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Estimates on some power nonlinear Volterra-Fredholm type dynamic integral inequalities on time scales

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Abstract
In this paper, some explicit bounds on solutions to a class of new power nonlinear Volterra-Fredholm type dynamic integral inequalities on time scales are established, which can be used as effective tools in the study of certain dynamic equations. Application examples are also given.

Keywords: time scale; Volterra-Fredholm type; power nonlinear dynamic integral inequalities; explicit bounds; dynamic equations

1 Introduction

The calculus on time scales, which was initiated by Hilger in 1990 [1], has received considerable attention in recent years due to its broad applications in economics, population's models, quantum physics and other science fields.

In the past 20 years, there has been much research activity concerning Volterra integral equations and the dynamic integral inequalities on time scales which usually can be used as handy tools to study the qualitative theory of dynamic integral equations and dynamic equations on time scales. We refer the reader to [2–21] and the references therein. However, nonlinear Volterra-Fredholm type dynamic integral inequalities on time scales have been paid little attention to. To the best of our knowledge, Meng and Shao [22] and Gu and Meng [23] have established the linear Volterra-Fredhlom type dynamic integral inequalities on time scales. On the other hand, various Volterra-Fredholm type inequalities including continuous and discrete versions have been established. For example, Pachpatte [24, 25] has established the useful linear Volterra-Fredholm type continuous and discrete integral inequalities. Ma [26–28] has established some nonlinear Volterra-Fredholm type continuous and discrete integral inequalities. Liu and Meng [29] have investigated some new generalized Volterra-Fredholm type discrete fractional sum inequalities.

The aim of this paper is to give some explicit bounds to some new power nonlinear Volterra-Fredhlom type dynamic integral inequalities on time scales, which can be used as handy and effective tools in the study of Volterra-Fredhlom type dynamic equations on time scales.

Throughout this paper, a knowledge and understanding of time scales and time scale notations is assumed. For an excellent introduction to the calculus on time scales, we refer the reader to [30, 31].

2 Preliminaries

In what follows, $\mathbf{T}$ is an arbitrary time scale, C_{rd} denotes the set of rd continuous functions. $\mathcal{R}$ denotes the set of all regressive and rd continuous functions, $\mathcal{R}^+ = \{P \in \mathcal{R}, 1 + \mu(t)P(t) > 0, t \in \mathbf{T}\}$. $\mathbf{R}$ denotes the set of real numbers, $\mathbf{R}_+ = [0, +\infty)$, while $\mathbf{Z}$ denotes the set of integers.

In the rest of this paper, for the convenience of notation, we always assume that $I = [t_0, \alpha] \cap \mathbf{T}$, where $t_0 \in \mathbf{T}$, $\alpha \in \mathbf{T}$, $\alpha > t_0$.

Lemma 2.1 ([30]) *Suppose* $u, b \in C_{rd}(I)$, $a \in \mathbf{R}_+$. *Then*

$$u^{\Delta}(t) \le a(t)u(t) + b(t), \quad t \in \mathbf{T},$$

implies

$$u(t) \le u(t_0)e_a(t, t_0) + \int_{t_0}^{t} e_a\bigl(t, \sigma(\tau)\bigr)b(\tau)\Delta\tau, \quad t \in \mathbf{T}.$$

Lemma 2.2 ([32]) *Let* $a \ge 0$, $p \ge q \ge 0$, *then*

$$a^{\frac{q}{p}} \le \frac{q}{p}k^{(q-p)/p}a + \frac{p-q}{p}k^{q/p},$$

for any $k > 0$.

Theorem 2.1 *Assume that* $u, a, f, g, l, h_1, h_2, h_3, h_4, h_5 \in C_{rd}(I)$, $u(t)$, $a(t)$, $f(t)$, $g(t)$, $l(t)$, $h_1(t)$, $h_2(t)$, $h_3(t)$, $h_4(t)$, $h_5(t)$ *are nonnegative. Suppose that* $u(t)$ *satisfies*

$$\begin{aligned} u^p(t) \le{}& a(t) + \int_{t_0}^{t} f(\tau)\biggl[u^q(\tau) + \int_{t_0}^{\tau}\biggl[g(s)u^r(s) + \int_{t_0}^{s} l(\nu)u^n(\nu)\Delta\nu\biggr]\Delta s\biggr]\Delta\tau \\ &+ \sum_{i=1}^{3}\int_{t_0}^{\alpha} h_i(\tau)u^{m_i}(\tau)\Delta\tau + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)u^{m_4}(s)\Delta s\Delta\tau, \quad t \in I, \end{aligned} \tag{1}$$

where p, q, r, n *and* m_i $(i = 1,2,3,4)$ *are constants with* $p \ge q > 0$, $p \ge r > 0$, $p \ge n > 0$, $p \ge m_i > 0$ $(i = 1,2,3,4)$. *If*

$$\begin{aligned} \lambda_{pqrnm_1m_2m_3m_4} ={}& \sum_{i=1}^{3}\frac{m_i}{p}k^{\frac{m_i-p}{p}}\int_{t_0}^{\alpha} h_i(\tau)e_{B_{pqrn}}(\tau, t_0)\Delta\tau \\ &+ \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau}\frac{m_4}{p}k^{(m_4-p)/p}h_5(s)e_{B_{pqrn}}(s, t_0)\Delta s\Delta\tau \\ <{}& 1, \end{aligned} \tag{2}$$

then

$$u(t) \le \biggl[a(t) + \frac{C_{pm_1m_2m_3m_4} + A_{pqrn}(t)}{1 - \lambda_{pqrnm_1m_2m_3m_4}}e_{B_{pqrn}}(t, t_0)\biggr]^{\frac{1}{p}} \tag{3}$$

for $t \in I$ and for any $k > 0$, where

$$\begin{aligned} C_{pm_1m_2m_3m_4} = {} & \sum_{i=1}^{3} \int_{t_0}^{\alpha} h_i(\tau)\left[\frac{m_i}{p}k^{(m_i-p)/p}a(\tau) + \frac{p-m_i}{p}k^{m_i/p}\right]\Delta\tau \\ & + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)\left[\frac{m_4}{p}k^{(m_4-p)/p}a(s) + \frac{p-m_4}{p}k^{m_4/p}\right]\Delta s\Delta\tau, \end{aligned} \tag{4}$$

$$\begin{aligned} A_{pqrn}(t) = {} & \int_{t_0}^{t} f(\tau)\left\{\frac{q}{p}k^{(q-p)/p}a(\tau) + \frac{p-q}{p}k^{q/p}\right. \\ & + \int_{t_0}^{\tau}\left[g(s)\left[\frac{r}{p}k^{(r-p)/p}a(s) + \frac{p-r}{p}k^{r/p}\right]\right. \\ & \left.\left. + \int_{t_0}^{s} l(\nu)\left[\frac{n}{p}k^{(n-p)/p}a(\nu) + \frac{p-n}{p}k^{n/p}\right]\Delta\nu\right]\Delta s\right\}\Delta\tau, \end{aligned} \tag{5}$$

and

$$B_{pqrn}(t) = f(t)\left[\frac{q}{p}k^{(q-p)/p} + \int_{t_0}^{t}\left[\frac{r}{p}k^{(r-p)/p}g(s) + \int_{t_0}^{s}\frac{n}{p}k^{(n-p)/p}l(\nu)\Delta\nu\right]\Delta s\right]. \tag{6}$$

Proof Define a function $z(t)$, $t \in I$, by

$$\begin{aligned} z(t) = {} & \int_{t_0}^{t} f(\tau)\left[u^q(\tau) + \int_{t_0}^{\tau}\left[g(s)u^r(s) + \int_{t_0}^{s} l(\nu)u^n(\nu)\Delta\nu\right]\Delta s\right]\Delta\tau \\ & + \sum_{i=1}^{3}\int_{t_0}^{\alpha} h_i(\tau)u^{m_i}(\tau)\Delta\tau + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)u^{m_4}(s)\Delta s\Delta\tau, \end{aligned} \tag{7}$$

then

$$u^p(t) \le a(t) + z(t) \quad \text{or} \quad u(t) \le \big(a(t) + z(t)\big)^{\frac{1}{p}}. \tag{8}$$

By Lemma 2.2 and (8) for any $k > 0$, we have

$$\begin{aligned} & u^q(t) \le \big[a(t) + z(t)\big]^{\frac{q}{p}} \le \frac{q}{p}k^{(q-p)/p}\big[a(t) + z(t)\big] + \frac{p-q}{p}k^{q/p}, \\ & u^r(t) \le \big[a(t) + z(t)\big]^{\frac{r}{p}} \le \frac{r}{p}k^{(r-p)/p}\big[a(t) + z(t)\big] + \frac{p-r}{p}k^{r/p}, \\ & u^n(t) \le \big[a(t) + z(t)\big]^{\frac{n}{p}} \le \frac{n}{p}k^{(n-p)/p}\big[a(t) + z(t)\big] + \frac{p-n}{p}k^{n/p}, \\ & u^{m_i}(t) \le \big[a(t) + z(t)\big]^{\frac{m_i}{p}} \le \frac{m_i}{p}k^{(m_i-p)/p}\big[a(t) + z(t)\big] + \frac{p-m_i}{p}k^{m_i/p}, \quad i = 1, 2, 3, 4. \end{aligned}$$

Substituting the last seven inequalities into (7) we have

$$\begin{aligned} z(t) \le {} & \int_{t_0}^{t} f(\tau)\left\{\left[\frac{q}{p}k^{(q-p)/p}\big[a(\tau) + z(\tau)\big] + \frac{p-q}{p}k^{q/p}\right]\right. \\ & + \int_{t_0}^{\tau}\left[g(s)\left[\frac{r}{p}k^{(r-p)/p}\big[a(s) + z(s)\big] + \frac{p-r}{p}k^{r/p}\right]\right. \end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^{s} l(\nu)\left[\frac{n}{p}k^{(n-p)/p}\big[a(\nu)+z(\nu)\big]+\frac{p-n}{p}k^{n/p}\right]\Delta\nu\Bigg]\Delta s\Bigg\}\Delta\tau \\
&+\sum_{i=1}^{3}\int_{t_0}^{\alpha} h_i(\tau)\left[\frac{m_i}{p}k^{(m_i-p)/p}\big[a(\tau)+z(\tau)\big]+\frac{p-m_i}{p}k^{m_i/p}\right]\Delta\tau \\
&+\int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)\left[\frac{m_4}{p}k^{(m_4-p)/p}\big[a(s)+z(s)\big]+\frac{p-m_4}{p}k^{m_4/p}\right]\Delta s\Delta\tau \\
=\ & C_{pm_1m_2m_3m_4}+A_{pqrn}(t) \\
&+\int_{t_0}^{t} f(\tau)\left[\frac{q}{p}k^{(q-p)/p}z(\tau)+\int_{t_0}^{\tau}\left[\frac{r}{p}k^{(r-p)/p}g(s)z(s)\right.\right. \\
&\left.\left.+\int_{t_0}^{s}\frac{n}{p}k^{(n-p)/p}l(\nu)z(\nu)\Delta\nu\right]\Delta s\right]\Delta\tau \\
&+\sum_{i=1}^{3}\int_{t_0}^{\alpha}\frac{m_i}{p}k^{(m_i-p)/p}h_i(\tau)z(\tau)\Delta\tau \\
&+\int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)\left[\frac{m_4}{p}k^{(m_4-p)/p}z(s)\right]\Delta s\Delta\tau.
\end{aligned}
$$

Fix any arbitrary $\widetilde{t}\in I$. Since $A_{pqrn}(t)$ is nondecreasing for each $t\in I$, then, for $t\in\widetilde{I}$, where $\widetilde{I}=[t_0,\widetilde{t}]\cap\mathbf{T}$, from the above inequality we have

$$
\begin{aligned}
z(t)\le\ & C_{pm_1m_2m_3m_4}+A_{pqrn}(\widetilde{t}) \\
&+\int_{t_0}^{t} f(\tau)\left[\frac{q}{p}k^{(q-p)/p}z(\tau)\right. \\
&\left.+\int_{t_0}^{\tau}\left[\frac{r}{p}k^{(r-p)/p}g(s)z(s)+\int_{t_0}^{s}\frac{n}{p}k^{(n-p)/p}l(\nu)z(\nu)\Delta\nu\right]\Delta s\right]\Delta\tau \\
&+\sum_{i=1}^{3}\int_{t_0}^{\alpha}\frac{m_i}{p}k^{(m_i-p)/p}h_i(\tau)z(\tau)\Delta\tau \\
&+\int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)\left[\frac{m_4}{p}k^{(m_4-p)/p}z(s)\right]\Delta s\Delta\tau,\quad t\in\widetilde{I}.
\end{aligned}
\tag{9}
$$

Let

$$
\begin{aligned}
N=\ & C_{pm_1m_2m_3m_4}+A_{pqrn}(\widetilde{t})+\sum_{i=1}^{3}\int_{t_0}^{\alpha}\frac{m_i}{p}k^{(m_i-p)/p}h_i(\tau)z(\tau)\Delta\tau \\
&+\int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)\left[\frac{m_4}{p}k^{(m_4-p)/p}z(s)\right]\Delta s\Delta\tau,
\end{aligned}
\tag{10}
$$

then (9) can be restated as

$$
\begin{aligned}
z(t)\le\ & N+\int_{t_0}^{t} f(\tau)\left[\frac{q}{p}k^{(q-p)/p}z(\tau)+\int_{t_0}^{\tau}\left[\frac{r}{p}k^{(r-p)/p}g(s)z(s)\right.\right. \\
&\left.\left.+\int_{t_0}^{s}\frac{n}{p}k^{(n-p)/p}l(\nu)z(\nu)\Delta\nu\right]\Delta s\right]\Delta\tau,
\end{aligned}
\tag{11}
$$

for $t \in \widetilde{I}$. Since $z(t)$ is nondecreasing, by (11) we have

$$z(t) \le N + \int_{t_0}^{t} f(\tau)\left[\frac{q}{p}k^{(q-p)/p} + \int_{t_0}^{\tau}\left[\frac{r}{p}k^{(r-p)/p}g(s) + \int_{t_0}^{s}\frac{n}{p}k^{(n-p)/p}l(\nu)\Delta\nu\right]\Delta s\right]z(\tau)\Delta\tau. \tag{12}$$

Set

$$w(t) = N + \int_{t_0}^{t} f(\tau)\left[\frac{q}{p}k^{(q-p)/p} + \int_{t_0}^{\tau}\left[\frac{r}{p}k^{(r-p)/p}g(s) + \int_{t_0}^{s}\frac{n}{p}k^{(n-p)/p}l(\nu)\Delta\nu\right]\Delta s\right]z(\tau)\Delta\tau, \tag{13}$$

then

$$\begin{aligned} w^{\Delta}(t) &= f(t)\left[\frac{q}{p}k^{(q-p)/p} + \int_{t_0}^{t}\left[\frac{r}{p}k^{(r-p)/p}g(s) + \int_{t_0}^{s}\frac{n}{p}k^{(n-p)/p}l(\nu)\Delta\nu\right]\Delta s\right]z(t) \\ &\le f(t)\left[\frac{q}{p}k^{(q-p)/p} + \int_{t_0}^{t}\left[\frac{r}{p}k^{(r-p)/p}g(s) + \int_{t_0}^{s}\frac{n}{p}k^{(n-p)/p}l(\nu)\Delta\nu\right]\Delta s\right]w(t) \\ &= B_{pqrn}(t)w(t), \quad t \in \widetilde{I}, \end{aligned} \tag{14}$$

where $B_{pqrn}(t)$ is defined as in (6). Using Lemma 2.1, from (14), we get

$$w(t) \le Ne_{B_{pqrn}}(t, t_0), \quad t \in \widetilde{I}. \tag{15}$$

From (12), (13) and (15), we have

$$z(t) \le Ne_{B_{pqrn}}(t, t_0), \quad t \in \widetilde{I}. \tag{16}$$

Let $t = \widetilde{t}$ in the above inequality, we have

$$z(\widetilde{t}) \le Ne_{B_{pqrn}}(\widetilde{t}, t_0).$$

Since $\widetilde{t} \in I$ is arbitrary, from the above inequality with $\widetilde{t}$ replaced by t we get

$$z(t) \le Ne_{B_{pqrn}}(t, t_0), \quad t \in I. \tag{17}$$

Using (17) on the right side of (10) and according to (2) we get

$$N \le \frac{C_{pm_1m_2m_3m_4} + A_{pqrn}(\widetilde{t})}{1 - \lambda_{pqrnm_1m_2m_3m_4}}. \tag{18}$$

From (16) and (18) we have

$$z(t) \le \frac{C_{pm_1m_2m_3m_4} + A_{pqrn}(\widetilde{t})}{1 - \lambda_{pqrnm_1m_2m_3m_4}} e_{B_{pqrn}}(t, t_0), \quad t \in \widetilde{I}. \tag{19}$$

Since $\widetilde{t} \in I$ is arbitrary, from (19) with $\widetilde{t}$ replaced by t we get

$$z(t) \le \frac{C_{pm_1m_2m_3m_4} + A_{pqrn}(t)}{1 - \lambda_{pqrnm_1m_2m_3m_4}} e_{B_{pqrn}}(t, t_0), \quad t \in I. \tag{20}$$

Now the desired inequality in (3) follows by using (20) and combining with (8). This completes the proof of Theorem 2.1. □

When $p = 2$, $q = r = n = m_1 = 1$, $h_2(t) \equiv 0$, $h_3(t) \equiv 0$, $h_4(t) \equiv 0$ in Theorem 2.1 we get a new Volterra-Fredholm-Ou-Iang type inequality as follows.

Corollary 2.2 *Let $u(t)$, $a(t)$, $f(t)$, $g(t)$, $l(t)$, $h_1(t)$ and α be defined as in Theorem* 2.1. *If $u(t)$ satisfies*

$$\begin{aligned} u^2(t) \le a(t) &+ \int_{t_0}^{t} f(\tau)\left[u(\tau) + \int_{t_0}^{\tau}\left[g(s)u(s) + \int_{t_0}^{s} l(v)u(v)\Delta v\right]\Delta s\right]\Delta\tau \\ &+ \int_{t_0}^{\alpha} h_1(\tau)u(\tau)\Delta\tau, \quad t \in I, \end{aligned} \tag{21}$$

and

$$\lambda_{21111000} = \frac{1}{2}k^{\frac{-1}{2}} \int_{t_0}^{\alpha} h_1(\tau)e_{B_{2111}}(\tau, t_0)\Delta\tau < 1, \tag{22}$$

then

$$u(t) \le \left[a(t) + \frac{C_{21000} + A_{2111}(t)}{1 - \lambda_{21111000}} e_{B_{2111}}(t, t_0)\right]^{\frac{1}{2}} \tag{23}$$

for $t \in I$ and for any $k > 0$, where

$$C_{21000} = \int_{t_0}^{\alpha} h_1(\tau)\left[\frac{1}{2}k^{-1/2}a(\tau) + \frac{1}{2}k^{1/2}\right]\Delta\tau, \tag{24}$$

$$\begin{aligned} A_{2111}(t) = \int_{t_0}^{t} f(\tau)\Bigg\{&\frac{1}{2}k^{-1/2}a(\tau) + \frac{1}{2}k^{1/2} \\ &+ \int_{t_0}^{\tau}\left[g(s)\left[\frac{1}{2}k^{-1/2}a(s) + \frac{1}{2}k^{1/2}\right]\right. \\ &\left.+ \int_{t_0}^{s} l(v)\left[\frac{1}{2}k^{-1/2}a(v) + \frac{1}{2}k^{1/2}\right]\Delta v\right]\Delta s\Bigg\}\Delta\tau, \end{aligned} \tag{25}$$

and

$$B_{2111}(t) = f(t)\left[\frac{1}{2}k^{-1/2} + \int_{t_0}^{t}\left[\frac{1}{2}k^{-1/2}g(s) + \int_{t_0}^{s}\frac{1}{2}k^{-1/2}l(v)\Delta v\right]\Delta s\right]. \tag{26}$$

When $p = 1$ in Theorem 2.1, we get the following inequality.

Corollary 2.3 *Let $u(t)$, $a(t)$, $f(t)$, $g(t)$, $l(t)$, $h_i(t)$ $(i = 1, 2, 3, 4, 5)$, q, r, n, m_i $(i = 1, 2, 3, 4)$ and α be as in Theorem 2.1. If $u(t)$ satisfies*

$$u(t) \le a(t) + \int_{t_0}^{t} f(\tau)\left[u^q(\tau) + \int_{t_0}^{\tau}\left[g(s)u^r(s) + \int_{t_0}^{s} l(v)u^n(v)\Delta v\right]\Delta s\right]\Delta\tau + \sum_{i=1}^{3}\int_{t_0}^{\alpha} h_i(\tau)u^{m_i}(\tau)\Delta\tau + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)u^{m_4}(s)\Delta s\Delta\tau, \quad t \in I, \tag{27}$$

and

$$\lambda_{1qrnm_1m_2m_3m_4} = \sum_{i=1}^{3} m_i k^{m_i-1}\int_{t_0}^{\alpha} h_i(\tau)e_{B_{1qrn}}(\tau, t_0)\Delta\tau + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} m_4 k^{m_4-1}h_5(s)e_{B_{pqrn}}(s, t_0)\Delta s\Delta\tau < 1, \tag{28}$$

then

$$u(t) \le a(t) + \frac{C_{1m_1m_2m_3m_4} + A_{1qrn}(t)}{1 - \lambda_{1qrnm_1m_2m_3m_4}} e_{B_{1qrn}}(t, t_0) \tag{29}$$

for $t \in I$ and for any $k > 0$, where

$$C_{1m_1m_2m_3m_4} = \sum_{i=1}^{3}\int_{t_0}^{\alpha} h_i(\tau)\left[m_i k^{m_i-1}a(\tau) + (1 - m_i)k^{m_i}\right]\Delta\tau + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)\left[m_4 k^{m_4-1}a(s) + (1 - m_4)k^{m_4}\right]\Delta s\Delta\tau, \tag{30}$$

$$A_{1qrn}(t) = \int_{t_0}^{t} f(\tau)\left\{qk^{q-1}a(\tau) + (1 - q)k^q + \int_{t_0}^{\tau}\left[g(s)\left[rk^{r-1}a(s) + (1 - r)k^r\right] + \int_{t_0}^{s} l(v)\left[nk^{n-1}a(v) + (1 - n)k^n\right]\Delta v\right]\Delta s\right\}\Delta\tau, \tag{31}$$

and

$$B_{1qrn}(t) = f(t)\left[qk^{q-1} + \int_{t_0}^{t}\left[rk^{r-1}g(s) + \int_{t_0}^{s} nk^{n-1}l(v)\Delta v\right]\Delta s\right]. \tag{32}$$

When $p = 1$, $q = r = n = m_1 = m_4 = 1$, $h_2(t) \equiv 0$, $h_3(t) \equiv 0$ in Theorem 2.1, we get the following inequality.

Corollary 2.4 *Let $u(t)$, $a(t)$, $f(t)$, $g(t)$, $l(t)$, $h_1(t)$, $h_4(t)$, $h_5(t)$ and α be defined as in Theorem* 2.1. *If $u(t)$ satisfies*

$$u(t) \le a(t) + \int_{t_0}^{t} f(\tau)\left[u(\tau) + \int_{t_0}^{\tau}\left[g(s)u(s) + \int_{t_0}^{s} l(v)u(v)\Delta v\right]\Delta s\right]\Delta\tau \\ + \int_{t_0}^{\alpha} h_1(\tau)u(\tau)\Delta\tau + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)u(s)\Delta s\Delta\tau, \quad t \in I, \tag{33}$$

and

$$\lambda_{11111001} = \int_{t_0}^{\alpha} h_1(\tau)e_{B_{1111}}(\tau, t_0)\Delta\tau \\ + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)e_{B_{1111}}(s, t_0)\Delta s\Delta\tau \\ < 1, \tag{34}$$

then

$$u(t) \le a(t) + \frac{C_{11001} + A_{1111}(t)}{1 - \lambda_{11111001}} e_{B_{1111}}(t, t_0) \tag{35}$$

for $t \in I$ and for any $k > 0$, where

$$C_{11001} = \int_{t_0}^{\alpha} h_1(\tau)a(\tau)\Delta\tau + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)a(s)\Delta s\Delta\tau, \tag{36}$$

$$A_{1111}(t) = \int_{t_0}^{t} f(\tau)\left\{a(\tau) + \int_{t_0}^{\tau}\left[g(s)a(s) + \int_{t_0}^{s} l(v)a(v)\Delta v\right]\Delta s\right\}\Delta\tau, \tag{37}$$

and

$$B_{1111}(t) = f(t)\left[1 + \int_{t_0}^{t}\left[g(s) + \int_{t_0}^{s} l(v)\Delta v\right]\Delta s\right]. \tag{38}$$

Remark 2.1 When $a(t) = u_0$ (u_0 is a constant), $l(t) \equiv 0$ and $h_4(t) \equiv 0$, (35) will deduce inequality (2.1) given in [16], so Corollary 2.4 can be taken as a generalization of Theorem 2.1 given in [16].

Remark 2.2 Though the inequalities discussed in Theorem 2.1 and its corollaries belong to a class of nonlinear Volterra-Fredholm type dynamic integral inequalities on time scales, the estimates obtained in (3), (23), (29) and (35) cannot be derived by some known results given in [16, 17].

Using procedures similar to the proof of Theorem 2.1, we can get a more general result as follows.

Theorem 2.5 *Suppose that* $u(t)$, $a(t)$, $f_i(t)$, $g_i(t)$ $(i = 1,2,\dots,n)$ *and* $h_j(t) \in C_{rd}(I)$ $(j = 1,2,\dots,l)$ *are nonnegative* (*n and l are some positive integers*). *If* $u(t)$ *satisfies*

$$u^p(t) \le a(t) + \sum_{i=1}^{n}\int_{t_0}^{t} f_i(\tau)\left[u^{q_i}(\tau) + \int_{t_0}^{\tau} g_i(s)u^{r_i}(s)\Delta s\right]\Delta\tau + \sum_{j=1}^{l}\int_{t_0}^{\alpha} h_j(\tau)u^{m_j}(\tau)\Delta\tau, \tag{39}$$

for $t \in I$, *where* $p \ge q_i > 0$, $p \ge r_i > 0$, $p \ge m_j > 0$ *and* $k > 0$ *are constants, and*

$$\lambda = \sum_{j=1}^{l}\frac{m_j}{p}k^{(m_j-p)/p}\int_{t_0}^{\alpha} h_j(\tau)e_B(\tau,t_0)\Delta\tau < 1, \tag{40}$$

then

$$u(t) \le \left[a(t) + \frac{C + A(t)}{1-\lambda}e_B(t,t_0)\right]^{\frac{1}{p}} \tag{41}$$

for $t \in I$, *where*

$$C = \sum_{j=1}^{l}\int_{t_0}^{\alpha} h_j(\tau)\left[\frac{m_j}{p}k^{(m_j-p)/p}a(\tau) + \frac{p-m_j}{p}k^{m_j/p}\right]\Delta\tau, \tag{42}$$

$$A(t) = \sum_{i=1}^{n}\int_{t_0}^{t} f_i(\tau)\left\{\frac{q_i}{p}k^{(q_i-p)/p}a(\tau) + \frac{p-q_i}{p}k^{q_i/p} + \int_{t_0}^{\tau} g(s)\left[\frac{r_i}{p}k^{(r_i-p)/p}a(s) + \frac{p-r_i}{p}k^{r_i/p}\right]\Delta s\right\}\Delta\tau, \tag{43}$$

and

$$B(t) = \sum_{i=1}^{n} f_i(t)\left[\frac{q_i}{p}k^{(q_i-p)/p} + \int_{t_0}^{t}\frac{r_i}{p}k^{(r_i-p)/p}g_i(s)\Delta s\right]. \tag{44}$$

Theorem 2.6 *Let* $u(t), f(t), g(t), a(t)$ *be as in Theorem* 2.1 *and* $h_i(t)$ $(i = 1,2,\dots,l) \in C_{rd}(I)$ *are nonnegative. If* $u(t)$ *satisfies*

$$u^p(t) \le a(t) + \int_{t_0}^{t} f(\tau)\left[u^q(\tau) + \int_{t_0}^{\tau} g(s)u^r(s)\Delta s\right]\Delta\tau + \sum_{i=1}^{l}\int_{t_0}^{\alpha} h_i(\tau)H_i\big(\tau,u(\tau)\big)\Delta\tau, \tag{45}$$

for $t \in I$, *where* p, q *and* r *are constants with* $p \ge 1$, $p \ge q > 0$, $p \ge r > 0$ *and* $H_i, L_i : I \times \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ *satisfying*

$$0 \le H_i(t,u) - H_i(t,v) \le L_i(t,v)(u-v), \quad i = 1,2,\dots,l, \tag{46}$$

for $u \ge v \ge 0$ *and*

$$\lambda^*_{pqr} = \frac{k^{(1-p)/p}}{p} \sum_{i=1}^{l} \int_{t_0}^{\alpha} h_i(\tau) L_i\left(\tau, \frac{p-1}{p} k^{1/p} + \frac{k^{(1-p)/p}}{p}\right) e_{B^*_{pqr}}(\tau, t_0) \Delta\tau < 1, \tag{47}$$

then

$$u(t) \le \left[a(t) + \frac{C^*_p + A^*_{pqr}(t)}{1 - \lambda^*_{pqr}} e_{B^*_{pqr}}(t, t_0)\right]^{\frac{1}{p}} \tag{48}$$

for $t \in I$ *and for any* $k > 0$, *where*

$$C^*_p = \sum_{i=1}^{l} \int_{t_0}^{\alpha} h_i(\tau) H_i\left(\tau, \frac{p-1}{p} k^{1/p} + \frac{k^{(1-p)/p}}{p} a(\tau)\right) \Delta\tau, \tag{49}$$

$$\begin{aligned} A^*_{pqr}(t) = {} & \int_{t_0}^{t} f(\tau) \left\{ \frac{q}{p} k^{(q-p)/p} a(\tau) + \frac{p-q}{p} k^{q/p} \right. \\ & \left. + \int_{t_0}^{\tau} g(s) \left[\frac{r}{p} k^{(r-p)/p} a(s) + \frac{p-r}{p} k^{r/p}\right] \Delta s \right\} \Delta\tau, \end{aligned} \tag{50}$$

and

$$B^*_{pqr}(t) = f(t) \left[\frac{q}{p} k^{(q-p)/p} + \int_{t_0}^{t} \frac{r}{p} k^{r-p/p} g(s) \Delta s\right]. \tag{51}$$

Proof Define a function $\overline{z}(t)$ by

$$\overline{z}(t) = \int_{t_0}^{t} f(\tau) \left[u^q(\tau) + \int_{t_0}^{\tau} g(s) u^r(s) \Delta s\right] \Delta\tau + \sum_{i=1}^{l} \int_{t_0}^{\alpha} h_i(\tau) H_i\big(\tau, u(\tau)\big) \Delta\tau, \quad t \in I. \tag{52}$$

Then

$$u^p(t) \le a(t) + \overline{z}(t) \quad \text{or} \quad u(t) \le \big(a(t) + \overline{z}(t)\big)^{\frac{1}{p}}. \tag{53}$$

By Lemma 2.2, for any $k > 0$, we have

$$\begin{aligned} u(t) &\le \big[a(t) + \overline{z}(t)\big]^{\frac{1}{p}} \le \frac{1}{p} k^{(1-p)/p} \big[a(t) + \overline{z}(t)\big] + \frac{p-1}{p} k^{1/p}, \\ u^q(t) &\le \big[a(t) + \overline{z}(t)\big]^{\frac{q}{p}} \le \frac{q}{p} k^{(q-p)/p} \big[a(t) + \overline{z}(t)\big] + \frac{p-q}{p} k^{q/p}, \\ u^r(t) &\le \big[a(t) + \overline{z}(t)\big]^{\frac{r}{p}} \le \frac{r}{p} k^{(r-p)/p} \big[a(t) + \overline{z}(t)\big] + \frac{p-r}{p} k^{r/p}. \end{aligned}$$

Substituting the last relations into (52) and using (46), it follows that

$$\begin{aligned} \overline{z}(t) \le {} & \int_{t_0}^{t} f(\tau) \left\{ \left[\frac{q}{p} k^{q-p/p} \big[a(\tau) + \overline{z}(\tau)\big] + \frac{p-q}{p} k^{q/p}\right] \right. \\ & \left. + \int_{t_0}^{\tau} g(s) \left[\frac{r}{p} k^{(r-p)/p} \big[a(s) + \overline{z}(s)\big] + \frac{p-r}{p} k^{r/p}\right] \Delta s \right\} \Delta\tau \end{aligned}$$

$$+\sum_{i=1}^{l}\int_{t_0}^{\alpha}h_i(\tau)H_i\left(\tau,\frac{1}{p}k^{(1-p)/p}\left[a(\tau)+\overline{z}(\tau)\right]+\frac{p-1}{p}k^{1/p}\right)\Delta\tau$$
$$-\sum_{i=1}^{l}\int_{t_0}^{\alpha}h_i(\tau)H_i\left(\tau,\frac{1}{p}k^{(1-p)/p}a(\tau)+\frac{p-1}{p}k^{1/p}\right)\Delta\tau$$
$$+\sum_{i=1}^{l}\int_{t_0}^{\alpha}h_i(\tau)H_i\left(\tau,\frac{1}{p}k^{(1-p)/p}a(\tau)+\frac{p-1}{p}k^{1/p}\right)\Delta\tau$$
$$=C_p^*+A_{pqr}^*(t)+\int_{t_0}^{t}f(\tau)\left[\frac{q}{p}k^{(q-p)/p}\overline{z}(\tau)+\int_{t_0}^{\tau}g(s)\frac{r}{p}k^{(r-p)/p}\overline{z}(s)\Delta s\right]\Delta\tau$$
$$+\sum_{i=1}^{l}\int_{t_0}^{\alpha}h_i(\tau)\frac{1}{p}k^{(1-p)/p}L_i\left(\tau,\frac{1}{p}k^{(1-p)/p}a(\tau)+\frac{p-1}{p}k^{1/p}\right)\overline{z}(\tau)\Delta\tau,$$

i.e.,

$$\overline{z}(t)\le C_p^*+A_{pqr}^*(t)+\int_{t_0}^{t}f(\tau)\left[\frac{q}{p}k^{(q-p)/p}\overline{z}(\tau)+\int_{t_0}^{\tau}g(s)\frac{r}{p}k^{(r-p)/p}\overline{z}(s)\Delta s\right]\Delta\tau$$
$$+\sum_{i=1}^{l}\int_{t_0}^{\alpha}h_i(\tau)\frac{1}{p}k^{(1-p)/p}L_i\left(\tau,\frac{1}{p}k^{(1-p)/p}a(\tau)+\frac{p-1}{p}k^{1/p}\right)\overline{z}(\tau)\Delta\tau,\quad t\in I. \tag{54}$$

Here C_p^* and A_{pqr}^* and defined in (49) and (50), respectively. From (54) and employing a procedure similar to (9)-(20), we obtain the desired inequality (48). □

3 Applications

In this section, we apply our results to study the boundedness, uniqueness and continuous dependence of the solutions of certain Volterra-Fredholm type dynamic integral equations of the form

$$u^p(t)=a(t)+\int_{t_0}^{t}F\left(\tau,u(\tau),\int_{t_0}^{\tau}G\big(s,u(s)\big)\Delta s\right)\Delta\tau$$
$$+\int_{t_0}^{\alpha}H\left(\tau,u(\tau),\int_{t_0}^{\tau}L\big(s,u(s)\big)\Delta s\right)\Delta\tau, \tag{55}$$

for $t\in I$, where $u,a:I\to\mathbf{R}$, $F,H:I\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, G and $L:I\times\mathbf{R}\to\mathbf{R}$ and $p>0$ is a constant. The following theorem gives a bound on the solutions of equation (55).

Theorem 3.1 *Assume that the functions F, G, H and L in* (55) *satisfy the conditions*

$$\big|F(t,u,v)\big|\le f(t)\big(|u|^q+|v|\big), \tag{56}$$
$$\big|G(t,u)\big|\le g(t)|u|^r, \tag{57}$$

and

$$\big|H(t,u,v)\big|\le\sum_{i=1}^{3}h_i(t)|u|^{m_i}+h_4(t)|v|,\qquad\big|L(t,u)\big|\le h_5(t)|u|^{m_4}, \tag{58}$$

for $t \in I$, $u, v \in \mathbf{R}$, *where* $f(t)$, $g(t)$, $h_i(t)$ $(i = 1,2,3,4,5)$, m_i $(i = 1,2,3,4)$, p, q *and* r *are the same as in Theorem* 2.1, *if*

$$\begin{aligned}\lambda_{pqrm_1m_2m_3m_4} &= \sum_{i=1}^{3} \frac{m_i}{p} k^{\frac{m_i-p}{p}} \int_{t_0}^{\alpha} h_i(\tau) e_{B_{pqr}(\tau)}(\tau, t_0) \Delta\tau \\ &\quad + \int_{t_0}^{\alpha} h_4(\tau) \int_{t_0}^{\tau} \frac{m_4}{p} k^{(m_4-p)/p} h_5(s) e_{B_{pqr}}(s, t_0) \Delta s \Delta\tau \\ &< 1, \end{aligned} \tag{59}$$

then all solutions of equation (55) *satisfy*

$$u(t) \leq \left[a(t) + \frac{C_{pm_1m_2m_3m_4} + A_{pqr}(t)}{1 - \lambda_{pqrm_1m_2m_3m_4}} e_{B_{pqr}}(t, t_0) \right]^{\frac{1}{p}} \tag{60}$$

for $t \in I$ *and for any* $k > 0$, *where*

$$\begin{aligned}C_{pm_1m_2m_3m_4} &= \sum_{i=1}^{3} \int_{t_0}^{\alpha} h_i(\tau) \left[\frac{m_i}{p} k^{(m_i-p)/p} a(\tau) + \frac{p - m_i}{p} k^{m_i/p} \right] \Delta\tau \\ &\quad + \int_{t_0}^{\alpha} h_4(\tau) \int_{t_0}^{\tau} h_5(s) \left[\frac{m_4}{p} k^{(m_4-p)/p} a(s) + \frac{p - m_4}{p} k^{m_4/p} \right] \Delta s \Delta\tau, \end{aligned} \tag{61}$$

$$\begin{aligned}A_{pqr}(t) &= \int_{t_0}^{t} f(\tau) \left\{ \frac{q}{p} k^{(q-p)/p} a(\tau) + \frac{p-q}{p} k^{q/p} \right. \\ &\quad \left. + \int_{t_0}^{\tau} g(s) \left[\frac{r}{p} k^{(r-p)/p} a(s) + \frac{p-r}{p} k^{r/p} \right] \Delta s \right\} \Delta\tau, \end{aligned} \tag{62}$$

and

$$B_{pqr}(t) = f(t) \left[\frac{q}{p} k^{(q-p)/p} + \int_{t_0}^{t} \left[\frac{r}{p} k^{(r-p)/p} g(s) \right] \Delta s \right]. \tag{63}$$

Proof From (55) and the conditions (56)-(58), we have

$$\begin{aligned}\left|u(t)\right| &\leq \left|a(t)\right| + \int_{t_0}^{t} f(\tau) \left[\left|u(\tau)\right|^q + \int_{t_0}^{\tau} g(s) \left|u(s)\right|^r \Delta s \right] \Delta\tau \\ &\quad + \sum_{i=1}^{3} \int_{t_0}^{\alpha} h_i(\tau) \left|u(\tau)\right|^{m_i} \Delta\tau + \int_{t_0}^{\alpha} h_4(\tau) \int_{t_0}^{\tau} h_5(s) \left|u(s)\right|^{m_4} \Delta s \Delta\tau, \end{aligned} \tag{64}$$

for $t \in I$. By a suitable application of Theorem 2.1 to $|u(t)|$ in the last inequality follows the desired (60) immediately. □

Secondly, we consider the uniqueness of the solutions of equation (55).

Theorem 3.2 *Assume that the function* F, G, H *and* L *in* (55) *satisfy the conditions*:

$$\left|F(t, u_1, v_1) - F(t, u_2, v_2)\right| \leq f(t) \left(\left|u_1^p - u_2^p\right| + |v_1 - v_2| \right), \tag{65}$$

$$\left|G(t, u) - G(t, v)\right| \leq g(t) \left|u^p - v^p\right|, \tag{66}$$

$$\begin{aligned} &\bigl|H(t,u_1,v_1)-H(t,u_2,v_2)\bigr| \le h_1(t)\bigl|u_1^p-u_2^p\bigr|+h_4(t)|v_1-v_2|, \\ &\bigl|L(t,u)-L(t,v)\bigr| \le h_5(t)\bigl|u^p-v^p\bigr|, \end{aligned} \tag{67}$$

where f, g, h_1, h_4 *and* h_5 *are the same as in Theorem* 2.1, *and if*

$$\lambda=\int_{t_0}^{\alpha}h_1(\tau)e_B(\tau,t_0)\Delta\tau+\int_{t_0}^{\alpha}h_4(\tau)\int_{t_0}^{\tau}h_5(s)e_B(s,t_0)\Delta s\Delta\tau<1,$$

where

$$B(t)=f(t)\biggl[1+\int_{t_0}^{t}g(s)\Delta s\biggr],$$

then if $p=m/n$ $(m,n\in N)$ *and* m *is odd*, (55) *has at most one solution on* I.

Proof Let $u(t)$ and $v(t)$ be two solutions of equation (55) on I. From (55) and conditions (65), (66) and (67), we have

$$\begin{aligned} &\bigl|u^p(t)-v^p(t)\bigr| \\ &\quad\le \int_{t_0}^{t}\biggl[F\biggl(\tau,u(\tau),\int_{t_0}^{\tau}G\bigl(s,u(s)\bigr)-F\biggl(\tau,v(\tau),\int_{t_0}^{\tau}G\bigl(s,v(s)\bigr)\biggr]\Delta s\biggr)\Delta\tau \\ &\qquad+\int_{t_0}^{\alpha}\biggl[H\biggl(\tau,u(\tau),\int_{t_0}^{\tau}L\bigl(s,u(s)\bigr)\Delta s\biggr)-H\biggl(\tau,v(\tau),\int_{t_0}^{\tau}L\bigl(s,v(s)\bigr)\Delta s\biggr)\biggr]\Delta\tau, \\ &\quad\le \int_{t_0}^{t}f(\tau)\biggl[\bigl|u^p(\tau)-v^p(\tau)\bigr|+\int_{t_0}^{\tau}\bigl|G\bigl(s,u(s)\bigr)-G\bigl(s,v(s)\bigr)\bigr|\Delta s\biggr]\Delta\tau\Delta s \\ &\qquad+\int_{t_0}^{\alpha}h_1(\tau)\bigl|u^p(\tau)-v^p(\tau)\bigr|\Delta\tau+\int_{t_0}^{\alpha}h_4(\tau)\int_{t_0}^{\tau}\bigl|L\bigl(s,u(s)\bigr)-L\bigl(s,v(s)\bigr)\bigr|\Delta s\Delta\tau \\ &\quad\le \int_{t_0}^{t}f(\tau)\biggl[\bigl|u^p(\tau)-v^p(\tau)\bigr|+\int_{t_0}^{\tau}g(s)\bigl|u^p(s)-v^p(s)\bigr|\Delta s\biggr]\Delta\tau \\ &\qquad+\int_{t_0}^{\alpha}h_1(\tau)\bigl|u^p(\tau)-v^p(\tau)\bigr|\Delta\tau+\int_{t_0}^{\alpha}h_4(\tau)\int_{t_0}^{\tau}h_5(s)\bigl|u^p(s)-v^p(s)\bigr|\Delta s\Delta\tau,\quad t\in I. \end{aligned} \tag{68}$$

An application of Corollary 2.4 (with $a(t)=0$) to the function $|u^p(t)-v^p(t)|$ in (68) yields

$$\bigl|u^p(t)-v^p(t)\bigr|\le 0$$

for all $t\in I$. Hence $u^p(t)=v^p(t)$ on I. This completes the proof of Theorem 3.2. □

The next result deal with the continuous dependence of the solutions of (55) on the functions F, G, H and L. For this purpose we consider the following variation of (55):

$$\begin{aligned} u^p(t)={}&\overline{a}(t)+\int_{t_0}^{t}\overline{F}\biggl(\tau,u(\tau),\int_{t_0}^{\tau}\overline{G}\bigl(s,u(s)\bigr)\Delta s\biggr)\Delta\tau \\ &+\int_{t_0}^{\alpha}\overline{H}\biggl(\tau,u(\tau),\int_{t_0}^{\tau}\overline{L}\bigl(s,u(s)\bigr)\Delta s\biggr)\Delta\tau, \end{aligned} \tag{$\overline{55}$}$$

for $t \in I$, where $\overline{F}, \overline{H} : I \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$, $\overline{G}, \overline{L} : I \times \mathbf{R} \longrightarrow \mathbf{R}$ and $p > 0$ is a constant as in (55).

Theorem 3.3 *Consider* (55) *and* ($\overline{55}$). *If*

(i)

$$\begin{aligned}
&\left|F(t,u_1,v_1) - F(t,u_2,v_2)\right| \le f(t)\left(\left|u_1^p - u_2^p\right| + |v_1 - v_2|\right),\\
&\left|G(t,u) - G(t,v)\right| \le g(t)\left|u^p - v^p\right|,\\
&\left|H(t,u_1,v_1) - H(t,u_2,v_2)\right| \le h_1(t)\left|u_1^p - u_2^p\right| + h_4(t)|v_1 - v_2|,\\
&\left|L(t,u) - L(t,v)\right| \le h_5(t)\left|u^p - v^p\right|;
\end{aligned}$$

(ii) $|a(t) - \overline{a}(t)| \le \varepsilon/2$;

(iii) $\lambda = \int_{t_0}^{\alpha} h_1(\tau)e_B(\tau,t_0)\Delta\tau + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)e_B(s,t_0)\Delta s\Delta\tau < 1$, *where* $B(t) = f(t)[1 + \int_{t_0}^{t} g(s)\Delta s]$;

(iv) *for all solutions* $\overline{u}$ *of* ($\overline{55}$),

$$\int_{t_0}^{t}\left|F\left(\tau,\overline{u}(\tau),\int_{t_0}^{\tau} G\left(s,\overline{u}(s)\right)\Delta s\right) - \overline{F}\left(\tau,\overline{u}(\tau),\int_{t_0}^{\tau} \overline{G}\left(s,\overline{u}(s)\right)\Delta s\right)\right|\Delta\tau \le \varepsilon/4$$

and

$$\int_{t_0}^{\alpha}\left|H\left(\tau,\overline{u}(\tau),\int_{t_0}^{\tau} L\left(s,\overline{u}(s)\right)\Delta s\right) - \overline{H}\left(\tau,\overline{u}(\tau),\int_{t_0}^{\tau} \overline{L}\left(s,\overline{u}(s)\right)\Delta s\right)\right|\Delta\tau < \varepsilon/4,$$

for $t \in I$ *and* $u_1, u_2, v_1, v_2 \in \mathbf{R}$, *where* $\varepsilon > 0$ *is an arbitrary constant, then*

$$\left|u^p(t) - \overline{u}^p(t)\right| \le \varepsilon\left[1 + \frac{C + A(t)}{1-\lambda}e_B(t,t_0)\right], \tag{69}$$

for $t \in I$ *where*

$$C = \int_{t_0}^{\alpha} h_1(\tau)\Delta\tau + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)\Delta s\Delta\tau$$

and

$$A(t) = \int_{t_0}^{t} f(\tau)\left[1 + \int_{t_0}^{\tau} g(s)\Delta s\right]\Delta\tau.$$

Hence $u^p(t)$ *depends continuously on* F, G, H *and* L. *In particular, if* u *does not change sign, it depends continuously on* F, G, H *and* L.

Proof Let $u(t)$ and $\overline{u}(t)$ ba solutions of (55) and ($\overline{55}$), respectively. Then from (55) and ($\overline{55}$), we have

$$\begin{aligned}
&\left|u^p(t) - \overline{u}^p(t)\right|\\
&\quad \le \left|a(t) - \overline{a}(t)\right| + \int_{t_0}^{t}\left|F\left(\tau,u(\tau),\int_{t_0}^{\tau} G\left(s,u(s)\right)\Delta s\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&- \overline{F}\left(\tau, \overline{u}(\tau), \int_{t_0}^{\tau} \overline{G}(s, \overline{u}(s))\Delta s\right)\Big|\Delta\tau \\
&+ \int_{t_0}^{\alpha}\Big|H\left(\tau, u(\tau), \int_{t_0}^{\tau} L(s, u(s))\Delta s\right) - \overline{H}\left(\tau, \overline{u}(\tau), \int_{t_0}^{\tau} \overline{L}(s, \overline{u}(s))\Delta s\right)\Big|\Delta\tau \\
\le{}& \varepsilon/2 + \int_{t_0}^{t}\Big|F\left(\tau, u(\tau), \int_{t_0}^{\tau} G(s, u(s))\Delta s\right) - F\left(\tau, \overline{u}(\tau), \int_{t_0}^{\tau} G(s, \overline{u}(s))\Delta s\right)\Big|\Delta\tau \\
&+ \int_{t_0}^{t}\Big|F\left(\tau, \overline{u}(\tau), \int_{t_0}^{\tau} G(s, \overline{u}(s))\Delta s\right) - \overline{F}\left(\tau, \overline{u}(\tau), \int_{t_0}^{\tau} \overline{G}(s, \overline{u}(s))\Delta s\right)\Big|\Delta\tau \\
&+ \int_{t_0}^{\alpha}\Big|H\left(\tau, u(\tau), \int_{t_0}^{\tau} L(s, u(s))\Delta s\right) - H\left(\tau, \overline{u}(\tau), \int_{t_0}^{\tau} L(s, \overline{u}(s))\Delta s\right)\Big|\Delta\tau \\
&+ \int_{t_0}^{\alpha}\Big|H\left(\tau, \overline{u}(\tau), \int_{t_0}^{\tau} L(s, \overline{u}(s))\Delta s\right) - \overline{H}\left(\tau, \overline{u}(\tau), \int_{t_0}^{\tau} \overline{L}(s, \overline{u}(s))\Delta s\right)\Big|\Delta\tau \\
\le{}& \varepsilon + \int_{t_0}^{t} f(\tau)\left[\left|u^p(\tau) - \overline{u}^p(\tau)\right| + \int_{t_0}^{\tau} g(s)\left|u^p(s) - \overline{u}^p(s)\right|\Delta s\right]\Delta\tau \\
&+ \int_{t_0}^{\alpha} h_1(\tau)\left|u^p(\tau) - \overline{u}^p(\tau)\right|\Delta\tau + \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)\left|u^p(s) - \overline{u}^p(s)\right|\Delta s\Delta\tau,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\left|u^p(t) - \overline{u}^p(t)\right| \le{}& \varepsilon + \int_{t_0}^{t} f(\tau)\left[\left|u^p(\tau) - \overline{u}^p(\tau)\right| + \int_{t_0}^{\tau} g(s)\left|u^p(s) - \overline{u}^p(s)\right|\Delta s\right]\Delta\tau \\
&+ \int_{t_0}^{\alpha} h_1(\tau)\left|u^p(\tau) - \overline{u}^p(\tau)\right|\Delta\tau \\
&+ \int_{t_0}^{\alpha} h_4(\tau)\int_{t_0}^{\tau} h_5(s)\left|u^p(s) - \overline{u}^p(s)\right|\Delta s\Delta\tau, \quad t \in I. \qquad (70)
\end{aligned}
$$

Now by applying Corollary 2.4 (with $a(t) = \varepsilon$) to the function $|u^p(t) - \overline{u}^p(t)|$, the last inequality provides the desired inequality (69). Evidently, if the function $A(t)$ and $e_B(t, t_0)$ are bounded on I,

$$
\left|u^p(t) - \overline{u}^p(t)\right| \le \varepsilon M
$$

for some $M > 0$ and $t \in I$. Hence u^p depends continuously on F, G, H and L. This completes the proof of Theorem 3.3. □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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