda_1\lambda_2, \qquad \mathcal{E} = C^2[0,1].$$

We need the following assumption throughout in this paper:

(A_1) $\lambda > 0$ and $0 < \lambda_1, \lambda_2 < 1$.

Definition 2.3 $(\underline{w}, \underline{z}) \in \mathcal{E} \times \mathcal{E}$ is called the lower system of solutions of the fractional differential system (1), if

$$\begin{cases} {}^c\mathcal{D}^\alpha \underline{w}(t) + \Theta(t, \underline{w}(t), \underline{z}(t)) \geq 0; \quad t \in (0,1), 1 < \alpha \leq 2, \\ {}^c\mathcal{D}^\beta \underline{z}(t) + \Phi(t, \underline{w}(t), \underline{z}(t)) \geq 0; \quad t \in (0,1), 1 < \beta \leq 2, \\ \underline{w}(0) \leq 0, \qquad \underline{z}(0) \leq 0, \qquad \underline{w}(1) \leq \int_0^1 \underline{z}(t)\phi(t)\,dt, \qquad \underline{z}(1) \leq \int_0^1 \underline{w}(t)\varphi(t)\,dt. \end{cases}$$

Similarly $(\overline{w}, \overline{z}) \in \mathcal{E} \times \mathcal{E}$ is called an upper system of solutions for the fractional differential system (1), if

$$\begin{cases} {}^c\mathcal{D}^\alpha \overline{w}(t) + \Theta(t, \overline{w}(t), \overline{z}(t)) \leq 0, \qquad {}^c\mathcal{D}^\beta \overline{z}(t) + \Phi(t, \overline{w}(t), \overline{z}(t)) \leq 0; \quad t \in (0,1), \\ \overline{w}(0) \geq 0, \qquad \overline{z}(0) \geq 0, \qquad \overline{w}(1) \geq \int_0^1 \overline{z}(t)\phi(t)\,dt, \qquad \overline{z}(1) \geq \int_0^1 \overline{w}(t)\varphi(t)\,dt. \end{cases}$$

Assume that

$$\underline{w}(t) \leq \overline{w}(t), \qquad \underline{z}(t) \leq \overline{z}(t), \quad t \in [0,1]. \tag{6}$$

We define the ordered sector as

$$\mathbb{S} = [\underline{w}, \overline{w}] \times [\underline{z}, \overline{z}] = \big\{(w,z) \in \mathcal{E} \times \mathcal{E} : (\underline{w}, \underline{z}) \leq (w,z) \leq (\overline{w}, \overline{z})\big\}. \tag{7}$$

We recall the following lemma.

Lemma 2.2 ([30]) *Let $w \in \mathcal{E}$, $1 < \alpha \leq 2$, attain its minimum at $t_0 \in (0,1)$, then*

$${}^c\mathcal{D}^\alpha w(t_0) \geq \frac{1}{\Gamma(2-\alpha)}\big[(\alpha-1)t_0^{-\alpha}\big(w(0) - w(t_0)\big) - t_0^{1-\alpha}w'(0)\big], \quad \textit{for all } 1 < \alpha \leq 2. \tag{8}$$

Then ${}^c\mathcal{D}^\alpha w(t_0) \geq 0$, for all $1 < \alpha \leq 2$.

Lemma 2.3 ([30]) *Assume that $w \in \mathcal{E}$, $1 < \alpha \leq 2$, attains its minimum at $t_0 \in (0,1)$ and if $w'(0) \leq 0$. Then ${}^c\mathcal{D}^\alpha w(t_0) \geq 0$, for all $1 < \alpha \leq 2$.*

We extend this inequity for our coupled system (1) in the following theorem.

Theorem 2.4 *Let $1 - \lambda_1 > 0$, $1 - \lambda_2 > 0$ hold and assume that $w, z \in \mathcal{E}$, $\Theta(t,w,z), \Phi(t,w,z) \in C([0,1] \times \mathbb{R}^2)$ such that $\Theta(t,w,z) < 0$, $\Phi(t,w,z) < 0$ for all $t \in (0,1)$. If $w(t)$, $z(t)$ satisfy the following inequalities*:

$$\begin{cases} {}^c\mathcal{D}^\alpha w(t) + \Theta(t,w,z) \leq 0, \qquad {}^c\mathcal{D}^\beta z(t) + \Phi(t,w,z) \leq 0; \quad t \in (0,1), 1 < \alpha, \beta \leq 2, \\ w(0) \geq 0, \qquad z(0) \geq 0, \qquad w(1) \geq \int_0^1 z(t)\phi(t)\,dt, \qquad z(1) \geq \int_0^1 w(t)\varphi(t)\,dt, \end{cases} \tag{9}$$

then $w(t) \geq 0$, $z(t) \geq 0$, for all $t \in [0,1]$.

Proof Assume that the conclusion is not true, then $w(t)$ and $z(t)$ have absolute minima at some t_0 with $w(t_0) < 0$ and $z(t_0) < 0$. If $t_0 \in (0,1)$, then $w'(t_0) = 0$, $z'(t_0) = 0$. Therefore, we prove that

$$^{c}\mathcal{D}^{\alpha} w(t_0) \geq 0, \qquad ^{c}\mathcal{D}^{\beta} w(t_0) \geq 0.$$

In view of Lemma 2.2, we have

$$\begin{aligned} ^{c}\mathcal{D}^{\alpha} w(t_0) &\geq \frac{1}{\Gamma(2-\alpha)}\big[(\alpha-1)t_0^{-\alpha}\big(w(0)-w(t_0)\big) - t_0^{1-\alpha}w'(0)\big], \quad \text{for all } 1 < \alpha \leq 2, \\ ^{c}\mathcal{D}^{\beta} z(t_0) &\geq \frac{1}{\Gamma(2-\beta)}\big[(\beta-1)t_0^{-\beta}\big(z(0)-z(t_0)\big) - t_0^{1-\beta}z'(0)\big], \quad \text{for all } 1 < \beta \leq 2. \end{aligned} \tag{10}$$

Since $w(t_0) \leq w(0)$, $z(t_0) \leq z(0)$, $t_0 > 0$, and $w'(0) \leq 0$, $z'(0) \leq 0$, by Lemma 2.3, from the first inequality of (10) and boundary condition $w(0) \geq 0$, we have

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)}\big[(\alpha-1)t_0^{-\alpha}\big(w(0)-w(t_0)\big) - t_0^{1-\alpha}w'(0)\big] \\ &\quad \geq \frac{t_0^{-\alpha}}{\Gamma(2-\alpha)}\big[(\alpha-1)\big(w(0)-w(0)\big) - t_0 w'(0)\big] \\ &\quad \geq \frac{t_0^{-\alpha}}{\Gamma(2-\alpha)}\big[-t_0 w'(0)\big] \geq 0, \quad \text{as } w'(0) \leq 0. \end{aligned}$$

Thus $^{c}\mathcal{D}^{\alpha} w(t_0) \geq 0$ and in a similar way we can prove that $^{c}\mathcal{D}^{\beta} z(t_0) \geq 0$.

If $w'(0) > 0$ and $z'(0) > 0$, then by similar way as in [15], we can obtain the same results by using Lemma 2.2. Hence in both cases, we concluded that $w(t) \geq 0$ and $z(t) \geq 0$ for all $t \in [0,1]$. □

We need the following assumptions:

(A_2) The nonlinear function $\Theta(t,w,z)$ is strictly decreasing in w;
(A_3) the nonlinear function $\Phi(t,w,z)$ is strictly decreasing in z.

Lemma 2.4 *Under the assumptions* (A_1)-(A_3), *let* $(\underline{w},\underline{z})$ *and* $(\overline{w},\overline{z})$ *be ordered lower and upper solutions such that* $\Theta(t,w,z)$ *is strictly decreasing with respect to* w *and* $\Phi(t,w,z)$ *is strictly deceasing with respect to* z. *Then*

$$(\underline{w},\underline{z}) \leq (\overline{w},\overline{z}), \quad \textit{for } t \in [0,1].$$

Proof In view of the definition of the lower and upper solutions, we have

$$\begin{cases} ^{c}\mathcal{D}^{\alpha}\underline{w}(t) + \Theta(t,\underline{w}(t),\underline{z}(t)) \geq 0, & ^{c}\mathcal{D}^{\beta}\underline{z}(t) + \Phi(t,\underline{w},\underline{z}) \geq 0; \quad t \in (0,1), \\ \underline{w}(0) \leq 0, \quad \underline{z}(0) \leq 0, & \underline{w}(1) \leq \int_0^1 \underline{z}(t)\phi(t)\,dt, \quad \underline{z}(1) \leq \int_0^1 \underline{w}(t)\varphi(t)\,dt, \end{cases} \tag{11}$$

and

$$\begin{cases} ^{c}\mathcal{D}^{\alpha}\overline{w}(t) + \Theta(t,\overline{w}(t),\overline{z}(t)) \leq 0, & ^{c}\mathcal{D}^{\beta}\overline{z}(t) + \Phi(t,\overline{w},\overline{z}) \leq 0; \quad t \in (0,1), \\ \overline{w}(0) \geq 0, \quad \overline{z}(0) \geq 0, & \overline{w}(1) \geq \int_0^1 \overline{z}(t)\phi(t)\,dt, \quad \overline{z}(1) \geq \int_0^1 \overline{w}(t)\varphi(t)\,dt. \end{cases} \tag{12}$$

From (11) and (12), we have

$$
\begin{aligned}
&{}^c\mathcal{D}^\alpha(\overline{w}-\underline{w}) + \Theta\big(t,\overline{w}(t),\overline{z}(t)\big) - \Theta\big(t,\underline{w}(t),\underline{z}(t)\big) \le 0,\\
&{}^c\mathcal{D}^\beta(\overline{z}-\underline{z}) + \Phi\big(t,\overline{w}(t),\overline{z}(t)\big) - \Phi\big(t,\underline{w}(t),\underline{z}(t)\big) \le 0.
\end{aligned}
$$

Using the mean value theorem and taking $u = \overline{w}-\underline{w}$, $v = \overline{z}-\underline{z}$, we have

$$
\begin{aligned}
&{}^c\mathcal{D}^\alpha u + \frac{\partial\Theta}{\partial w}(\xi) \le 0, \quad \text{where } \xi = \delta\overline{w} + (1-\delta)\underline{w}, \xi \in [0,1],\\
&{}^c\mathcal{D}^\beta v + \frac{\partial\Phi}{\partial z}(\eta) \le 0, \quad \text{where } \eta = \delta\overline{w} + (1-\delta)\underline{w}, \eta \in [0,1],\\
&u(0) \ge 0, \qquad v(0) \ge 0, \qquad u(1) \ge \int_0^1 v(t)\phi(t)\,dt, \qquad v(1) \ge \int_0^1 u(t)\varphi(t)\,dt.
\end{aligned}
$$

Since Θ, Φ are strictly decreasing with respect to w, z, respectively,

$$
\frac{\partial\Theta}{\partial w}(\xi) < 0, \qquad \frac{\partial\Phi}{\partial z}(\eta) < 0.
$$

Hence in view of Theorem 2.4, we have $u \ge 0$, $v \ge 0$. Therefore $\overline{w} \ge \underline{w}$ and $\overline{z} \ge \underline{z}$ imply that $(\underline{w},\underline{z}) \le (\overline{w},\overline{z})$. □

Let $y(t), x(t) \in C[0,1]$, then we shall consider the linear fractional differential system with coupled integral boundary conditions given by

$$
\begin{cases}
-{}^c\mathcal{D}^\alpha w(t) = y(t), \qquad -{}^c\mathcal{D}^\beta z(t) = x(t); \quad t \in [0,1],\\
w(0) = z(0) = 0, \qquad w(1) = \int_0^1 z(t)\phi(t)\,dt, \qquad z(1) = \int_0^1 w(t)\varphi(t)\,dt.
\end{cases}
\tag{13}
$$

3 Main results

This part of the manuscript is devoted to the main results. We obtained an equivalent system of Hammerstein integral equations to our system of coupled integral boundary conditions.

Theorem 3.1 *Under the assumption* (A_1), *if* $(w,z) \in \mathcal{E} \times \mathcal{E}$ *is a system of solutions of the coupled system* (13) *if and only if* $(w,z) \in \mathcal{E} \times \mathcal{E}$ *is a system of solutions of the following coupled system of Hammerstein integral equations namely*:

$$
\begin{cases}
w(t) = \int_0^1 \mathbb{G}_{11}(t,s)y(s)\,ds + \int_0^1 \mathbb{G}_{12}(t,s)x(s)\,ds,\\
z(t) = \int_0^1 \mathbb{G}_{21}(t,s)x(s)\,ds + \int_0^1 \mathbb{G}_{22}(t,s)y(s)\,ds,
\end{cases}
\tag{14}
$$

where

$$
\begin{aligned}
&\mathbb{G}_{11}(t,s) = \frac{t\lambda_1}{\lambda}\int_0^1 \phi(x)\mathcal{K}_1(s,v)\,dv + \mathcal{K}_1(t,s), \qquad \mathbb{G}_{12} = \frac{t}{\lambda}\int_0^1 \phi(v)\mathcal{K}_2(s,v)\,dv,\\
&\mathbb{G}_{21}(t,s) = \frac{t\lambda_2}{\lambda}\int_0^1 \varphi(v)\mathcal{K}_2(s,v)\,dv + \mathcal{K}_2(t,s), \qquad \mathbb{G}_{22} = \frac{t}{\lambda}\int_0^1 \varphi(v)\mathcal{K}_1(s,v)\,dv,
\end{aligned}
$$

and $\mathcal{K}_i(t,s)$, $i = 1,2$, *are Green's functions given in* (15) *and* (16), *respectively,*

$$\mathcal{K}_1(t,s) = \begin{cases} \frac{t(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \tag{15}$$

$$\mathcal{K}_2(t,s) = \begin{cases} \frac{t(1-s)^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le t \le 1, \\ \frac{t(1-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le t \le s \le 1. \end{cases} \tag{16}$$

Proof Applying $\mathcal{I}^\alpha$, $\mathcal{I}^\beta$ on both sides of the coupled system of the FDE (1) corresponding to the boundary conditions $w(0) = z(0) = w(1) = z(1) = 0$, the coupled system (13) is equivalent to the following system of integral equations:

$$w(t) = tw(1) + \int_0^1 \mathcal{K}_1(t,s)y(s)\,ds, \quad t \in [0,1], \tag{17}$$

$$z(t) = tz(1) + \int_0^1 \mathcal{K}_2(t,s)x(s)\,ds, \quad t \in [0,1]. \tag{18}$$

Upon multiplication of (17) by $\varphi(t)$ and (18) by $\phi(t)$ and integrating with respect to t on $[0,1]$ we have

$$\begin{aligned} \int_0^1 w(t)\varphi(t)\,dt &= w(1)\int_0^1 t\varphi(t)\,dt + \int_0^1 \varphi(t)\int_0^1 \mathcal{K}_1(t,s)y(s)\,ds\,dt, \\ \int_0^1 z(t)\phi(t)\,dt &= z(1)\int_0^1 t\phi(t)\,dt + \int_0^1 \phi(t)\int_0^1 \mathcal{K}_2(t,s)x(s)\,ds\,dt. \end{aligned}$$

This implies that

$$\begin{aligned} z(1) - w(1)\lambda_2 &= \int_0^1 \phi(t)\int_0^1 \mathcal{K}_1(t,s)y(s)\,ds\,dt, \\ w(1) - z(1)\lambda_1 &= \int_0^1 \varphi(t)\int_0^1 \mathcal{K}_2(t,s)x(s)\,ds\,dt. \end{aligned} \tag{19}$$

By simple calculation from (19), we have

$$\begin{pmatrix} w(1) \\ z(1) \end{pmatrix} = \frac{1}{\lambda}\begin{pmatrix} \lambda_2 & 1 \\ 1 & \lambda_1 \end{pmatrix}\begin{pmatrix} \int_0^1 \phi(t)\int_0^1 \mathcal{K}_1(t,s)y(s)\,ds\,dt \\ \int_0^1 \varphi(t)\int_0^1 \mathcal{K}_2(t,s)x(s)\,ds\,dt \end{pmatrix}. \tag{20}$$

This produces

$$w(1) = \frac{\lambda_2}{\lambda}\int_0^1 \phi(t)\int_0^1 \mathcal{K}_1(t,s)y(s)\,ds\,dt + \frac{1}{\lambda}\int_0^1 \varphi(t)\int_0^1 \mathcal{K}_2(t,s)x(s)\,ds\,dt \tag{21}$$

and

$$z(1) = \frac{\lambda_1}{\lambda}\int_0^1 \varphi(t)\int_0^1 \mathcal{K}_2(t,s)x(s)\,ds\,dt + \frac{1}{\lambda}\int_0^1 \phi(t)\int_0^1 \mathcal{K}_1(t,s)y(s)\,ds\,dt. \tag{22}$$

Using (21) in (17) and (22) in (18), we get

$$
\begin{aligned}
w(t) &= t\left[\frac{\lambda_2}{\lambda}\int_0^1 \phi(t)\int_0^1 \mathcal{K}_1(t,s)y(s)\,ds\,dt + \frac{1}{\lambda}\int_0^1 \varphi(t)\int_0^1 \mathcal{K}_2(t,s)x(s)\,ds\,dt\right] \\
&\quad + \int_0^1 \mathcal{K}_1(t,s)y(s)\,ds, \quad t\in[0,1], \\
z(t) &= t\left[\frac{\lambda_1}{\lambda}\int_0^1 \varphi(t)\int_0^1 \mathcal{K}_2(t,s)x(s)\,ds\,dt + \frac{1}{\lambda}\int_0^1 \phi(t)\int_0^1 \mathcal{K}_1(t,s)y(s)\,ds\,dt\right] \\
&\quad + \int_0^1 \mathcal{K}_2(t,s)x(s)\,ds, \quad t\in[0,1],
\end{aligned}
$$

which is equivalent to the system (14).

For the converse, let $(w,z)\in\mathcal{E}\times\mathcal{E}$ is a system of solutions of integral equations (14), then upon fractional differentiation of corresponding order of (14) yield

$$
-{}^c\mathcal{D}^\alpha w(t) = y(t), \qquad -{}^c\mathcal{D}^\beta z(t) = x(t).
$$

Further, using the fact $\mathcal{K}_i(0,s) = \mathcal{K}_i(1,s) = 0$ $(i=1,2)$, for $s\in[0,1]$. Hence we get $w(0) = z(0) = 0$. Moreover, on simple computations, one can easily verify that

$$
w(1) = \int_0^1 z(t)\phi(t)\,dt, \qquad z(1) = \int_0^1 w(t)\varphi(t)\,dt.
$$

This proves that $(w,z)\in\mathcal{E}\times\mathcal{E}$ is a system of solutions of our considered coupled system (1). □

In view of Theorem 2.4 and by means of monotone iterative technique, we derive our main result concerning the existence of a system of solutions of the considered system (1). For ordered lower and upper solutions $(\underline{w},\underline{z})$ and $(\overline{w},\overline{z})$, respectively, we have defined the set $\mathbb{S}$ in (7). Further under the assumptions (A_2) and (A_3), let $\frac{\partial\Theta}{\partial w}(t,\xi,z)$, $\frac{\partial\Phi}{\partial z}(t,w,\eta)$ be bounded below, that is, there exist constants c, d such that

$$
-c \le \frac{\partial\Theta}{\partial w}(t,\xi,z) < 0, \qquad -d \le \frac{\partial\Phi}{\partial z}(t,w,\eta) < 0, \quad \text{for all } \xi,\eta\in[0,1]. \tag{23}
$$

In the following theorem, we construct monotone sequences which describe lower and upper solutions of BVP (1).

Theorem 3.2 *Assume the hypotheses* (A_1)*-*(A_3) *together with the initial approximation* $(\underline{w}^{(0)},\underline{z}^{(0)})$ *and* $(\overline{w}_0,\overline{z}_0)$ *of the ordered lower and upper system of solutions for the coupled system* (1)*, respectively, in* $\mathbb{S}$*. Let* $\{(\underline{w}^{(n)},\underline{z}^{(n)})\}$ *and* $\{(\overline{w}_n,\overline{z}_n)\}$*,* $n\ge 1$*, be the solutions of*

$$
\begin{cases}
-{}^c\mathcal{D}^\alpha \underline{w}^{(n)}(t) + c\underline{w}^{(n)} = c\underline{w}^{(n-1)} + \Theta(t,\underline{w}^{(n-1)}(t),\underline{z}^{(n-1)}(t)); \quad t\in(0,1), 1<\alpha\le 2, \\
-{}^c\mathcal{D}^\beta \underline{z}^{(n)}(t) + d\underline{z}^{(n)} = d\underline{z}^{(n-1)} + \Phi(t,\underline{w}^{(n-1)}(t),\underline{z}^{(n-1)}(t)); \quad t\in(0,1), 1<\beta\le 2, \\
\underline{w}^{(n)}(0) = \underline{w}_0^{(n)} \ge \underline{w}^{(n-1)}(0), \qquad \underline{w}^{(n)}(1) = \underline{w}_1^{(n)}(1) \ge \underline{w}^{(n-1)}(1), \\
\underline{z}^{(n)}(0) = \underline{z}_0^{(n)} \ge \underline{z}^{(n-1)}(0), \qquad \underline{z}^{(n)}(1) = \underline{z}_1^{(n)}(1) \ge \underline{z}^{(n-1)}(1),
\end{cases} \tag{24}
$$

and

$$\begin{cases} -{}^c\mathcal{D}^{\alpha}\overline{w}_n(t)+c\overline{w}_n=c\overline{w}_{n-1}+\Theta(t,\overline{w}_{n-1}(t),\overline{z}_{n-1}(t)); & t\in(0,1),\\ -{}^c\mathcal{D}^{\beta}\overline{z}_n(t)+d\overline{z}_n=d\overline{z}_{n-1}+\Phi(t,\overline{w}_{n-1}(t),\overline{z}_{n-1}(t)); & t\in(0,1),\\ \overline{w}_n(0)=\overline{w}_n^{(0)}(0)\geq\overline{w}_{n-1}(0), & \overline{w}_n(1)=\overline{w}_n^{(1)}(1)\geq\overline{w}_{n-1}(1),\\ \overline{z}_n(0)=\overline{z}_n^{(0)}(0)\geq\overline{z}_{n-1}(0), & \overline{z}_n(1)=\overline{z}_n^{(1)}(1)\geq\overline{z}_{n-1}(1). \end{cases} \tag{25}$$

Then we have

(i) *The sequence* $(\underline{w}^{(n)},\underline{z}^{(n)})$, $n\geq 1$, *is an increasing sequence of lower solutions of BVP* (1);

(ii) *the sequence* $(\overline{w}_n,\overline{z}_n)$, $n\geq 1$ *is a decreasing sequence of upper solutions of BVP* (1).

Further

(iii) $(\underline{w}^{(n)},\underline{w}^{(n)})\leq(\overline{w}_n,\overline{z}_n)$, *for all* $n\geq 1$.

Proof To prove (i), we need to show that

(a) $\underline{w}^{(n)}-\underline{w}^{(n-1)}\geq 0$, and $\underline{z}^{(n)}-\underline{z}^{(n-1)}\geq 0$, for each $n\geq 1$;

(b) $(\underline{w}^{(n)},\underline{z}^{(n)})$ is a lower solution for each $n\geq 1$.

Thanks to induction, taking $n=1$, from (24), we have

$$\begin{cases} -{}^c\mathcal{D}^{\alpha}\underline{w}^{(1)}+c\underline{w}^{(1)}=c\underline{w}^{(0)}+\Theta(t,\underline{w}^{(0)},\underline{z}^{(0)}); & t\in(0,1),\\ -{}^c\mathcal{D}^{\beta}\underline{z}^{(1)}+d\underline{z}^{(1)}=d\underline{z}^{(0)}+\Phi(t,\underline{w}^{(0)},\underline{z}^{(0)}); & t\in(0,1),\\ \underline{w}^{(1)}(0)=\underline{w}_0^{(1)}\geq\underline{w}^{(0)}(0), & \underline{w}^{(1)}(1)=\underline{w}_1^{(1)}(1)\geq\underline{w}^{(0)}(1),\\ \underline{z}^{(1)}(0)=\underline{z}_0^{(1)}\geq\underline{z}^{(0)}(0), & \underline{z}^{(1)}(1)=\underline{z}_1^{(1)}(1)\geq\underline{z}^{(0)}(1). \end{cases} \tag{26}$$

Since $(\underline{w}^{(0)},\underline{z}^{(0)})$ is a lower solution,

$$\begin{cases} {}^c\mathcal{D}^{\alpha}\underline{w}^{(0)}+\Theta(t,\underline{w}^{(0)},\underline{z}^{(0)})\geq 0,\\ {}^c\mathcal{D}^{\beta}\underline{z}^{(0)}+\Phi(t,\underline{w}^{(0)},\underline{z}^{(0)})\geq 0. \end{cases} \tag{27}$$

Adding the corresponding equations of the system (26) and (27), we get

$$\begin{cases} {}^c\mathcal{D}^{\alpha}(\underline{w}^{(1)}-\underline{w}^{(0)})-c(\underline{w}^{(1)}-\underline{w}^{(0)})\leq 0,\\ {}^c\mathcal{D}^{\beta}(\underline{z}^{(1)}-\underline{z}^{(0)})-d(\underline{z}^{(1)}-\underline{z}^{(0)})\leq 0. \end{cases} \tag{28}$$

Using $u=\underline{w}^{(1)}-\underline{w}^{(0)}$, $v=\underline{z}^{(1)}-\underline{z}^{(0)}$. Then (u,v) satisfies

$$\begin{cases} {}^c\mathcal{D}^{\alpha}u-cu\leq 0, & {}^c\mathcal{D}^{\beta}v-dv\leq 0,\\ u(0)\geq 0, \quad v(0)\geq 0, & u(1)\geq 0, \quad v(1)\geq 0. \end{cases} \tag{29}$$

Since $c<0$, $d<0$ by using Theorem 2.4, we have $u\geq 0$, $v\geq 0$. Therefore we have $(\underline{w}^{(0)},\underline{z}^{(0)})\leq(\underline{w}^{(1)},\underline{z}^{(1)})$. Hence the result is true for $n=1$.

Let the result be true for $m\leq n$ and we will derive the result for $m=n+1$. From the system (24), we have

$$\begin{aligned} &-{}^c\mathcal{D}^{\alpha}\big(\underline{w}^{(n+1)}-\underline{w}^{(n)}\big)+c\big(\underline{w}^{(n+1)}-\underline{w}^{(n)}\big)\\ &\quad=c\big(\underline{w}^{(n)}-\underline{w}^{(n-1)}\big)+\Theta\big(t,\underline{w}^{(n)},\underline{z}^{(n)}\big)-\Theta\big(t,\underline{w}^{(n-1)},\underline{z}^{(n-1)}\big),\\ &-{}^c\mathcal{D}^{\beta}\big(\underline{z}^{(n+1)}-\underline{z}^{(n)}\big)+d\big(\underline{z}^{(n+1)}-\underline{z}^{(n)}\big)\\ &\quad=d\big(\underline{z}^{(n)}-\underline{z}^{(n-1)}\big)+\Phi\big(t,\underline{w}^{(n)},\underline{z}^{(n)}\big)-\Phi\big(t,\underline{w}^{(n-1)},\underline{z}^{(n-1)}\big). \end{aligned} \tag{30}$$

Use $u = (\underline{w}^{(n+1)} - \underline{w}^{(n)})$, $v = (\underline{z}^{(n+1)} - \underline{z}^{(n)})$ and apply the mean value theorem together with $(\underline{w}^{(n-1)}, \underline{w}^{(n-1)}) \leq (\underline{w}^{(n)}, \underline{w}^{(n)})$. We have

$$ {}^{c}\mathcal{D}^{\alpha} u - cu \leq 0, \qquad {}^{c}\mathcal{D}^{\beta} v - cv \leq 0. \tag{31}$$

Hence in view of Theorem 2.4, $u \geq 0$, $v \geq 0$, which yield $(\underline{w}^{(n)}, \underline{z}^{(n)}) \leq (\underline{w}^{(n+1)}, \underline{z}^{(n+1)})$. Thus the result is proved for $m = n + 1$ and, therefore, we have

$$\bigl(\underline{w}^{(n-1)}, \underline{z}^{(n-1)}\bigr) \leq \bigl(\underline{w}^{(n)}, \underline{z}^{(n)}\bigr), \quad \text{for each } n \geq 1.$$

This proves (a).

To derive (b), upon subtracting $\Theta(t, \underline{w}^{(n)}, \underline{z}^{(n)})$ from the first equation and $\Phi(t, \underline{w}^{(n)}, \underline{z}^{(n)})$ from the second equation of the system (24) and rearranging the terms and applying the mean value theorem, we arrive at

$$\begin{aligned} &{}^{c}\mathcal{D}^{\alpha} \underline{w}^{(n)}(t) + \Theta\bigl(t, \underline{w}^{(n)}(t), \underline{z}^{(n)}(t)\bigr) \geq 0; \quad t \in (0,1), \\ &{}^{c}\mathcal{D}^{\beta} \underline{z}^{(n)}(t) + \Phi\bigl(t, \underline{w}^{(n)}(t), \underline{z}^{(n)}(t)\bigr) \geq 0; \quad t \in (0,1). \end{aligned} \tag{32}$$

Therefore $(\underline{w}^{(n)}, \underline{z}^{(n)})$ for each $n \geq 1$ is a lower solution of BVP (1) which proves (b).

The proof of (ii) is similar to the proof of (i).

By (i) and (ii) $(\underline{w}^{(n)}, \underline{z}^{(n)})$ and $(\overline{w}_n, \overline{z}_n)$ are lower and upper solutions of BVP (1). Therefore in view of Theorem 2.4, (iii) immediately follows. □

Theorem 3.3 *Under the assumptions* (A_2), (A_3) *and condition* (23), *let* $(\underline{w}^{(n)}, \underline{z}^{(n)})$ *and* $(\overline{w}_{(n)}, \overline{z}_{(n)})$ *be lower and upper solutions of BVP* (1) *as defined in Theorem* 3.2. *Then the sequences* $(\underline{w}^{(n)}, \underline{z}^{(n)})$ *and* $(\overline{w}_{(n)}, \overline{z}_{(n)})$, $n \geq 0$, *converge uniformly to* (w_*, z_*) *and* (w^*, z^*), *respectively, with* $(w_*, z_*) \leq (w^*, z^*)$.

Proof The sequence $u_n = (\underline{w}^{(n)}, \underline{z}^{(n)})$ is monotonically increasing and bounded above by $(\overline{w}_0, \overline{z}_0)$. The bounded monotonic increasing sequence shows convergence to its least upper bound, say (w_*, z_*). Along the same lines the sequence $v_n = (\overline{w}_n, \overline{z}_n)$ is monotonically decreasing and bounded below by $(\underline{w}^{(0)}, \underline{z}^{(0)})$, thus it is convergent to its greatest lower bound say (w^*, z^*). The sequences u_n and v_n are continuous functions defined on the compact square $[0,1] \times [0,1]$. Thus the convergence is uniform. Further, in view of Theorem 3.2, $u_n \leq v_n$ for each $n \geq 1$, so

$$u_* = \lim_{n\to\infty} u_n \leq \lim_{n\to\infty} v_n = v^*.$$ □

Theorem 3.4 *Under the assumptions* (A_2), (A_3), *BVP* (1) *has at most one solution.*

Proof Let (w_1, z_1) and (w_2, z_2) be two solutions of the coupled system (1), then we have

$$\begin{aligned} &{}^{c}\mathcal{D}^{\alpha} w_1(t) + \Theta\bigl(t, w_1(t), z_1(t)\bigr) = 0; \quad t \in (0,1), 1 < \alpha \leq 2, \\ &{}^{c}\mathcal{D}^{\beta} z_1(t) + \Phi\bigl(t, w_1(t), z_1(t)\bigr) = 0; \quad t \in (0,1), 1 < \beta \leq 2, \\ &w_1(0) = z_1(0) = 0, \qquad w_1(1) = \int_0^1 z_1(t)\phi(t)\,dt, \qquad z_1(1) = \int_0^1 w_1(t)\varphi(t)\,dt, \end{aligned} \tag{33}$$

and

$$
\begin{aligned}
&{}^{c}\mathcal{D}^{\alpha} w_2(t) + \Theta\bigl(t, w_2(t), z_2(t)\bigr) = 0; \quad t \in (0,1), 1 < \alpha \le 2, \\
&{}^{c}\mathcal{D}^{\beta} z_2(t) + \Phi\bigl(t, w_2(t), z_2(t)\bigr) = 0; \quad t \in (0,1), 1 < \beta \le 2, \\
&w_2(0) = z_2(0) = 0, \qquad w_2(1) = \int_0^1 z_2(t)\phi(t)\,dt, \qquad z_2(1) = \int_0^1 w_2(t)\varphi(t)\,dt.
\end{aligned} \tag{34}
$$

Upon subtracting the first equation of (34) from the first equation of (33) and similarly the second equation of (34) from the second equation of (33), we have

$$
\begin{aligned}
&{}^{c}\mathcal{D}^{\alpha}(w_2 - w_1) + \Theta(t, w_2, z_2) - \Theta(t, w_1, z_1) = 0; \quad t \in (0,1), \\
&{}^{c}\mathcal{D}^{\beta}(z_2 - z_1) + \Phi(t, w_2, z_2) - \Phi(t, w_1, z_1) = 0; \quad t \in (0,1).
\end{aligned} \tag{35}
$$

Using $u = w_2 - w_1$, $v = z_2 - z_1$ and applying the mean value theorem, we get

$$
\begin{aligned}
&{}^{c}\mathcal{D}^{\alpha} u + u\frac{\partial \Theta}{\partial w}(\xi) = 0, \quad \text{where } \xi \in [0,1], \\
&{}^{c}\mathcal{D}^{\beta} v + v\frac{\partial \Phi}{\partial z}(\eta) = 0, \quad \text{where } \eta \in [0,1],
\end{aligned} \tag{36}
$$

with $u(0) = v(0) = 0$ and $u(1) = \int_0^1 v(t)\phi(t)\,dt$, $v(1) = \int_0^1 u(t)\varphi(t)\,dt$. By Theorem 2.4, we have $u \ge 0$, $v \ge 0$. Also the system (36) is satisfied by using $-u, -v$, therefore again by Theorem 2.4, we have $-u \ge 0$, $-v \ge 0$. Thus $u = 0$, $v = 0$, which implies that $w_1 = w_2$, $z_1 = z_2$. Hence $(w_1, z_1) = (w_2, z_2)$. Thus the coupled system (1) has at most one solution. □

4 Examples

Example 1 Consider the following coupled system of coupled integral boundary values problem:

$$
\begin{cases}
{}^{c}\mathcal{D}^{\frac{3}{2}} w(t) - w^3(t) + z(t) + 1 = 0; \quad t \in (0,1), \\
{}^{c}\mathcal{D}^{\frac{3}{2}} z(t) + w(t) - z^3(t) + 1 = 0; \quad t \in (0,1), \\
w(0) = z(0) = 0, \qquad w(1) = \int_0^1 tz(t)\,dt, \qquad z(1) = \int_0^1 tw(t)\,dt.
\end{cases} \tag{37}
$$

From the above system (37), we see

$$
\Theta\bigl(t, w(t), z(t)\bigr) = -w^3(t) + z(t) + 1, \qquad \Phi\bigl(t, w(t), z(t)\bigr) = w(t) - z^3(t) + 1. \tag{38}
$$

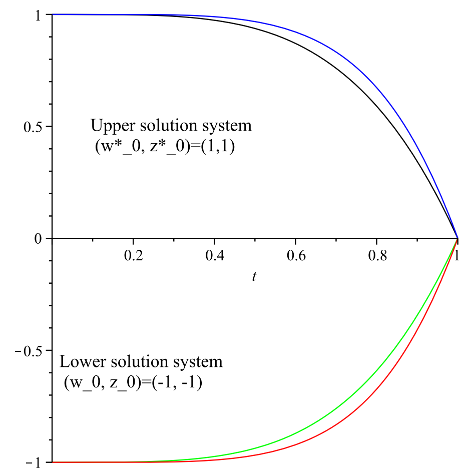
Here $\phi(t) = t$, $\varphi(t) = t$. Also $\lambda_1 = \lambda_2 = \frac{1}{3}$, $\lambda = \frac{8}{9}$. Take $(0,0) = (\underline{w}^{(0)}, \underline{z}^{(0)})$ and $(1,1) = (\overline{w}_0, \overline{z}_0)$ as the initial approximation of the system of lower and upper solutions, respectively. Further, the function $\Theta(t, w, z)$ is strictly decreasing with

$$
-3 \le \frac{\partial \Theta(t, w, z)}{\partial w} = -3w^2 < 0
$$

and $\Phi(t, w, z)$ is strictly decreasing with

$$
-3 \le \frac{\partial \Phi(t, w, z)}{\partial z} = -3z^2 < 0
$$

Figure 1 Plot of upper and lower solutions of coupled system of Example 1.



for all $(w,z) \in [\underline{w}^{(0)}, \underline{z}^{(0)}] \times [\overline{w}_0, \overline{z}_0]$. Here the constants c, d of the procedure are $c = 3$, $d = 3$. Thus $(0,0)$ and $(1,1)$ are the initial approximations of the lower and upper solutions, respectively, for the coupled system (37). In Figure 1, we have ploted upper and lower solutions for the given system (37).

Example 2 For more explanation, we give another example of FDEs subject to the coupled integral boundary conditions:

$$\begin{cases} {}^c\mathcal{D}^{\frac{7}{4}} w(t) - w(t)\exp(w(t)) + z(t) = 0; & t \in (0,1), \\ {}^c\mathcal{D}^{\frac{7}{4}} z(t) + w(t) - z(t)\exp(z(t)) = 0; & t \in (0,1), \\ w(0) = z(0) = 0, \qquad w(1) = \int_0^1 t^2 z(t)\,dt, & z(1) = \int_0^1 t^2 w(t)\,dt. \end{cases} \tag{39}$$

From the above system (39), we see

$$\Theta\bigl(t, w(t), z(t)\bigr) = -w(t)\exp\bigl(w(t)\bigr) + z(t), \qquad \Phi\bigl(t, w(t), z(t)\bigr) = w(t) - z(t)\exp\bigl(z(t)\bigr). \tag{40}$$

Here

$$\phi(t) = t^2, \qquad \varphi(t) = t^2, \quad \text{also} \quad \lambda_1 = \lambda_2 = \frac{1}{4}, \qquad \lambda = \frac{15}{16}.$$

Take $(0,0) = (\underline{w}^{(0)}, \underline{z}^{(0)})$ and $(2,2) = (\overline{w}_0, \overline{z}_0)$ as the initial approximation of the lower and upper solutions, respectively. Then from (40), we see that the function $\Theta(t,w,z)$ is strictly decreasing with

$$-3\exp(2) \leq \frac{\partial\Theta(t,w,z)}{\partial w} = -\exp(w)(w+1) < 0$$

and $\Phi(t,w,z)$ is strictly decreasing with

$$-3\exp(2) \leq \frac{\partial\Phi(t,w,z)}{\partial z} = -\exp(z)(z+1) < 0$$

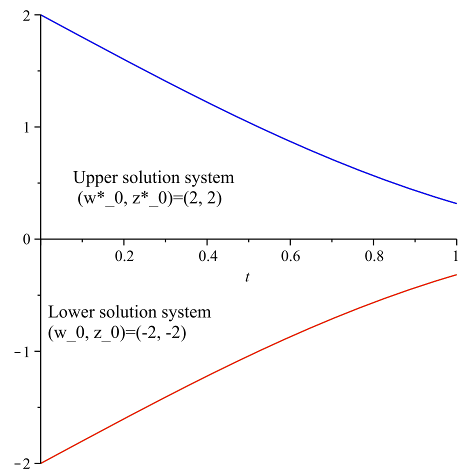


Figure 2 Plot of upper and lower solutions of coupled system of Example 2.

for all $(w,z) \in [\underline{w}^{(0)}, \underline{z}^{(0)}] \times [\overline{w}_0, \overline{z}_0]$. Here the constants c, d of the method are $c = d = 3\exp(2)$. Thus $(0,0)$ and $(1,1)$ are the initial approximations of the lower and upper solutions, respectively, for the coupled system (39). Further, in Figure 2, we have ploted upper and lower solutions for the system (39).

5 Conclusion

By the use of a monotone iterative technique, we successfully developed a scheme for order upper and lower solutions to the coupled system of highly nonlinear fractional order differential equations with coupled integral boundary conditions. We have introduced an algorithm to construct a convergent increasing sequence of lower solutions as well as a convergent decreasing sequence of upper solutions. Further we have proved that the constructed sequences converge uniformly to the unique solution of the considered problem. Moreover, the results are justified by some suitable numerical examples.

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Competing interests
It is declared that we have no conflict of interest.

Authors' contributions
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