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# Sums of finite products of Bernoulli functions

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**Abstract**
In this paper, we consider three types of functions given by sums of finite products of Bernoulli functions and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

**MSC:** 11B68; 42A16

**Keywords:** Fourier series; sums of finite products of Bernoulli functions

## 1 Introduction

As is well known, the *Bernoulli polynomials* $B_m(x)$ are given by the generating function

$$\frac{t}{e^t-1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!} \quad \text{(see [1–10]).} \tag{1.1}$$

When $x = 0$, $B_m = B_m(0)$ are called Bernoulli numbers. For any real number $x$, we let

$$\langle x\rangle = x - [x] \in [0,1) \tag{1.2}$$

denote the fractional part of $x$.

Fourier series expansion of higher-order Bernoulli functions was treated in the recent paper [11]. Here we will consider the following three types of functions given by sums of finite products of Bernoulli functions and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

(1) $\alpha_m(\langle x\rangle) = \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 0} B_{c_1}(\langle x\rangle)B_{c_2}(\langle x\rangle)\cdots B_{c_r}(\langle x\rangle)$ $(m \geq 1)$;
(2) $\beta_m(\langle x\rangle) = \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 0} \frac{1}{c_1!c_2!\cdots c_r!}B_{c_1}(\langle x\rangle)B_{c_2}(\langle x\rangle)\cdots B_{c_r}(\langle x\rangle)$ $(m \geq 1)$;
(3) $\gamma_{r,m}(\langle x\rangle) = \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{1}{c_1 c_2\cdots c_r}B_{c_1}(\langle x\rangle)B_{c_2}(\langle x\rangle)\cdots B_{c_r}(\langle x\rangle)$ $(m \geq r)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [12, 13]).

As to $\beta_m(\langle x\rangle)$, we note that the next polynomial identity follows immediately from Theorems 3.1 and 3.2, which is in turn derived from the Fourier series expansion of $\beta_m(\langle x\rangle)$:

$$\sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!}B_{c_1}(x)B_{c_2}(x)\cdots B_{c_r}(x) = \frac{1}{r}\Omega_{m+1} + \sum_{j=1}^{m}\frac{r^{j-1}}{j!}\Omega_{m-j+1}B_j(x),$$

where

$$\Omega_l = \sum_{\max\{0,r-l\}\le a\le r-1} \binom{r}{a} \sum_{c_1+c_2+\cdots+c_a=l+a-r} \frac{B_{c_1}B_{c_2}\cdots B_{c_a}}{c_1!c_2!\cdots c_a!}. \tag{1.3}$$

The obvious polynomial identities can be derived also for $\alpha_m(\langle x\rangle)$ and $\gamma_m(\langle x\rangle)$ from Theorems 2.1 and 2.2, and Theorems 4.1 and 4.2, respectively. It is remarkable that from the Fourier series expansion of the function $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(\langle x\rangle)B_{m-k}(\langle x\rangle)$ we can derive the Faber-Pandharipande-Zagier identity (see [14–16]) and the Miki identity (see [15–19]).

## 2 The function $\alpha_m(\langle x\rangle)$

Let $\alpha_m(x) = \sum_{c_1+c_2+\cdots+c_r=m} B_{c_1}(x)B_{c_2}(x)\cdots B_{c_r}(x)$ ($m \ge 1$). Here the sum runs over all nonnegative integers $c_1, c_2, \ldots, c_r$ with $c_1 + c_2 + \cdots + c_r = m$ ($r \ge 1$). Then we will consider the function

$$\alpha_m\bigl(\langle x\rangle\bigr) = \sum_{c_1+c_2+\cdots+c_r=m} B_{c_1}\bigl(\langle x\rangle\bigr)B_{c_2}\bigl(\langle x\rangle\bigr)\cdots B_{c_r}\bigl(\langle x\rangle\bigr), \tag{2.1}$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\alpha_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi inx}, \tag{2.2}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m\bigl(\langle x\rangle\bigr)e^{-2\pi inx}\,dx \\ &= \int_0^1 \alpha_m(x)e^{-2\pi inx}\,dx. \end{aligned} \tag{2.3}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned} \alpha_m'(x) &= \sum_{c_1+c_2+\cdots+c_r=m} \bigl(c_1B_{c_1-1}(x)B_{c_2}(x)\cdots B_{c_r}(x) \\ &\quad + \cdots + c_rB_{c_1}(x)B_{c_2}(x)\cdots B_{c_{r-1}}(x)B_{c_r-1}(x)\bigr) \\ &= \sum_{c_1+c_2+\cdots+c_r=m,c_1\ge 1} c_1B_{c_1-1}(x)B_{c_2}(x)\cdots B_{c_r}(x) \\ &\quad + \cdots + \sum_{c_1+c_2+\cdots+c_r=m,c_r\ge 1} c_rB_{c_1-1}(x)B_{c_2}(x)\cdots B_{c_r}(x) \\ &= (m+r-1)\sum_{c_1+c_2+\cdots+c_r=m-1} B_{c_1}(x)B_{c_2}(x)\cdots B_{c_r}(x) \\ &= (m+r-1)\alpha_{m-1}(x). \end{aligned} \tag{2.4}$$

From this, we have

$$\left(\frac{\alpha_{m+1}(x)}{m+r}\right)' = \alpha_m(x) \tag{2.5}$$

and

$$\int_0^1 \alpha_m(x)\,dx = \frac{1}{m+r}\big(\alpha_{m+1}(1) - \alpha_{m+1}(0)\big). \tag{2.6}$$

For $m \geq 1$, we put

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{c_1+c_2+\cdots+c_r=m} \big(B_{c_1}(1)B_{c_2}(1)\cdots B_{c_r}(1) - B_{c_1}B_{c_2}\cdots B_{c_r}\big) \\
&= \sum_{c_1+c_2+\cdots+c_r=m} \big((B_{c_1} + \delta_{1,c_1})\cdots(B_{c_r} + \delta_{1,c_r}) - B_{c_1}B_{c_2}\cdots B_{c_r}\big) \\
&= \sum_{\substack{0\leq a\leq r\\ a\geq r-m}} \binom{r}{a} \sum_{c_1+c_2+\cdots+c_a=m+a-r} B_{c_1}B_{c_2}\cdots B_{c_a} - \sum_{c_1+c_2+\cdots+c_r=m} B_{c_1}B_{c_2}\cdots B_{c_r} \\
&= \sum_{\max\{0,r-m\}\leq a\leq r-1} \binom{r}{a} \sum_{c_1+c_2+\cdots+c_a=m+a-r} B_{c_1}B_{c_2}\cdots B_{c_a},
\end{aligned} \tag{2.7}$$

where we understand that, for $r - m \leq 0$ and $a = 0$, the inner sum is $\delta_{m,r}$.

Observe here that the sum over all $c_1 + c_2 + \cdots + c_r = m$ of any term with $a$ of $B_{c_e}$ and $b$ of $\delta_{1,c_f}$ $(1 \leq e, f \leq r,\ a + b = r)$, all give the same sum

$$\begin{aligned}
&\sum_{c_1+c_2+\cdots+c_r=m} B_{c_1}\cdots B_{c_a}\delta_{1,c_{a+1}}\cdots\delta_{1,c_{a+b}} \\
&\quad= \sum_{c_1+c_2+\cdots+c_a=m+a-r} B_{c_1}B_{c_2}\cdots B_{c_a},
\end{aligned} \tag{2.8}$$

which is not an empty sum as long as $m + a - r \geq 0$, i.e., $a \geq r - m$.

Thus

$$\alpha_m(1) = \alpha_m(0) \quad\Longleftrightarrow\quad \Delta_m = 0 \tag{2.9}$$

and

$$\int_0^1 \alpha_m(x)\,dx = \frac{1}{m+r}\Delta_{m+1}. \tag{2.10}$$

Now, we are ready to determine the Fourier coefficients $A_n^{(m)}$.

*Case* $1: n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}\,dx \\
&= -\frac{1}{2\pi in}\big[\alpha_m(x)e^{-2\pi inx}\big]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha'_m(x)e^{-2\pi inx}\,dx \\
&= \frac{m+r-1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m \\
&= \frac{m+r-1}{2\pi in}\left(\frac{m+r-2}{2\pi in}A_n^{(m-2)} - \frac{1}{2\pi in}\Delta_{m-1}\right) - \frac{1}{2\pi in}\Delta_m
\end{aligned}$$

$$= \frac{(m+r-1)_2}{(2\pi in)^2} A_n^{(m-2)} - \sum_{j=0}^{2} \frac{(m+r-1)_{j-1}}{(2\pi in)^j} \Delta_{m-j+1}$$

$$= \cdots$$

$$= \frac{(m+r-1)_m}{(2\pi in)^m} A_n^{(0)} - \sum_{j=1}^{m} \frac{(m+r-1)_{j-1}}{(2\pi in)^j} \Delta_{m-j+1}$$

$$= -\frac{1}{m+r} \sum_{j=1}^{m} \frac{(m+r)_j}{(2\pi in)^j} \Delta_{m-j+1}, \tag{2.11}$$

where $A_n^{(0)} = \int_0^1 e^{-2\pi inx}\, dx = 0$.

*Case* 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)\, dx = \frac{1}{m+r} \Delta_{m+1}. \tag{2.12}$$

Let us recall the following facts about Bernoulli functions $B_m(\langle x \rangle)$:

(a) for $m \geq 2$,

$$B_m\big(\langle x \rangle\big) = -m! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}; \tag{2.13}$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(\langle x \rangle) & \text{for } x \notin \mathbb{Z}, \\ 0 & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.14}$$

$\alpha_m(\langle x \rangle)$ ($m \geq 1$) is piecewise $C^\infty$. Moreover, $\alpha_m(\langle x \rangle)$ is continuous for those positive integers $m$ with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\Delta_m \neq 0$.

Assume first that $m$ is a positive integer with $\Delta_m = 0$. Then $\alpha_m(1) = \alpha_m(0)$. Hence $\alpha_m(\langle x \rangle)$ is piecewise $C^\infty$ and continuous. Thus the Fourier series of $\alpha_m(\langle x \rangle)$ converges uniformly to $\alpha_m(\langle x \rangle)$, and

$$\begin{aligned}
\alpha_m\big(\langle x \rangle\big) &= \frac{1}{m+r} \Delta_{m+1} \\
&\quad + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+r} \sum_{j=1}^{m} \frac{(m+r)_j}{(2\pi in)^j} \Delta_{m-j+1} \right) e^{2\pi inx} \\
&= \frac{1}{m+r} \Delta_{m+1} + \frac{1}{m+r} \sum_{j=1}^{m} \binom{m+r}{j} \Delta_{m-j+1} \left( -j! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^j} \right) \\
&= \frac{1}{m+r} \Delta_{m+1} + \frac{1}{m+r} \sum_{j=2}^{m} \binom{m+r}{j} \Delta_{m-j+1} B_j\big(\langle x \rangle\big) \\
&\quad + \Delta_m \times \begin{cases} B_1(\langle x \rangle) & \text{for } x \notin \mathbb{Z}, \\ 0 & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{2.15}$$

We can now state our first result.

**Theorem 2.1** *For each positive integer $l$, we let*

$$\Delta_l = \sum_{\max\{0,r-l\}\le a\le r-1} \binom{r}{a} \sum_{c_1+c_2+\cdots+c_a=l+a-r} B_{c_1}B_{c_2}\cdots B_{c_a}.$$

*Assume that $\Delta_m = 0$ for a positive integer $m$. Then we have the following.*

(a) $\sum_{c_1+c_2+\cdots+c_r=m} B_{c_1}(\langle x\rangle)B_{c_2}(\langle x\rangle)\cdots B_{c_r}(\langle x\rangle)$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{c_1+c_2+\cdots+c_r=m} B_{c_1}\bigl(\langle x\rangle\bigr)B_{c_2}\bigl(\langle x\rangle\bigr)\cdots B_{c_r}\bigl(\langle x\rangle\bigr)\\ &\quad= \frac{1}{m+r}\Delta_{m+1} - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{1}{m+r}\sum_{j=1}^{m}\frac{(m+r)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},\end{aligned}$$

*for all $x\in\mathbb{R}$, where the convergence is uniform.*

(b)

$$\begin{aligned}&\sum_{c_1+c_2+\cdots+c_r=m} B_{c_1}\bigl(\langle x\rangle\bigr)B_{c_2}\bigl(\langle x\rangle\bigr)\cdots B_{c_r}\bigl(\langle x\rangle\bigr)\\ &\quad= \frac{1}{m+r}\Delta_{m+1} + \frac{1}{m+r}\sum_{j=2}^{m}\binom{m+r}{j}\Delta_{m-j+1}B_j\bigl(\langle x\rangle\bigr),\end{aligned}$$

*for all $x\in\mathbb{R}$, where $B_j(\langle x\rangle)$ is the Bernoulli function.*

Assume next that $\Delta_m \neq 0$ for a positive integer $m$. Then $\alpha_m(1)\neq\alpha_m(0)$. Hence $\alpha_m(\langle x\rangle)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(\langle x\rangle)$ converges pointwise to $\alpha_m(\langle x\rangle)$ for $x\notin\mathbb{Z}$ and converges to

$$\frac{1}{2}\bigl(\alpha_m(0)+\alpha_m(1)\bigr) = \alpha_m(0) + \frac{1}{2}\Delta_m \tag{2.16}$$

for $x\in\mathbb{Z}$.

Now, we can state our second result.

**Theorem 2.2** *For each positive integer $l$, we let*

$$\Delta_l = \sum_{\max\{0,r-l\}\le a\le r-1} \binom{r}{a} \sum_{c_1+c_2+\cdots+c_a=l+a-r} B_{c_1}B_{c_2}\cdots B_{c_a}.$$

*Assume that $\Delta_m \neq 0$ for a positive integer $m$. Then we have the following.*

(a)
$$\begin{aligned}&\frac{1}{m+r}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+r}\sum_{j=1}^{m}\frac{(m+r)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\ &\quad= \begin{cases}\sum_{c_1+c_2+\cdots+c_r=m} B_{c_1}(\langle x\rangle)B_{c_2}(\langle x\rangle)\cdots B_{c_r}(\langle x\rangle) & \text{for } x\notin\mathbb{Z},\\ \sum_{c_1+c_2+\cdots+c_r=m} B_{c_1}B_{c_2}\cdots B_{c_r} + \frac{1}{2}\Delta_m & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned}$$

(b) $$\frac{1}{m+r}\Delta_{m+1}+\frac{1}{m+r}\sum_{j=1}^{m}\binom{m+r}{j}\Delta_{m-j+1}B_j\bigl(\langle x\rangle\bigr)$$
$$=\sum_{c_1+c_2+\cdots+c_r=m}B_{c_1}\bigl(\langle x\rangle\bigr)B_{c_2}\bigl(\langle x\rangle\bigr)\cdots B_{c_r}\bigl(\langle x\rangle\bigr)\quad \textit{for } x\notin\mathbb{Z};$$
$$\frac{1}{m+r}\Delta_{m+1}+\frac{1}{m+r}\sum_{j=2}^{m}\binom{m+r}{j}\Delta_{m-j+1}B_j\bigl(\langle x\rangle\bigr)$$
$$=\sum_{c_1+c_2+\cdots+c_r=m}B_{c_1}B_{c_2}\cdots B_{c_r}+\frac{1}{2}\Delta_m\quad \textit{for } x\in\mathbb{Z}.$$

## 3 The function $\beta_m(\langle x\rangle)$

Let $\beta_m(x)=\sum_{c_1+c_2+\cdots+c_r=m}\frac{1}{c_1!c_2!\cdots c_r!}B_{c_1}(x)B_{c_2}(x)\cdots B_{c_r}(x)$ ($m\geq 1$). Here the sum runs over all nonnegative integers $c_1,c_2,\ldots,c_r$ with $c_1+c_2+\cdots+c_r=m$ ($r\geq 1$). Then we will consider the function

$$\beta_m\bigl(\langle x\rangle\bigr)=\sum_{c_1+c_2+\cdots+c_r=m}\frac{1}{c_1!c_2!\cdots c_r!}B_{c_1}\bigl(\langle x\rangle\bigr)B_{c_2}\bigl(\langle x\rangle\bigr)\cdots B_{c_r}\bigl(\langle x\rangle\bigr),\tag{3.1}$$

defined on $(-\infty,\infty)$, which is periodic with period 1. The Fourier series of $\beta_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},\tag{3.2}$$

where

$$B_n^{(m)}=\int_0^1\beta_m\bigl(\langle x\rangle\bigr)e^{-2\pi inx}\,dx=\int_0^1\beta_m(x)e^{-2\pi inx}\,dx.\tag{3.3}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned}\beta_m'(x)&=\sum_{c_1+c_2+\cdots+c_r=m}\biggl(\frac{c_1}{c_1!c_2!\cdots c_r!}B_{c_1-1}(x)B_{c_2}(x)\cdots B_{c_r}(x)\\&\quad+\cdots+\frac{c_r}{c_1!c_2!\cdots c_r!}B_{c_1}(x)B_{c_2}(x)\cdots B_{c_r-1}(x)\biggr)\\&=\sum_{c_1+c_2+\cdots+c_r=m,c_1\geq 1}\frac{1}{(c_1-1)!c_2!\cdots c_r!}B_{c_1-1}(x)B_{c_2}(x)\cdots B_{c_r}(x)\\&\quad+\cdots+\sum_{c_1+c_2+\cdots+c_r=m,c_r\geq 1}\frac{1}{c_1!c_2!\cdots (c_r-1)!}B_{c_1}(x)B_{c_2}(x)\cdots B_{c_r-1}(x)\\&=r\sum_{c_1+c_2+\cdots+c_r=m-1}\frac{1}{c_1!c_2!\cdots c_r!}B_{c_1}(x)B_{c_2}(x)\cdots B_{c_r}(x)\\&=r\beta_{m-1}(x).\end{aligned}\tag{3.4}$$

From this, we have

$$\left(\frac{\beta_{m+1}(x)}{r}\right)'=\beta_m(x)\tag{3.5}$$

and

$$\int_0^1 \beta_m(x)\,dx = \frac{1}{r}\bigl(\beta_{m+1}(1) - \beta_{m+1}(0)\bigr). \tag{3.6}$$

Let

$$\begin{aligned}
\Omega_m &= \beta_m(1) - \beta_m(0) \\
&= \sum_{c_1+c_2+\cdots+c_r=m} \frac{B_{c_1}(1)B_{c_2}(1)\cdots B_{c_r}(1)}{c_1!c_2!\cdots c_r!} - \sum_{c_1+c_2+\cdots+c_r=m} \frac{B_{c_1}B_{c_2}\cdots B_{c_r}}{c_1!c_2!\cdots c_r!} \\
&= \sum_{c_1+c_2+\cdots+c_r=m} \frac{(B_{c_1}+\delta_{1,c_1})(B_{c_2}+\delta_{1,c_2})\cdots(B_{c_r}+\delta_{1,c_r})}{c_1!c_2!\cdots c_r!} \\
&\quad - \sum_{c_1+c_2+\cdots+c_r=m} \frac{B_{c_1}B_{c_2}\cdots B_{c_r}}{c_1!c_2!\cdots c_r!} \\
&= \sum_{\max\{0,r-m\}\le a\le r-1} \binom{r}{a} \sum_{c_1+c_2+\cdots+c_a=m+a-r} \frac{B_{c_1}B_{c_2}\cdots B_{c_a}}{c_1!c_2!\cdots c_a!},
\end{aligned} \tag{3.7}$$

where we understand that, for $r-m \le 0$ and $a=0$, the inner sum is $\delta_{m,r}$.

Observe here that the sum over all $c_1+c_2+\cdots+c_r=m$ of any term with $a$ of $B_{c_e}$ and $b$ of $\delta_{1,c_f}$ $(1 \le e, f \le r, a+b=r)$, all give the same sum

$$\begin{aligned}
&\sum_{c_1+c_2+\cdots+c_r=m} \frac{B_{c_1}\cdots B_{c_a}\delta_{1,c_{a+1}}\cdots\delta_{1,c_{a+b}}}{c_1!c_2!\cdots c_r!} \\
&\quad = \sum_{c_1+c_2+\cdots+c_a=m+a-r} \frac{B_{c_1}B_{c_2}\cdots B_{c_r}}{c_1!c_2!\cdots c_r!},
\end{aligned} \tag{3.8}$$

which is not an empty sum as long as $m+a-r \ge 0$, i.e., $a \ge r-m$.

Also, we have

$$\beta_m(1) = \beta_m(0) \quad \Leftrightarrow \quad \Omega_m = 0 \tag{3.9}$$

and

$$\int_0^1 \beta_m(x)\,dx = \frac{1}{r}\Omega_{m+1}. \tag{3.10}$$

Now, we would like to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \ne 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}\,dx \\
&= -\frac{1}{2\pi in}\bigl[\beta_m(x)e^{-2\pi inx}\bigr]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}\,dx \\
&= -\frac{1}{2\pi in}\bigl(\beta_m(1) - \beta_m(0)\bigr) + \frac{r}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{r}{2\pi in} B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m \\
&= \frac{r}{2\pi in}\left(\frac{r}{2\pi in} B_n^{(m-2)} - \frac{1}{2\pi in}\Omega_{m-1}\right) - \frac{1}{2\pi in}\Omega_m \\
&= \left(\frac{r}{2\pi in}\right)^2 B_n^{(m-2)} - \sum_{j=1}^{2} \frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \\
&= \cdots \\
&= \left(\frac{r}{2\pi in}\right)^m B_n^{(0)} - \sum_{j=1}^{m} \frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \\
&= -\sum_{j=1}^{m} \frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1},
\end{aligned}
\tag{3.11}
$$

where $B_n^{(0)} = \int_0^1 e^{-2\pi inx}\,dx = 0$.

Case 2: $n = 0$.

$$
B_0^{(m)} = \int_0^1 \beta_m(x) = \frac{1}{r}\Omega_{m+1}.
\tag{3.12}
$$

$\beta_m(\langle x\rangle)$ $(m \geq 1)$ is piecewise $C^\infty$. Moreover, $\beta_m(\langle x\rangle)$ is continuous for those positive integers $m$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$ for a positive integer $m$. Then $\beta_m(1) = \beta_m(0)$. Hence $\beta_m(\langle x\rangle)$ is piecewise $C^\infty$ and continuous. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges uniformly to $\beta_m(\langle x\rangle)$, and

$$
\begin{aligned}
\beta_m\bigl(\langle x\rangle\bigr) &= \frac{1}{r}\Omega_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx} \\
&= \frac{1}{r}\Omega_{m+1} + \sum_{j=1}^{m}\frac{r^{j-1}}{j!}\Omega_{m-j+1} \times \left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\
&= \frac{1}{r}\Omega_{m+1} + \sum_{j=2}^{m}\frac{r^{j-1}}{j!}\Omega_{m-j+1}B_j\bigl(\langle x\rangle\bigr) \\
&\quad + \Omega_m \times \begin{cases} B_1(\langle x\rangle) & \text{for } x \notin \mathbb{Z}, \\ 0 & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}
\tag{3.13}
$$

Now, we can state our first result.

**Theorem 3.1** *For each positive integer l, we let*

$$
\Omega_l = \sum_{\max\{0,r-l\}\leq a\leq r-1}\binom{r}{a}\sum_{c_1+c_2+\cdots+c_a=l+a-r}\frac{B_{c_1}B_{c_2}\cdots B_{c_a}}{c_1!c_2!\cdots c_a!}.
\tag{3.14}
$$

*Assume that $\Omega_m = 0$ for a positive integer $m$. Then we have the following.*

(a) $\sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} B_{c_1}(\langle x\rangle)B_{c_2}(\langle x\rangle)\cdots B_{c_r}(\langle x\rangle)$ *has the Fourier series expansion*

$$\sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} B_{c_1}\bigl(\langle x\rangle\bigr)B_{c_2}\bigl(\langle x\rangle\bigr)\cdots B_{c_r}\bigl(\langle x\rangle\bigr)$$
$$= \frac{1}{r}\Omega_{m+1} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( \sum_{j=1}^{m} \frac{r^{j-1}}{(2\pi in)^j} \Omega_{m-j+1} \right) e^{2\pi inx}, \tag{3.15}$$

*for all* $x \in (-\infty, \infty)$, *where the convergence is uniform.*

(b)

$$\sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} B_{c_1}\bigl(\langle x\rangle\bigr)B_{c_2}\bigl(\langle x\rangle\bigr)\cdots B_{c_r}\bigl(\langle x\rangle\bigr)$$
$$= \frac{1}{r}\Omega_{m+1} + \sum_{j=2}^{m} \frac{r^{j-1}}{j!} \Omega_{m-j+1} B_j\bigl(\langle x\rangle\bigr), \tag{3.16}$$

*for all* $x \in (-\infty, \infty)$, *where* $B_j(\langle x\rangle)$ *is the Bernoulli function.*

Assume next that $m$ is a positive integer with $\Omega_m \neq 0$. Then $\beta_m(1) \neq \beta_m(0)$. Hence $\beta_m(\langle x\rangle)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges pointwise to $\beta_m(\langle x\rangle)$ for $x \notin \mathbb{Z}$ and converges to

$$\frac{1}{2}\bigl(\beta_m(0) + \beta_m(1)\bigr) = \beta_m(0) + \frac{1}{2}\Omega_m$$
$$= \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} B_{c_1}B_{c_2}\cdots B_{c_r} + \frac{1}{2}\Omega_m \tag{3.17}$$

for $x \in \mathbb{Z}$.

Now, we can state our second result.

**Theorem 3.2** *For each positive integer* $l$, *let*

$$\Omega_l = \sum_{\max\{0,r-l\}\leq a\leq r-1} \binom{r}{a} \sum_{c_1+c_2+\cdots+c_a=l+a-r} \frac{B_{c_1}B_{c_2}\cdots B_{c_a}}{c_1!c_2!\cdots c_a!}. \tag{3.18}$$

*Assume that* $\Omega_m \neq 0$ *for a positive integer* $m$. *Then we have the following.*

(a) $$\frac{1}{r}\Omega_{m+1} - \sum_{n=-\infty,n\neq 0}^{\infty} \left( \sum_{j=1}^{m} \frac{r^{j-1}}{(2\pi in)^j} \Omega_{m-j+1} \right) e^{2\pi inx}$$
$$= \begin{cases} \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} B_{c_1}(\langle x\rangle)B_{c_2}(\langle x\rangle)\cdots B_{c_r}(\langle x\rangle) & \text{for } x \notin \mathbb{Z}, \\ \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} B_{c_1}B_{c_2}\cdots B_{c_r} + \frac{1}{2}\Omega_m & \text{for } x \in \mathbb{Z}. \end{cases}$$

*Here the convergence is pointwise.*

(b) $$\frac{1}{r}\Omega_{m+1} + \sum_{j=1}^{m} \frac{r^{j-1}}{j!} \Omega_{m-j+1} B_j\bigl(\langle x\rangle\bigr)$$
$$= \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!} B_{c_1}\bigl(\langle x\rangle\bigr)B_{c_2}\bigl(\langle x\rangle\bigr)\cdots B_{c_r}\bigl(\langle x\rangle\bigr) \quad \text{for } x \notin \mathbb{Z},$$

$$\frac{1}{r}\Omega_{m+1} + \sum_{j=2}^{m} \frac{r^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)$$
$$= \sum_{c_1+c_2+\cdots+c_r=m} \frac{1}{c_1!c_2!\cdots c_r!}B_{c_1}B_{c_2}\cdots B_{c_r} + \frac{1}{2}\Omega_m \quad \textit{for } x \in \mathbb{Z}.$$

*Here $B_j(\langle x\rangle)$ is the Bernoulli function.*

## 4 The function $\gamma_{r,m}(\langle x\rangle)$

Let $\gamma_{r,m}(x) = \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{1}{c_1c_2\cdots c_r}B_{c_1}(x)B_{c_2}(x)\cdots B_{c_r}(x)$ $(m \geq r \geq 1)$. Here the sum is over all positive integers $c_1, c_2, \ldots, c_r$ with $c_1 + c_2 + \cdots + c_r = m$.

$$\begin{aligned}
\gamma'_{r,m}(x) &= \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{1}{c_2\cdots c_r}B_{c_1-1}(x)B_{c_2}(x)\cdots B_{c_r}(x) \\
&\quad + \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{1}{c_1c_3\cdots c_r}B_{c_1}(x)B_{c_2-1}(x)\cdots B_{c_r}(x) \\
&\quad + \cdots + \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{1}{c_1c_2\cdots c_{r-1}}B_{c_1}(x)B_{c_2}(x)\cdots B_{c_r-1}(x) \\
&= \sum_{c_2+\cdots+c_r=m-1, c_2,\ldots,c_r\geq 1} \frac{1}{c_2\cdots c_r}B_{c_2}(x)\cdots B_{c_r}(x) \\
&\quad + \sum_{c_1+\cdots+c_r=m-1, c_1,\ldots,c_r\geq 1} \frac{1}{c_2\cdots c_r}B_{c_1}(x)\cdots B_{c_r}(x) \\
&\quad + \cdots + \sum_{c_1+c_2+\cdots+c_{r-1}=m-1, c_1,\ldots,c_{r-1}\geq 1} \frac{1}{c_1c_2\cdots c_{r-1}}B_{c_1}(x)\cdots B_{c_{r-1}}(x) \\
&\quad + \sum_{c_1+c_2+\cdots+c_r=m-1, c_1,\ldots,c_r\geq 1} \frac{1}{c_1c_2\cdots c_{r-1}}B_{c_1}(x)\cdots B_{c_r}(x) \\
&= r\gamma_{r-1,m-1}(x) + (m-1)\gamma_{r,m-1}(x).
\end{aligned} \tag{4.1}$$

Thus,

$$\gamma'_{r,m}(x) = r\gamma_{r-1,m-1}(x) + (m-1)\gamma_{r,m-1}(x) \quad (m \geq r), \tag{4.2}$$

with $\gamma_{r,r-1}(x) = 0$.

Replacing $m$ by $m+1$, we get

$$m\gamma_{r,m}(x) = \gamma'_{r,m+1}(x) - r\gamma_{r-1,m}(x). \tag{4.3}$$

Denoting $\int_0^1 \gamma_{r,m}(x)\,dx$ by $a_{r,m}$, we have

$$a_{r,m} = -\frac{r}{m}a_{r-1,m} + \frac{1}{m}\Lambda_{r,m+1}, \tag{4.4}$$

where $\Lambda_{r,m} = \gamma_{r,m}(1) - \gamma_{r,m}(0)$. From the recurrence relation (4.4), we can easily show that

$$\int_0^1 \gamma_{r,m}(x)\,dx = \sum_{j=1}^{r-1} (-1)^{j-1} \frac{(r)_{j-1}}{m^j}\Lambda_{r-j+1,m+1}, \tag{4.5}$$

$$\begin{aligned}
\Lambda_{r,m} &= \gamma_{r,m}(1) - \gamma_{r,m}(0) \\
&= \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{B_{c_1}(1)\cdots B_{c_r}(1)}{c_1\cdots c_r} \\
&\quad - \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{B_{c_1}\cdots B_{c_r}}{c_1\cdots c_r} \\
&= \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} (B_{c_1}+\delta_{1,c_1})\cdots(B_{c_r}+\delta_{1,c_r}) \\
&\quad - \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{B_{c_1}\cdots B_{c_r}}{c_1\cdots c_r} \\
&= \sum_{0\leq a\leq r-1} \binom{r}{a} \sum_{c_1+c_2+\cdots+c_a=m+a-r, c_1,\ldots,c_a\geq 1} \frac{B_{c_1}B_{c_2}\cdots B_{c_a}}{c_1c_2\cdots c_a}.
\end{aligned} \tag{4.6}$$

Observe here that the sum over all positive integers $c_1,\ldots,c_r$ satisfying $c_1+c_2+\cdots+c_r=m$ of any term with $a$ of $B_{c_e}$ and $b$ of $\delta_{1,c_f}$ $(1\leq e,f\leq r,\ a+b=r)$, all give the same sum

$$\begin{aligned}
&\sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_a\geq 1} \frac{B_{c_1}\cdots B_{c_a}\delta_{1,c_{a+1}}\cdots\delta_{1,c_{a+b}}}{c_1c_2\cdots c_r} \\
&\quad = \sum_{c_1+c_2+\cdots+c_a=m+a-r, c_1,\ldots,c_a\geq 1} \frac{B_{c_1}B_{c_2}\cdots B_{c_a}}{c_1c_2\cdots c_a},
\end{aligned} \tag{4.7}$$

and that, as $m+a-r\geq a$, there are no empty sums.

Here we note that, for $a=0$, the inner sum is $\delta_{m,r}$ since it corresponds to the sums

$$\sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{\delta_{1,c_1}\delta_{1,c_2}\cdots\delta_{1,c_r}}{c_1c_2\cdots c_r}. \tag{4.8}$$

Also, $\gamma_{r,m}(1)=\gamma_{r,m}(0)\Leftrightarrow\Lambda_{r,m}=0$.

Now, we would like to consider the function

$$\gamma_{r,m}(\langle x\rangle) = \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{1}{c_1c_2\cdots c_r} B_{c_1}(\langle x\rangle)B_{c_2}(\langle x\rangle)\cdots B_{c_r}(\langle x\rangle), \tag{4.9}$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\gamma_{r,m}(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(r,m)} e^{2\pi inx}, \tag{4.10}$$

where

$$C_n^{(r,m)} = \int_0^1 \gamma_{r,m}(\langle x\rangle) e^{-2\pi inx}\,dx = \int_0^1 \gamma_{r,m}(x) e^{-2\pi inx}\,dx. \tag{4.11}$$

Now, we are going to determine the Fourier coefficients $C_n^{(r,m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} C_n^{(r,m)} &= \int_0^1 \gamma_{r,m}(x) e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\big[\gamma_{r,m}(x)e^{-2\pi inx}\big]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma'_{r,m}(x)e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\big(\gamma_{r,m}(1)-\gamma_{r,m}(0)\big) + \frac{1}{2\pi in}\int_0^1 \big(r\gamma_{r-1,m-1}(x)+(m-1)\gamma_{r,m-1}(x)\big) \\ &\quad \times e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\Lambda_{r,m} + \frac{1}{2\pi in}C_n^{(r,m-1)} + \frac{r}{2\pi in}C_n^{(r-1,m-1)}. \end{aligned} \tag{4.12}$$

From this, we obtain

$$\begin{aligned} C_n^{(r,m)} &= \frac{m-1}{2\pi in}C_n^{(r,m-1)} + \frac{r}{2\pi in}C_n^{(r-1,m-1)} - \frac{1}{2\pi in}\Lambda_{r,m} \\ &= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in}C_n^{(r,m-2)} + \frac{r}{2\pi in}C_n^{(r-1,m-2)} - \frac{1}{2\pi in}\Lambda_{r,m-1}\right) \\ &\quad + \frac{r}{2\pi in}C_n^{(r-1,m-1)} - \frac{1}{2\pi in}\Lambda_{r,m} \\ &= \frac{(m-1)_2}{(2\pi in)^2}C_n^{(r,m-2)} + \sum_{j=1}^{2}\frac{r(m-1)_{j-1}}{(2\pi in)^j}C_n^{(r-1,m-j)} - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{r,m-j+1} \\ &= \cdots \\ &= \frac{(m-1)_{m-r}}{(2\pi in)^{m-r}}C_n^{(r,r)} + \sum_{j=1}^{m-r}\frac{r(m-1)_{j-1}}{(2\pi in)^j}C_n^{(r-1,m-j)} - \sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{r,m-j+1}. \end{aligned} \tag{4.13}$$

Here,

$$\begin{aligned} C_n^{(r,r)} &= \int_0^1 \left(x-\frac{1}{2}\right)^r e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\left[\left(x-\frac{1}{2}\right)^r e^{-2\pi inx}\right]_0^1 + \frac{r}{2\pi in}\int_0^1\left(x-\frac{1}{2}\right)^{r-1}e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\left(\left(\frac{1}{2}\right)^r - \left(-\frac{1}{2}\right)^r\right) + \frac{r}{2\pi in}C_n^{(r-1,r-1)}, \end{aligned} \tag{4.14}$$

and

$$\Lambda_{r,r} = \gamma_{r,r}(1) - \gamma_{r,r}(0) = \left(\frac{1}{2}\right)^r - \left(-\frac{1}{2}\right)^r. \tag{4.15}$$

Thus

$$C_n^{(r,r)} = -\frac{1}{2\pi in}\Lambda_{r,r} + \frac{r}{2\pi in}C_n^{(r-1,r-1)}. \tag{4.16}$$

Finally, we obtain, for $n \neq 0$,

$$C_n^{(r,m)} = \sum_{j=1}^{m-r+1}\frac{r(m-1)_{j-1}}{(2\pi in)^j}C_n^{(r-1,m-j)} - \sum_{j=1}^{m-r+1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{r,m-j+1}. \tag{4.17}$$

Also, we note that, for $n \neq 0$,

$$C_n^{(1,m)} = \frac{1}{m}\int_0^1 B_m(x)e^{-2\pi inx}\,dx = -\frac{(m-1)!}{(2\pi in)^m}. \tag{4.18}$$

Thus, for $n \neq 0$, (4.17) together with (4.18) determine all $C_n^{(r,m)}$ recursively.
Case 2: $n = 0$.

$$C_0^{(r,m)} = \int_0^1 \gamma_{r,m}(x)\,dx = \sum_{j=1}^{r}(-1)^{j-1}\frac{(r)_{j-1}}{m^j}\Lambda_{r-j+1,m+1}. \tag{4.19}$$

$\gamma_{r,m}(\langle x\rangle)$ $(m \geq r \geq 1)$ is piecewise $C^\infty$. In addition, $\gamma_{r,m}(\langle x\rangle)$ is continuous for those positive integers $r$, $m$ with $\Lambda_{r,m} = 0$ and discontinuous with jump discontinuities at integers for those positive integers $r, m$ with $\Lambda_{r,m} \neq 0$.

Assume first that $\Lambda_{r,m} = 0$ for some integers $r$, $m$ with $m \geq r \geq 1$. Then $\gamma_{r,m}(1) = \gamma_{r,m}(0)$. Hence $\gamma_{r,m}(\langle x\rangle)$ is piecewise $C^\infty$ and continuous. Thus the Fourier series of $\gamma_m(\langle x\rangle)$ converges uniformly to $\gamma_m(\langle x\rangle)$, and

$$\gamma_m\big(\langle x\rangle\big) = C_0^{(r,m)} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} C_n^{(r,m)}e^{2\pi inx},$$

where $C_0^{(r,m)}$ is given by (4.19), and $C_n^{(r,m)}$, for each $n \neq 0$, are determined by relations (4.17) and (4.18).

Now, we are ready to state our first theorem.

**Theorem 4.1** *For all integers $s$, $l$, with $l \geq s \geq 1$, we let*

$$\begin{aligned}\Lambda_{s,l} &= \sum_{0\leq a\leq s-1}\binom{s}{a}\sum_{c_1+\cdots+c_a=l+a-s,c_1,\ldots,c_a\geq 1}\frac{B_{c_1}\cdots B_{c_a}}{c_1\cdots c_a}\\ &= \delta_{s,l} + \sum_{1\leq a\leq s-1}\binom{s}{a}\sum_{c_1+c_2+\cdots+c_a=l+a-s,c_1,\ldots,c_a\geq 1}\frac{B_{c_1}\cdots B_{c_a}}{c_1\cdots c_a}.\end{aligned} \tag{4.20}$$

*Assume that $\Lambda_{r,m} = 0$ for some integers $r, m$ with $m \geq r \geq 1$. Then we have the following. $\sum_{c_1+c_2+\cdots+c_r=m,c_1,\ldots,c_r\geq 1}\frac{1}{c_1\cdots c_r}B_{c_1}(\langle x\rangle)\cdots B_{c_r}(\langle x\rangle)$ has the Fourier series expansion*

$$\begin{aligned}&\sum_{c_1+c_2+\cdots+c_r=m,c_1,\ldots,c_r\geq 1}\frac{1}{c_1\cdots c_r}B_{c_1}\big(\langle x\rangle\big)\cdots B_{c_r}\big(\langle x\rangle\big)\\ &\quad= C_0^{(r,m)} + \sum_{n=-\infty,n\neq 0}^{\infty} C_n^{(r,m)}e^{2\pi inx},\end{aligned}$$

*where $C_0^{(r,m)} = \sum_{j=1}^{r-1}(-1)^{j-1}\frac{(r)_{j-1}}{m^j}\Lambda_{r-j+1,m+1}$, with $C_0^{(1,m)} = 0$, and $C_n^{(r,m)}$, for each $n \neq 0$, are determined recursively from*

$$C_n^{(r,m)} = \sum_{j=1}^{m-r+1}\frac{r(m-1)_{j-1}}{(2\pi in)^j}C_n^{(r-1,m-j)} - \sum_{j=1}^{m-r+1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{r,m-j+1}, \tag{4.21}$$

*and*

$$C_n^{(1,m)} = -\frac{(m-1)!}{(2\pi in)^m}. \tag{4.22}$$

*Here the convergence is uniform.*

Next, assume that $\Lambda_{r,m} \neq 0$ for some integers $r, m$ with $m \geq r \geq 1$. Then $\gamma_{r,m}(1) \neq \gamma_{r,m}(0)$. Hence $\gamma_{r,m}(\langle x\rangle)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Then the Fourier series of $\gamma_{r,m}(\langle x\rangle)$ converges pointwise to $\gamma_{r,m}(\langle x\rangle)$ for $x \notin \mathbb{Z}$ and converges to

$$\begin{aligned} \frac{1}{2}\bigl(\gamma_{r,m}(0) + \gamma_{r,m}(1)\bigr) &= \gamma_{r,m}(0) + \frac{1}{2}\Lambda_{r,m} \\ &= \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{1}{c_1\cdots c_r} B_{c_1}\cdots B_{c_r} + \frac{1}{2}\Lambda_{r,m} \end{aligned} \tag{4.23}$$

for $x \in \mathbb{Z}$.

Now, we can state our second result.

**Theorem 4.2** *For all integers $s$, $l$ with $l \geq s \geq 1$, we let*

$$\begin{aligned} \Lambda_{s,l} &= \sum_{0\leq a\leq s-1} \binom{s}{a} \sum_{c_1+c_2+\cdots+c_a=l+a-s, c_1,\ldots,c_a\geq 1} \frac{B_{c_1}\cdots B_{c_a}}{c_1\cdots c_a} \\ &= \delta_{s,l} + \sum_{1\leq a\leq s-1} \binom{s}{a} \sum_{c_1+c_2+\cdots+c_a=l+a-s, c_1,\ldots,c_a\geq 1} \frac{B_{c_1}\cdots B_{c_a}}{c_1\cdots c_a}. \end{aligned} \tag{4.24}$$

*Assume that $\Lambda_{r,m} \neq 0$ for some integers $r$, $m$ with $m \geq r \geq 1$. Let $C_0^{(r,m)}$, $C_n^{(r,m)}$ $(n \neq 0)$ be as in Theorem* 4.1. *Then we have the following.*

$$\begin{aligned} &C_0^{(r,m)} + \sum_{n=-\infty, n\neq 0}^{\infty} C_n^{(r,m)} e^{2\pi inx} \\ &\quad = \begin{cases} \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{1}{c_1\cdots c_r} B_{c_1}(x)\cdots B_{c_r}(x) & \text{for } x \notin \mathbb{Z}, \\ \sum_{c_1+c_2+\cdots+c_r=m, c_1,\ldots,c_r\geq 1} \frac{1}{c_1\cdots c_r} B_{c_1}\cdots B_{c_r} + \frac{1}{2}\Lambda_{r,m} & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{4.25}$$

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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