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Synchronization of fractional chaotic systems based on a simple Lyapunov function

Tianzeng Li[1,2*], Yu Wang[1,3] and Chao Zhao[4]

*Correspondence:
litianzeng27@163.com
[1]School of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, 643000, China
[2]Artificial Intelligence Key Laboratory of Sichuan Province, Zigong, 643000, China
Full list of author information is available at the end of the article

Abstract

In this paper the synchronization of fractional-order chaotic systems and a new property of fractional derivatives are studied. Then we propose a new fractional-order extension of Lyapunov direct method to control the fractional-order chaotic systems. A new synchronization method and a linear feedback controller are given to achieve the synchronization of fractional-order chaotic systems based on a simple Lyapunov candidate function. The proposed synchronization method can be applied to the synchronization of an arbitrary fractional-order chaotic system. This method is universal, simple, and theoretically rigorous. Numerical simulations of three fractional-order chaotic systems to verify the effectiveness and the universality of the proposed method.

Keywords: Lyapunov function; fractional order; synchronization; chaos

1 Introduction

Fractional calculus is a topic of more than 300 years old. The idea of fractional calculus has been known since the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695 where a half-order derivative was mentioned. As the generalization of integer-order dynamic systems, the fractional-order dynamic systems provide better mathematical models for some actual physical and engineering systems [1, 2]. The fractional-order nonlinear dynamic systems have many dynamic behaviors which are similar to the integer-order systems, such as chaos, bifurcation, and attractor [3–6]. The fractional-order chaotic systems are extensively studied due to their potential applications in biology, information, chemistry, physics, and other fields [7–10]. We can find numerous applications in electrochemistry, viscoelasticity, porous media, control, and electromagnetics [11–14].

As is well known, synchronization control of chaos is very important but also very difficult for the chaotic systems. There are some methods which are proposed to control fractional-order chaotic systems [5, 15–17], such as backstepping controller [15], linear feedback controller [5], the Lyapunov equation-based method [16, 18], sliding mode method [17], the iterative learning control method [19–21] and the monotone iterative method [22, 23]. In most of these cases, the stability of the whole controlled system has to be analyzed using the fractional-order techniques as well. For the fractional-order linear time invariant systems, the stability can easily be proved using the method proposed by Matignon [24]. The stability of a fractional-order nonlinear time varying system was proposed by Diethelm [25], but the result is valid only for scalar fractional-order systems. Hence, in order to prove the stability of fractional-order nonlinear and time varying systems, some other methods must be applied. The important one of these methods is the fractional-order extension of the Lyapunov direct method, which was proposed by Li *et al.* [26]. And in [27] the authors proposed the Lyapunov direct method to prove the stability of fractional-order nonlinear system with time delay. As is well known, it is difficult to find a suitable Lyapunov candidate function to prove the stability of fractional systems. Some authors have presented Lyapunov functions to prove the stability of fractional-order systems [28, 29]. And some Lyapunov functions have been proposed related to fractional-order sliding mode control [30]. However, we find that the Lyapunov functions which were proposed in these papers are not simple, and they are valid for fractional-order system with specific characteristics.

In this paper, we first propose a new property of fractional-order derivatives which can help us find a simple Lyapunov function. Then we propose a new fractional-order extension of Lyapunov direct method and a control method for synchronizing an arbitrary fractional-order chaotic system. Some sufficient conditions of synchronization for the fractional-order chaotic systems are proposed based on a simple Lyapunov function. The proposed approach is simple, universal and theoretically rigorous. Numerical simulation results of the synchronization of the fractional-order unified system, the fractional-order Liu's system, and the fractional-order Lorenz system demonstrate the effectiveness and the universality of the proposed method.

This paper is organized in the following manner: In Section 2 the preliminaries and some definitions are presented. Some synchronization criteria of fractional-order chaotic systems are proposed in Section 3. In Section 4, numerical simulation of three fractional chaotic systems shows the effectiveness and the universality of the control method. Finally, conclusions are in Section 5.

2 Preliminaries and definitions

2.1 Fractional derivative and numerical solution of differential equation

In the last decades, fractional calculus has been the subject of worldwide attention due to its broad range of applications in many fields, which indicates that it is an important role in modern science [1, 2]. Some definitions for fractional derivatives were studied in recent years. In this paper, two commonly used definitions of fractional derivatives, the Riemann-Liouville (RL) and the Caputo definition (C), are given as follows.

Definition 1 ([1, 2]) The fractional integral ${}_aD_t^{-\alpha}$ of the function $f(t)$ is given as follows:

$$ {}_aD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)\,d\tau, \tag{1}$$

where the fractional order $\alpha > 0$ and $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ is the gamma function.

Definition 2 ([1, 2]) The Caputo derivative of the function $f(t)$ with order α is defined as

$$ {}_a^CD_t^\alpha f(t) = {}_aD_t^{-(n-\alpha)}\frac{d^n}{dt^n}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)\,d\tau, \tag{2}$$

where $n-1 < \alpha < n$, $n \in Z^+$.

Definition 3 ([1, 2]) The Riemann-Liouville derivative of the function $f(t)$ with order α is given as

$$ {}_{a}^{RL}D_t^{\alpha} f(t) = \frac{d^n}{dt^n} {}_a D_t^{-(n-\alpha)} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \tag{3} $$

where $n-1 < \alpha < n$, $n \in Z^+$.

As is well known, various numerical methods have been applied to solve the fractional-order equation, such as the power series method [1], the Millin transform method [1], and the GMMP scheme (Gorenflo-Mainardi-Moretti-Paradisi) [31]. In this section, we adopt the improved version of Adams-Bashforth-Moulton algorithm [2, 32] to numerically solve the fractional differential equations, which is proposed based on the predictor-correctors scheme. For explaining this method, the following differential equation is considered:

$$ \begin{aligned} &{}_0D_t^{\alpha} y(t) = f\big(t, y(t)\big), \quad 0 \le t \le T, \\ &y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \ldots, m-1, \end{aligned} \tag{4} $$

where ${}_0D_t^{\alpha}$ denotes the fractional derivative of the Caputo (or Riemann-Liouville) definition. Equation (4) is equivalent to the Volterra integral equation:

$$ y(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, y(s)\big)\, ds. \tag{5} $$

Then let $h = T/N$, $t_n = nh$ $(n = 0, 1, \ldots, N)$. The integral equation (5) can be discretized as follows:

$$ y_h(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} y_0^{(k)} \frac{t^k}{k!} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f\big(t_{n+1}, y_h^p(t_{n+1})\big) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f\big(t_j, y_h(t_j)\big), \tag{6} $$

where

$$ y_h^p(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f\big(t_j, y_h(t_j)\big), \tag{7} $$

and

$$ a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n, \\ 1, & j = n+1, \end{cases} \tag{8} $$

$$ b_{j,n+1} = \frac{h^{\alpha}}{\alpha}\big((n+1-j)^{\alpha} - (n-j)^{\alpha}\big), \quad 0 \le j \le n. \tag{9} $$

The error of this approximation is given as follows:

$$ \max_{j=0,1,\ldots,N} \big|y(t_j) - y_h(t_j)\big| = O\big(h^p\big), \tag{10} $$

where $p = \min(2, 1+\alpha)$.

2.2 Some properties of the fractional derivative

In this paper we mainly consider the fractional order of the chaotic system to be $0 < \alpha < 1$. Then some general properties of the fractional-order derivative ${}_aD_t^\alpha$ (Riemann-Liouville and Caputo definition) are described as follows [1, 2].

Property 1 (Leibniz rule for fractional differentiation [1, 2]) The Leibniz rule for fractional differentiation is defined by

$$ {}_aD_t^\alpha\big(\varphi(t)f(t)\big) = \sum_{k=0}^{\infty}\binom{\alpha}{k}\varphi^{(k)}(t){}_aD_t^{\alpha-k}f(t), \tag{11}$$

if $f(t)$ and $\varphi(t)$ and all their derivatives are continuous in the interval $[a, t]$.

In the following, two new properties of fractional-order derivatives is proposed, which can help us find a simple Lyapunov function.

Property 2 (Caputo definition ${}_a^CD_t^\alpha$) Let $\boldsymbol{x}(t) = (x_1(t), \dots, x_n(t))^T \in \mathbb{R}^n$ have continuous derivatives in $[a, t]$, for any positive definite matrix P, then

$$ {}_a^CD_t^\alpha\left(\frac{1}{2}\boldsymbol{x}^T(t)P\boldsymbol{x}(t)\right) \leq \boldsymbol{x}^T(t)P{}_a^CD_t^\alpha\boldsymbol{x}(t), \quad \forall\alpha \in (0, 1), \tag{12}$$

where the ${}_a^CD_t^\alpha$ denotes the Caputo fractional derivative.

Proof Firstly, let

$$ f(t) = {}_a^CD_t^\alpha\left(\frac{1}{2}\boldsymbol{x}^T(t)P\boldsymbol{x}(t)\right) - \boldsymbol{x}^T(t)P{}_a^CD_t^\alpha\boldsymbol{x}(t), \tag{13}$$

then proving that equation (12) is true is equivalent to proving that

$$ f(t) = \frac{1}{2}{}_a^CD_t^\alpha\big(\boldsymbol{x}^T(t)P\boldsymbol{x}(t)\big) - \boldsymbol{x}^T(t)P{}_a^CD_t^\alpha\boldsymbol{x}(t) \leq 0. \tag{14}$$

Due to the Caputo definition (2), the function (13) can be written as

$$\begin{aligned} f(t) &= \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{\boldsymbol{x}^T(\tau)P\dot{\boldsymbol{x}}(\tau)}{(t-\tau)^\alpha}\,d\tau \\ &\quad - \frac{1}{\Gamma(1-\alpha)}\boldsymbol{x}^T(t)P\int_a^t \frac{\dot{\boldsymbol{x}}(\tau)}{(t-\tau)^\alpha}\,d\tau \\ &= \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{\boldsymbol{x}^T(\tau)P\dot{\boldsymbol{x}}(\tau) - \boldsymbol{x}^T(t)P\dot{\boldsymbol{x}}(\tau)}{(t-\tau)^\alpha}\,d\tau \\ &= \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{(\boldsymbol{x}^T(\tau) - \boldsymbol{x}^T(t))P\dot{\boldsymbol{x}}(\tau)}{(t-\tau)^\alpha}\,d\tau \\ &\overset{\boldsymbol{y}(\tau)=\boldsymbol{x}(\tau)-\boldsymbol{x}(t)}{=\!=\!=} \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{\boldsymbol{y}^T(\tau)P\dot{\boldsymbol{y}}(\tau)}{(t-\tau)^\alpha}\,d\tau \\ &\overset{\boldsymbol{y}^T(\tau)P\dot{\boldsymbol{y}}(\tau)\,d\tau=\frac{1}{2}d(\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau))}{=\!=\!=\!=\!=} \frac{1}{2\Gamma(1-\alpha)}\int_a^t (t-\tau)^{-\alpha}\,d\big(\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)\big). \end{aligned} \tag{15}$$

Let us integrate by parts equation (15), the function $f(t)$ can be rewritten as

$$f(t) = \frac{1}{2\Gamma(1-\alpha)} \frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^\alpha}\bigg|_a^t - \frac{\alpha}{2\Gamma(1-\alpha)} \int_a^t \frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^{\alpha+1}}\,d\tau$$
$$= \frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{2\Gamma(1-\alpha)(t-\tau)^\alpha}\bigg|_{\tau=t} - \frac{\boldsymbol{y}^T(a)P\boldsymbol{y}(a)}{2\Gamma(1-\alpha)(t-a)^\alpha} - \frac{\alpha}{2\Gamma(1-\alpha)} \int_a^t \frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^{\alpha+1}}\,d\tau. \quad (16)$$

Let us check the first term of equation (16), which has an indetermination at $\tau = t$, then we can analyze the corresponding limitation by the L'Hopital rule. We have

$$\lim_{\tau\to t} \frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^\alpha} = \lim_{\tau\to t} \frac{2\boldsymbol{y}^T(\tau)P\dot{\boldsymbol{y}}(\tau)}{-\alpha(t-\tau)^{\alpha-1}} = \lim_{\tau\to t} \frac{2\boldsymbol{y}^T(\tau)P\dot{\boldsymbol{y}}(\tau)(t-\tau)^{1-\alpha}}{-\alpha} = 0. \quad (17)$$

And the matrix P is positive definite, then

$$\frac{\boldsymbol{y}^T(a)P\boldsymbol{y}(a)}{2\Gamma(1-\alpha)(t-a)^\alpha} \geq 0 \quad (18)$$

and

$$\frac{\alpha}{2\Gamma(1-\alpha)} \int_a^t \frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^{\alpha+1}}\,d\tau \geq 0. \quad (19)$$

Hence, we obtain $f(t) \leq 0$, *i.e.* the conclusion (12) is clearly true. □

Property 3 (Riemann-Liouville definition ${}_a^R D_t^\alpha$) Let $\boldsymbol{x}(t) = (x_1(t),\ldots,x_n(t))^T \in \mathbb{R}^n$ have a continuous derivatives in $[a,t]$, for any positive definite matrix P, then

$${}_a^R D_t^\alpha\left(\frac{1}{2}\boldsymbol{x}^T(t)P\boldsymbol{x}(t)\right) \leq \boldsymbol{x}^T(t)P{}_a^R D_t^\alpha \boldsymbol{x}(t), \quad \forall\alpha \in (0,1), \quad (20)$$

where the ${}_a^R D_t^\alpha$ denotes the Riemann-Liouville fractional derivative.

Proof Firstly, let

$$f(t) = {}_a^R D_t^\alpha\left(\frac{1}{2}\boldsymbol{x}^T(t)P\boldsymbol{x}(t)\right) - \boldsymbol{x}^T(t)P{}_a^R D_t^\alpha \boldsymbol{x}(t), \quad (21)$$

then proving that equation (20) is true is equivalent to proving that

$$f(t) = \frac{1}{2}{}_a^R D_t^\alpha\left(\boldsymbol{x}^T(t)P\boldsymbol{x}(t)\right) - \boldsymbol{x}^T(t)P{}_a^R D_t^\alpha \boldsymbol{x}(t) \leq 0. \quad (22)$$

Due to the Riemann-Liouville definition (3), the function (21) can be written as

$$f(t) = \frac{1}{2\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{\boldsymbol{x}^T(\tau)P\boldsymbol{x}(\tau)}{(t-\tau)^\alpha}\,d\tau - \frac{1}{\Gamma(1-\alpha)}\boldsymbol{x}^T(t)P\frac{d}{dt}\int_a^t \frac{\boldsymbol{x}(\tau)}{(t-\tau)^\alpha}\,d\tau$$
$$= \frac{1}{\Gamma(1-\alpha)}\left\{\frac{1}{2}\frac{d}{dt}\int_a^t \frac{\boldsymbol{x}^T(\tau)P\boldsymbol{x}(\tau)}{(t-\tau)^\alpha}\,d\tau - \boldsymbol{x}^T(t)\frac{d}{dt}\int_a^t \frac{P\boldsymbol{x}(\tau)}{(t-\tau)^\alpha}\,d\tau\right\}. \quad (23)$$

Let

$$g(t) = \frac{1}{2}\frac{d}{dt}\int_a^t \frac{\boldsymbol{x}^T(\tau)P\boldsymbol{x}(\tau)}{(t-\tau)^\alpha}\,d\tau - \boldsymbol{x}^T(t)\frac{d}{dt}\int_a^t \frac{P\boldsymbol{x}(\tau)}{(t-\tau)^\alpha}\,d\tau, \tag{24}$$

then

$$\begin{aligned}
g(t) \quad &\overset{y=t-\tau}{=\!=\!=} \quad \frac{1}{2}\frac{d}{dt}\int_0^{t-a} \frac{\boldsymbol{x}^T(t-y)P\boldsymbol{x}(t-y)}{y^\alpha}\,dy - \boldsymbol{x}^T(t)\frac{d}{dt}\int_0^{t-a}\frac{P\boldsymbol{x}(t-y)}{y^\alpha}\,dy \\
&= \quad \frac{1}{2}\frac{\boldsymbol{x}^T(a)P\boldsymbol{x}(a)}{(t-a)^\alpha} + \int_0^{t-a}\frac{\boldsymbol{x}^T(t-y)P\dot{\boldsymbol{x}}(t-y)}{y^\alpha}\,dy \\
&\quad\quad - \boldsymbol{x}^T(t)\left\{\frac{P\boldsymbol{x}(a)}{(t-a)^\alpha} + \int_0^{t-a}\frac{P\dot{\boldsymbol{x}}(t-y)}{y^\alpha}\,dy\right\} \\
&= \quad \frac{1}{2}\frac{\boldsymbol{x}^T(a)P\boldsymbol{x}(a)}{(t-a)^\alpha} - \frac{\boldsymbol{x}^T(t)P\boldsymbol{x}(a)}{(t-a)^\alpha} + \int_0^{t-a}\frac{(\boldsymbol{x}^T(t-y)-\boldsymbol{x}^T(t))P\dot{\boldsymbol{x}}(t-y)}{y^\alpha}\,dy \\
&\overset{\tau=t-y}{=\!=\!=} \quad \frac{1}{2}\frac{\boldsymbol{x}^T(a)P\boldsymbol{x}(a)}{(t-a)^\alpha} - \frac{\boldsymbol{x}^T(t)P\boldsymbol{x}(a)}{(t-a)^\alpha} + \int_a^t \frac{(\boldsymbol{x}^T(\tau)-\boldsymbol{x}^T(t))P\dot{\boldsymbol{x}}(\tau)}{(t-\tau)^\alpha} \\
&\overset{\boldsymbol{y}(\tau)=\boldsymbol{x}(\tau)-\boldsymbol{x}(t)}{=\!=\!=\!=\!=} \quad \frac{1}{2}\frac{\boldsymbol{x}^T(a)P\boldsymbol{x}(a)}{(t-a)^\alpha} - \frac{\boldsymbol{x}^T(t)P\boldsymbol{x}(a)}{(t-a)^\alpha} + \int_a^t (t-\tau)^{-\alpha}\,d\left(\frac{1}{2}\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)\right).
\end{aligned} \tag{25}$$

Let us integrate by parts equation (25), the function $g(t)$ can be rewritten as

$$\begin{aligned}
g(t) &= \frac{1}{2}\frac{\boldsymbol{x}^T(a)P\boldsymbol{x}(a)}{(t-a)^\alpha} - \frac{\boldsymbol{x}^T(t)P\boldsymbol{x}(a)}{(t-a)^\alpha} + \frac{1}{2}\frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^\alpha}\bigg|_a^t - \frac{\alpha}{2}\int_a^t \frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^{\alpha+1}}\,d\tau \\
&= \frac{1}{2}\lim_{\tau\to t}\frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^\alpha} - \frac{\boldsymbol{x}^T(t)P\boldsymbol{x}(t)}{2(t-a)^\alpha} - \frac{\alpha}{2}\int_a^t \frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^{\alpha+1}}\,d\tau.
\end{aligned} \tag{26}$$

Let us check the first term of equation (26), which has an indetermination at $\tau = t$, then we can analyze the corresponding limitation by the L'Hopital rule:

$$\lim_{\tau\to t}\frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^\alpha} = \lim_{\tau\to t}\frac{2\boldsymbol{y}^T(\tau)P\dot{\boldsymbol{y}}(\tau)}{-\alpha(t-\tau)^{\alpha-1}} = \lim_{\tau\to t}\frac{2\boldsymbol{y}^T(\tau)P\dot{\boldsymbol{y}}(\tau)(t-\tau)^{1-\alpha}}{-\alpha} = 0. \tag{27}$$

And the matrix P is positive definite, then

$$\frac{\boldsymbol{x}^T(a)P\boldsymbol{x}(a)}{2(t-a)^\alpha} \geq 0 \tag{28}$$

and

$$\frac{\alpha}{2}\int_a^t \frac{\boldsymbol{y}^T(\tau)P\boldsymbol{y}(\tau)}{(t-\tau)^{\alpha+1}}\,d\tau \geq 0. \tag{29}$$

Hence, we obtain $g(t) \leq 0$, *i.e.* $f(t) \leq 0$, then the conclusion (20) is clearly true. □

Remark 1 If the positive definite matrix is an identity matrix, *i.e.* $P = I$, the property (2) and (3) can be written as

$${}_aD_t^\alpha\left(\frac{1}{2}\boldsymbol{x}^T(t)\boldsymbol{x}(t)\right) \leq \boldsymbol{x}^T(t)\,{}_aD_t^\alpha\boldsymbol{x}(t), \quad \forall\alpha \in (0,1). \tag{30}$$

2.3 Stability of fractional-order system

The fractional-order nonlinear system is given as

$$ {}_0D_t^{\alpha}\boldsymbol{x}(t) = \boldsymbol{f}\bigl(t,\boldsymbol{x}(\boldsymbol{t})\bigr), \quad \boldsymbol{x}(0) = \boldsymbol{c}, \tag{31} $$

where $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)^T$ is a vector function and f_i $(i = 1, 2, \ldots, n)$ is continuous differential nonlinear functions; α is the fractional order of the derivative; $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$; $\boldsymbol{x}(0) = (c_1, c_2, \ldots, c_n)^T$ is the initial value; ${}_0D_t^{\alpha}$ denotes the Caputo (or Riemann-Liouville) fractional-order derivative operator. The equilibrium points of this system are calculated by solving $\boldsymbol{f}(\boldsymbol{x}^*) = 0$. For this fractional-order nonlinear system, the fractional-order extension of Lyapunov direct method has been proposed, which is obtained as follows [26].

Theorem 1 *Let $\boldsymbol{x} = 0$ be an equilibrium point for the nonautonomous fractional-order system* (31). *Assume that there exists a Lyapunov function $V(t, \boldsymbol{x}(t))$ and class-K functions α_i $(i = 1, 2, 3)$ satisfying*

$$ \alpha_1\bigl(\|\boldsymbol{x}(t)\|\bigr) \le V\bigl(t, \boldsymbol{x}(t)\bigr) \le \alpha_2\bigl(\|\boldsymbol{x}(t)\|\bigr), \tag{32} $$

$$ {}_0D_t^{\beta} V\bigl(t, \boldsymbol{x}(t)\bigr) \le -\alpha_3\bigl(\|\boldsymbol{x}(t)\|\bigr), \tag{33} $$

where $\beta \in (0, 1)$. Then the equilibrium point of system (31) *is asymptotically stable.*

In the following, based on the fractional-order extension of the Lyapunov direct method and the new property of fractional derivatives, we will find a suitable Lyapunov function and propose the stability condition of the fractional chaotic system.

Theorem 2 *Take the fractional-order system*

$$ {}_0D_t^{\boldsymbol{\alpha}}\boldsymbol{x}(t) = \boldsymbol{f}\bigl(\boldsymbol{x}(t)\bigr), \tag{34} $$

where $\alpha \in (0, 1)$ and ${}_0D_t^{\boldsymbol{\alpha}}$ denotes the Caputo (*or Riemann-Liouville*) *derivative. Without loss of generality, let $\boldsymbol{x}^* = 0$ be the equilibrium point and $\boldsymbol{x}(t) \in \mathbb{R}^n$. If there exists a positive definite matrix P, which satisfies*

$$ \boldsymbol{x}^T(t)P\boldsymbol{f}\bigl(\boldsymbol{x}(t)\bigr) \le 0, \tag{35} $$

then the origin of the system (34) *is asymptotically stable.*

Proof Since P is a positive definite matrix, we introduce a Lyapunov function

$$ V\bigl(\boldsymbol{x}(t)\bigr) = \frac{1}{2}\boldsymbol{x}^T(t)P\boldsymbol{x}(t). \tag{36} $$

It follows from Property 2 and Property 3 that

$$ {}_0D_t^{\alpha} V\bigl(\boldsymbol{x}(t)\bigr) \le \boldsymbol{x}^T(t)P {}_0D_t^{\alpha}\boldsymbol{x}(t) = \boldsymbol{x}^T(t)P\boldsymbol{f}\bigl(\boldsymbol{x}(t)\bigr). \tag{37} $$

And since $\boldsymbol{x}^T(t)P\boldsymbol{f}(\boldsymbol{x}(t)) \leq 0$, the fractional-order derivative of the Lyapunov function is negative definite. Using the relation between positive definite functions and class-K functions in [33], it follows from Theorem 1 that the origin of the system (34) is asymptotically stable. □

3 Synchronization of fractional-order chaotic system

In the following, we would give the synchronization criterion of the fractional-order chaotic system. Firstly we rewrite the fractional-order chaotic system (34) as follows:

$$ {}_0D_t^\alpha \boldsymbol{x}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) = A\boldsymbol{x}(t) + \boldsymbol{g}\big(\boldsymbol{x}(t)\big)\boldsymbol{x}(t), \tag{38} $$

where $A\boldsymbol{x}(t)$ is the linear part of system (34), $\boldsymbol{g}(\boldsymbol{x}(t))\boldsymbol{x}(t)$ is the nonlinear part of system (34). This way of writing is very general and almost all fractional-order chaotic system can be written in the form (38). We consider the system (38) as the drive system, then the response system is given as

$$ {}_0D_t^\alpha \boldsymbol{y}(t) = \boldsymbol{f}\big(\boldsymbol{y}(t)\big) = A\boldsymbol{y}(t) + \boldsymbol{g}\big(\boldsymbol{y}(t)\big)\boldsymbol{y}(t). \tag{39} $$

To realize the synchronization of two system (38) and (39), we add a linear feedback control input to the response system (39). It is well known that the linear controller has many advantages: (1) it is very simple; (2) it is easily realized experimentally; (3) it is more suitable for engineering applications than other controllers.

Hence the controlled response system (39) with the linear feedback control input is given as

$$ {}_0D_t^\alpha \boldsymbol{y}(t) = A\boldsymbol{y}(t) + \boldsymbol{g}\big(\boldsymbol{y}(t)\big)\boldsymbol{y}(t) - K\big(\boldsymbol{y}(t) - \boldsymbol{x}(t)\big), \tag{40} $$

where $K(\boldsymbol{y}(t) - \boldsymbol{x}(t))$ is the linear feedback control input, and the feedback gain matrix $K \in \mathbb{R}^{n\times n}$ needs to be determined.

Let the synchronization error $\boldsymbol{e}(t) = \boldsymbol{y}(t) - \boldsymbol{x}(t)$, the error system from (38) and (40) is obtained:

$$ {}_0D_t^\alpha \boldsymbol{e}(t) = \boldsymbol{f}\big(\boldsymbol{y}(t)\big) - \boldsymbol{f}\big(\boldsymbol{x}(t)\big) = A\boldsymbol{e}(t) + B_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{e}(t) - K\boldsymbol{e}(t), \tag{41} $$

where $B_{\boldsymbol{x},\boldsymbol{y}}$ is a bounded matrix with its elements depending on $\boldsymbol{x}$ and $\boldsymbol{y}$.

Then we can easily see that systems (38) and (40) are synchronized if and only if the error system (41) is asymptotically stable at the origin. Therefore, our aim is to design a suitable feedback gain matrix K such that the error system (41) is asymptotically stable.

Theorem 3 *The controlled fractional-order error system* (41) *is asymptotically stable at the origin, i.e. the systems* (38) *and* (40) *are asymptotically synchronized, if the feedback gain matrix K makes the symmetric matrix*

$$ S = \frac{(PA + PB_{\boldsymbol{x},\boldsymbol{y}} - PK)^T + (PA + PB_{\boldsymbol{x},\boldsymbol{y}} - PK)}{2} \tag{42} $$

negative definite for all $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$, where P is a positive definite matrix.

Proof For the controlled error system (41), we introduce the Lyapunov function

$$V(\boldsymbol{e}) = \frac{1}{2}\boldsymbol{e}^T(t)P\boldsymbol{e}(t), \tag{43}$$

where P is a positive definite matrix. It follows from Properties 2, 3 that

$$\begin{aligned} {}_0D_t^\alpha V(\boldsymbol{e}) &= \frac{1}{2}{}_0D_t^\alpha\big(\boldsymbol{e}^T(t)P\boldsymbol{e}(t)\big) \\ &\leq \boldsymbol{e}^T(t)P_0D_t^\alpha\boldsymbol{e}(t) \\ &= \boldsymbol{e}^T(t)P\big(A\boldsymbol{e}(t) + B_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{e}(t) - K\boldsymbol{x}(t)\big) \\ &= \boldsymbol{e}^T(t)(PA + PB_{\boldsymbol{x},\boldsymbol{y}} - PK)\boldsymbol{e}(t) \\ &= \boldsymbol{x}^T(t)S\boldsymbol{x}(t), \end{aligned} \tag{44}$$

where

$$S = \frac{(PA + PB_{\boldsymbol{x},\boldsymbol{y}} - PK)^T + (PA + PB_{\boldsymbol{x},\boldsymbol{y}} - PK)}{2} \tag{45}$$

is a symmetric matrix. If S is negative definite for all $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$, we have

$${}_0D_t^\alpha V(\boldsymbol{e}) \leq \boldsymbol{e}^T(t)S\boldsymbol{e}(t) < 0. \tag{46}$$

It follows from Theorem 2 that the controller can make the error system asymptotically stable at the origin, *i.e.* the systems (38) and (40) are asymptotically synchronized.

In this paper, we mainly consider the synchronization of fractional-order chaotic system. It is well known that $\boldsymbol{x}(t)$ is bounded for the fractional-order chaotic system (38). It implies that there is a constant matrix C, satisfying

$$\boldsymbol{e}^T(t)B_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{e}(t) \leq \boldsymbol{e}^T(t)C\boldsymbol{e}(t). \tag{47}$$

Hence, we can give some corollaries which are simpler than Theorem 3. □

Corollary 1 *The controlled fractional-order error system* (41) *is asymptotically stable at the origin, i.e. the systems* (38) *and* (40) *are asymptotically synchronized, if the feedback gain matrix K makes the matrix*

$$S = \frac{(PA + PC - PK)^T + (PA + PC - PK)}{2} \tag{48}$$

negative definite, where P is a positive definite and C is defined as (47).

This corollary can easily be proved by Theorem 3 and inequality (47).

If the positive definite matrix is $P = I$, the constant matrix $C = cI$ and the feedback gain matrix $K = kI$, where I is identity matrix, the following corollary can be obtained.

Corollary 2 *The controlled fractional-order error system* (41) *is asymptotically stable at the origin, i.e. the systems* (38) *and* (40) *are asymptotically synchronized, if the feedback* *gain matrix $K = kI$ makes the matrix*

$$S = \frac{A^T + A}{2} + (c - k)I \tag{49}$$

negative definite. Particularly, let $\lambda_{\max}$ be the maximal eigenvalue of the matrix $\frac{A^T+A}{2}$, if $K = kI$ satisfies

$$\lambda_{\max} + c - k < 0, \tag{50}$$

the controlled error system (41) *is asymptotically stable at the origin.*

Remark 2 In these corollaries, these synchronization criteria are sufficient conditions for the synchronization of the fractional chaotic system. In the application, we only choose the feedback gain matrix $K = kI$, which satisfies $k > \lambda_{\max} + c$, then the controller can make the error system (41) asymptotically stable at the origin, *i.e.* it makes the systems (38) and (40) synchronize. This method is simple and universal.

Remark 3 In Corollary 2, we suppose that the constant matrix is $C = cI$ and the feedback gain matrix is $K = kI$, then the conclusion is obtained. If we suppose that C and K are diagonal matrices, *i.e.* $C = \mathrm{diag}(c_1, c_2, c_3)$ and $K = \mathrm{diag}(k_1, k_2, k_3)$. According to Theorem 3 and Corollary 2, we must find a suitable k_i which satisfies the condition. In many cases, some k_i and c_i are equal to zero, which makes the linear controller simpler.

4 Simulation and analysis

In this section, three 3D fractional-order chaotic systems are used as examples to show the validity and effectiveness of the synchronization criterion. For the fractional-order error system in the form (41), by adding a linear controller and letting the feedback gain matrix satisfy the conditions of Corollary 2, the error system can be stabilized to the equilibrium point, *i.e.* the drive system and response system are asymptotically synchronized.

4.1 Synchronization of the fractional-order unified system

Lü *et al.* introduced the unified system [34] in 2002, which can unify the Chen, Lü and Lorenz system. Its fractional expression is given as follows [35, 36]:

$$\begin{aligned}
{}_0D_t^{\alpha} x_1 &= (25a + 10)(x_2 - x_1), \\
{}_0D_t^{\alpha} x_2 &= (28 - 35a)x_1 + (29a - 1)x_2 - x_1 x_3, \\
{}_0D_t^{\alpha} x_3 &= x_1 x_2 - (a + 8)/3x_3,
\end{aligned} \tag{51}$$

where $a \in R$ and α is the fractional order of the derivative. If the parameter satisfies $a \in [0, 0.8)$, system (51) is the generalized Lorenz system; if parameter satisfies $a = 0.8$, the system becomes the fractional-order Lü system; and if parameter satisfies $a \in (0.8, 1]$, it becomes the fractional-order generalized Chen system. We select $a = 1$, there are three equilibrium points $(0, 0, 0)$, $(3\sqrt{7}, 3\sqrt{7}, 21)$ and $(-3\sqrt{7}, -3\sqrt{7}, 21)$ in this system (51). When we choose the fractional order $\alpha = 0.9$ and initial value $(2, 2, 1)$, the system (51) can generate chaos, which is shown in Figure 1.

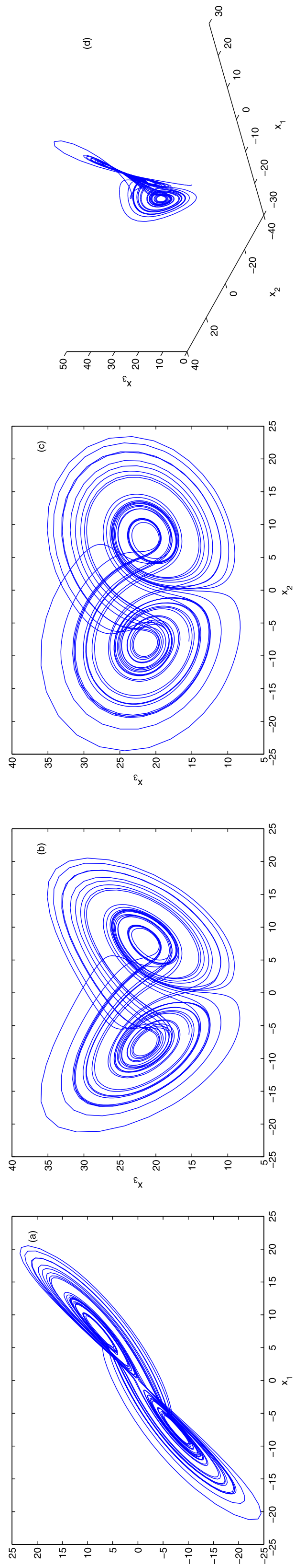

Figure 1 Chaotic attractors in the fractional-order unified system with $\alpha = 0.9$. The panels **(a)**, **(b)**, **(c)** and **(d)** show the $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$ and 3D views, respectively.

The fractional-order chaotic system (51) can be rewritten in the form of (38) as a drive system

$$ {}_0D_t^{\alpha}\boldsymbol{x}(t) = A\boldsymbol{x}(t) + \boldsymbol{g}\big(\boldsymbol{x}(t)\big)\boldsymbol{x}(t), \tag{52}$$

where

$$ A = \begin{pmatrix} -35 & 35 & 0 \\ 7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \qquad \boldsymbol{g}\big(\boldsymbol{x}(t)\big) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix}. \tag{53}$$

The controlled response system (51) can be rewritten in the form (40)

$$ {}_0D_t^{\alpha}\boldsymbol{y}(t) = A\boldsymbol{y}(t) + \boldsymbol{g}\big(\boldsymbol{y}(t)\big)\boldsymbol{y}(t) - K\big(\boldsymbol{y}(t) - \boldsymbol{x}(t)\big). \tag{54}$$

According to systems (52) and (54), the controlled error system is obtained

$$ {}_0D_t^{\boldsymbol{\alpha}}\boldsymbol{e}(t) = A\boldsymbol{e}(t) + B_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{e}(t) - K\boldsymbol{e}(t), \tag{55}$$

where $B_{\boldsymbol{x},\boldsymbol{y}}$ is a bounded matrix with its elements depending on $\boldsymbol{x}$ and $\boldsymbol{y}$.

Since the fractional system is chaotic, $\boldsymbol{x}(t)$ is bounded. Then we can easily obtain $\boldsymbol{e}^T(t)B_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{e}(t) = x_2e_1e_3 - x_3e_1e_2 < 25\boldsymbol{e}^T(t)\boldsymbol{e}(t)$ by calculating the eigenvalue of maximum, which implies $c \approx 25$. According to Corollary 2, if the matrix $S = \frac{A^T+A}{2} + (c-k)I$ is negative positive, the error system (55) is asymptotically stable, *i.e.* the systems (52) and (54) are asymptotically synchronized. And we can easily see that the maximal eigenvalue of matrix $\frac{A^T+A}{2}$ is $\lambda_{\max} \approx 31$. So if the feedback gain matrix $K = kI$ satisfies $k > 56$, the error system is asymptotically stable at the origin. When selecting $k = 57$, the numerical results, illustrated in Figure 2, show that the fractional-order error system (55) is driven to its equilibrium point $(0,0,0)$ asymptotically as $t \to \infty$, which implies that the drive system (52) and response system (54) are synchronized.

4.2 Synchronization of the fractional-order Liu system

Liu *et al.* introduced a three dimensional chaotic system in 2009, which is called Liu's system [37]. Gejji and Bhalekar [38] studied the fractional-order chaotic Liu system, which

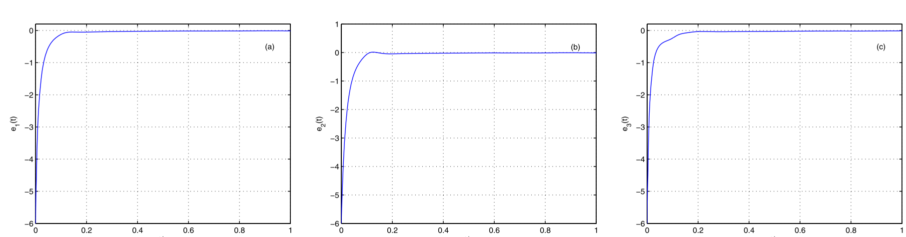

Figure 2 Time waveforms of synchronization errors e_1 (a), e_2 (b) and e_3 (c) between two fractional-order unified systems.

is described as

$$\begin{aligned} {}_0D_t^{\alpha} x_1(t) &= -ax_1 - ex_2^2, \\ {}_0D_t^{\alpha} x_2(t) &= bx_2 - kx_1x_3, \\ {}_0D_t^{\alpha} x_3(t) &= -cx_3 + mx_1x_2, \end{aligned} \tag{56}$$

where $a, b, c, e, k, m \in R$. Its parameters are chosen as $a = e = 1$, $b = 2.5$, $k = m = 4$, $c = 5$, the system has five equilibrium points. Two of them are complex and three are real equilibriums $(0, 0, 0)$, $(-0.884, 0.940, -0.665)$ and $(-0.884, -0.940, -0.665)$. On selecting the order $\alpha = 0.94$ and initial value $(1, 1, 1)$, the system (56) has chaotic behaviors, which is shown in Figure 3.

The fractional-order chaotic Liu's system (56) can be rewritten in the form of (38) as a drive system

$${}_0D_t^{\alpha} \boldsymbol{x}(t) = A\boldsymbol{x}(t) + \boldsymbol{g}\big(\boldsymbol{x}(t)\big)\boldsymbol{x}(t), \tag{57}$$

where

$$A = \begin{pmatrix} -a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -c \end{pmatrix}, \qquad \boldsymbol{g}\big(\boldsymbol{x}(t)\big) = \begin{pmatrix} 0 & -ex_2 & 0 \\ 0 & 0 & -kx_1 \\ 0 & mx_1 & 0 \end{pmatrix}. \tag{58}$$

The controlled response system (57) can be rewritten in the form (40)

$${}_0D_t^{\alpha} \boldsymbol{y}(t) = A\boldsymbol{y}(t) + \boldsymbol{g}\big(\boldsymbol{y}(t)\big)\boldsymbol{y}(t) - K\big(\boldsymbol{y}(t) - \boldsymbol{x}(t)\big). \tag{59}$$

According to the systems (58) and (60), the controlled error system is obtained:

$${}_0D_t^{\boldsymbol{\alpha}} \boldsymbol{e}(t) = A\boldsymbol{e}(t) + B_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{e}(t) - K\boldsymbol{e}(t), \tag{60}$$

where $B_{\boldsymbol{x},\boldsymbol{y}}$ is a bounded matrix with its elements depending on $\boldsymbol{x}$ and $\boldsymbol{y}$.

Since the fractional system is chaotic, $\boldsymbol{x}(t)$ is bounded. Then we can easily obtain $\boldsymbol{e}^T(t)B_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{e}(t) = -(x_2 + x_3 + y_2)e_1e_2 + x_2e_1e_3 < 3.5\boldsymbol{e}^T(t)\boldsymbol{e}(t)$ by calculating the eigenvalue of maximum. It implies $c = 3.5$. According to Corollary 2, if the matrix $S = \frac{A^T+A}{2} + (c - k)I$ is negative positive, the error system (60) is asymptotically stable, *i.e.* the systems (57) and (59) are asymptotically synchronized. And we can easily see that the maximal eigenvalue of matrix $\frac{A^T+A}{2}$ is $\lambda_{\max} = 2.5$. Hence, if the feedback gain matrix $K = kI$ satisfies $k > 6$, the system is asymptotically stable at the origin. When selecting $k = 7$, the numerical results, illustrated in Figure 4, show that the fractional-order error system (60) is driven to its equilibrium point $(0, 0, 0)$ asymptotically as $t \to \infty$, which implies that the drive system (57) and response system (59) are synchronized.

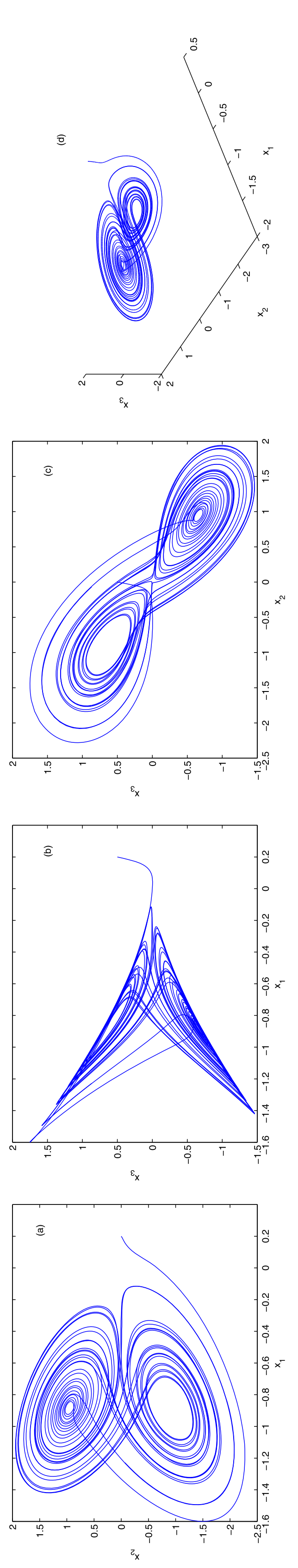

Figure 3 Chaotic attractors in the fractional-order Liu's system with $\alpha = 0.94$. The panels **(a)**, **(b)**, **(c)** and **(d)** show the $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$ and 3D views, respectively.

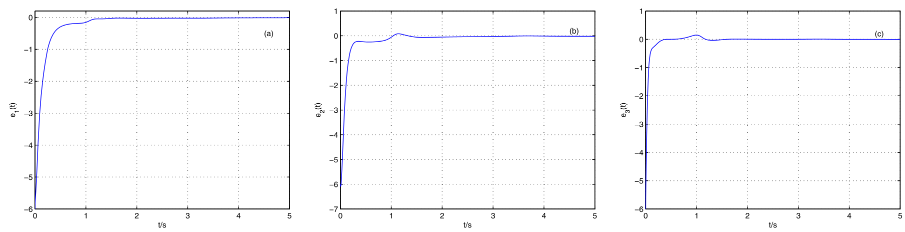

Figure 4 Time waveforms of synchronization errors e_1 (a), e_2 (b) and e_3 (c) between two fractional-order Liu systems.

4.3 Synchronization of the fractional-order Lorenz system

Lorenz introduced an integer-order dynamical system in 1963, which is call Lorenz system. It can be described as follows [39]:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= cx_1 - x_1x_3 - x_2, \\ \dot{x}_3 &= x_1x_2 - bx_3. \end{aligned} \tag{61}$$

When the parameters are chosen as $a = 10$, $b = 8/3$ and $c = 28$, this system (61) exhibits a chaotic behavior. Grigorenko and Grigorenko [40] studied the fractional version of this system which is given by

$$\begin{aligned} {}_0D_t^\alpha x_1 &= a(x_2 - x_1), \\ {}_0D_t^\alpha x_2 &= cx_1 - x_1x_3 - x_2, \\ {}_0D_t^\alpha x_3 &= x_1x_2 - bx_3, \end{aligned} \tag{62}$$

where α is the order of fractional derivative. With the order $\alpha = 0.995$ and initial value $(0.1, 0.2, 0.3)$, the system can generate chaotic behavior, as shown in Figure 5.

The fractional-order chaotic system (62) can be rewritten in the form of (38) as a drive system

$$ {}_0D_t^\alpha \boldsymbol{x}(t) = A\boldsymbol{x}(t) + \boldsymbol{g}\big(\boldsymbol{x}(t)\big)\boldsymbol{x}(t), \tag{63}$$

where

$$A = \begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \qquad \boldsymbol{g}\big(\boldsymbol{x}(t)\big) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix}. \tag{64}$$

The controlled response system (63) can be rewritten in the form of (40)

$$ {}_0D_t^\alpha \boldsymbol{y}(t) = A\boldsymbol{y}(t) + \boldsymbol{g}\big(\boldsymbol{y}(t)\big)\boldsymbol{y}(t) - K\big(\boldsymbol{y}(t) - \boldsymbol{x}(t)\big). \tag{65}$$

According to systems (63) and (65), the controlled error system is obtained

$$ {}_0D_t^{\boldsymbol{\alpha}} \boldsymbol{e}(t) = A\boldsymbol{e}(t) + B_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{e}(t) - K\boldsymbol{e}(t), \tag{66}$$

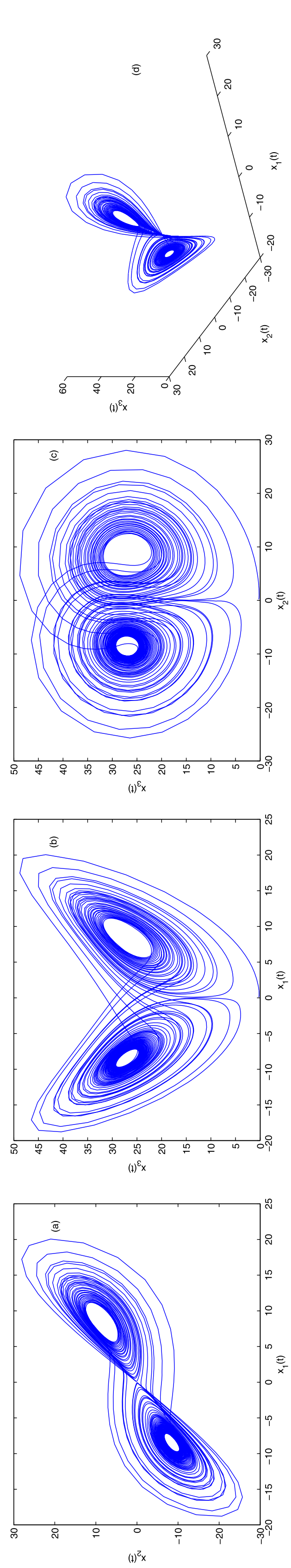

Figure 5 Chaotic attractors in the fractional-order Lorenz system with $\alpha = 0.995$. The panels **(a)**, **(b)**, **(c)** and **(d)** show the $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$ and 3D views, respectively.

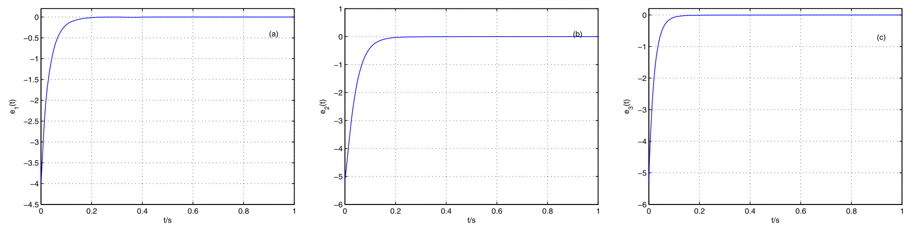

Figure 6 Time waveforms of synchronization errors e_1 (a), e_2 (b) and e_3 (c) between two fractional-order Lorenz systems.

where $B_{\boldsymbol{x},\boldsymbol{y}}$ is a bounded matrix with its elements depending on $\boldsymbol{x}$ and $\boldsymbol{y}$.

Since the fractional system is chaotic, $\boldsymbol{x}(t)$ is bounded. Then we can easily obtain $\mathbf{e}^T(t)B_{\boldsymbol{x},\boldsymbol{y}}\mathbf{e}(t) = -x_3e_1e_2 + x_2e_1e_3 \leq 25\mathbf{e}^T(t)\mathbf{e}(t)$ by numerical simulation, *i.e.* $c \approx 25$. Hence, according to Corollary 2, if the matrix $S = \frac{A^T+A}{2} + (c-k)I$ is negative positive, the error system (66) is asymptotically stable, *i.e.* the systems (63) and (65) are asymptotically synchronized. And we can easily see that the maximal eigenvalue of matrix $\frac{A^T+A}{2}$ is $\lambda_{\max} \approx 14$. So if the feedback gain matrix $K = kI$ satisfies $k > 39$, the system is asymptotically stable at the origin. When selecting $k = 40$, the numerical results, illustrated in Figure 6, show that the fractional-order unified system is driven to its equilibrium point $(0, 0, 0)$ asymptotically as $t \to \infty$, which implies that the drive system (63) and response system (65) are synchronized.

5 Conclusion

In this letter we proposed a new synchronization method for fractional-order chaotic systems based on a simple Lyapunov function. We also proposed some sufficient conditions of synchronization for the fractional-order chaotic systems. The proposed method is simple, universal and theoretically rigorous. Furthermore, we have implemented and verified our method for other fractional-order chaotic systems [2, 41–44], namely the Lü chaotic system, the fractional-order Newton-Leipnik system, the Rössler system, financial systems, etc. The numerical simulation results also indicate that the proposed controller can effectively make the fractional-order chaotic systems synchronized, and the proposed method provides a theoretical basis for the applications of synchronization method in fractional-order dynamic systems. In future work as regards this topic, we will consider whether the proposed synchronization method can be extended to control other complex chaotic systems, such as the networked fractional chaotic systems [45, 46] and fractional multi-scroll chaotic systems [47].

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, 643000, China. [2]Artificial Intelligence Key Laboratory of Sichuan Province, Zigong, 643000, China. [3]Key Laboratory of Higher Education of Sichuan Province for Enterprise Informationalization and Internet of Things, Zigong, 643000, China. [4]Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing, Zigong, 643000, China.

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