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Long-time behavior of a regime-switching Susceptible-Infective epidemic model with degenerate diffusion

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Abstract
In this paper, we consider a stochastic SI epidemic model with regime switching. The Markov semigroup theory is employed to obtain the existence of a unique stable stationary distribution. We prove that if $\mathcal{R}^s < 0$, then the disease becomes extinct exponentially; whereas if $\mathcal{R}^s > 0$ and $\beta(i) > \alpha(i)$, $i \in \mathbb{S}$, then the densities of the distributions of the solution can converge in L^1 to an invariant density.

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Keywords: stationary distribution; Markov semigroups; asymptotic stability

1 Introduction

A deterministic Susceptible-Infective model with standard incidence can be described by

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \frac{\beta S(t)I(t)}{S(t)+I(t)} - \mu S(t), \\ \frac{dI(t)}{dt} = \frac{\beta S(t)I(t)}{S(t)+I(t)} - (\alpha + \mu)I(t). \end{cases} \tag{1.1}$$

Here $S(t)$ and $I(t)$ denote the numbers of susceptible and infected individuals at time t, respectively, Λ is the influx of individuals into the susceptibles, β is the disease transmission coefficient, μ is the natural death rate, and α is the disease-related death rate.

Because of the existence of environmental noises, the parameters appearing in model (1.1) are emphatically not constants. In a simple case, environmental noises manifest themselves as white and color noises. We assume that white noise mainly affects the natural death rates, that is, $\mu \to \mu + \sigma \dot{B}(t)$, where σ represents the intensity of white noise, $B(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}(t)\}_{t\geq 0}, \mathbb{P})$ with filtration $\{\mathscr{F}(t)\}_{t\geq 0}$ satisfying the usual conditions (see [1]).

In real world, color noise can cause the population system to switch from one environmental regime to another. Such a switching is described by a continuous-time Markov chain $r(t)$, $t \geq 0$, with a finite state space $\mathbb{S} = \{1, 2, \dots, N\}$ and the generator $\Gamma = (\gamma_{ij})_{N\times N}$ of $r(t)$ given by

$$\mathbb{P}\{r(t+\delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\delta + o(\delta) & \text{if } i = j, \end{cases}$$

where $\gamma_{ij} \geq 0$ for $i,j = 1,\ldots,N$ with $j \neq i$ and $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$ for $i = 1,\ldots,N$. We assume that the Markov chain $r(t)$ is independent of Brownian motion. For convenience, throughout this paper, we assume that

$$\gamma_{ij} > 0 \quad \text{for } i,j = 1,\ldots,N \text{ with } j \neq i.$$

This assumption ensures that the Markov chain $r(t)$ is irreducible. Consequently, there exists a unique stationary distribution $\pi = \{\pi_1,\pi_2,\ldots,\pi_N\}$ of $r(t)$ that satisfies $\pi\Gamma = 0$, $\sum_{i=1}^N \pi_i = 1$, and $\pi_i > 0$ for all $i \in \mathbb{S}$.

Incorporating the two types of noises into system (1.1), we get a regime-switching diffusion model:

$$\begin{cases} dS(t) = (\Lambda(r(t)) - \frac{\beta(r(t))S(t)I(t)}{S(t)+I(t)} - \mu(r(t))S(t))\,dt + \sigma(r(t))S(t)\,dB(t), \\ dI(t) = (\frac{\beta(r(t))S(t)I(t)}{S(t)+I(t)} - (\alpha(r(t)) + \mu(r(t)))I(t))\,dt + \sigma(r(t))I(t)\,dB(t), \end{cases} \tag{1.2}$$

where the parameters $\Lambda(i), \beta(i), \mu(i), \alpha(i)$, and $\sigma(i)$, $i \in \mathbb{S}$ are all positive constants.

Recently, the long-time behavior of stochastic epidemic model under regime switching was considered, for example, in [2–4], where the uniform ellipticity condition was needed when proving the ergodicity of a stochastic system. However, in this paper the diffusion matrix of system (1.2) is given by

$$A_i = \sigma^2(i)\begin{pmatrix} S^2 & SI \\ SI & I^2 \end{pmatrix}, \quad i \in \mathbb{S}.$$

Since A_i is degenerate, the uniform ellipticity condition is not satisfied, and even a variable substitution cannot improve this situation. To our knowledge, rare works in this direction are known.

Throughout this paper, we denote by A' the transpose of a vector or matrix A; we set $\hat{g} = \min_{k\in\mathbb{S}}\{g(k)\}$ and $\check{g} = \max_{k\in\mathbb{S}}\{g(k)\}$ for any vector $g = (g(1),\ldots,g(N))$; moreover,

$$\mathcal{R}_i := \beta(i) - \mu(i) - \alpha(i) - \frac{\sigma^2(i)}{2}, \quad i \in \mathbb{S}, \quad \text{and} \quad \mathcal{R}^s = \sum_{i=1}^N \pi_i\mathcal{R}_i.$$

By similar arguments as in Theorem 2.1 of [5] it follows that, for any $(S(0),I(0),r(0)) \in \mathbb{R}_+^2 \times \mathbb{S}$, system (1.2) has a unique global solution, which remains in $\mathbb{R}_+^2$ with probability 1.

The aim of this paper is to consider the long-time behavior of system (1.2). We prove that the disease becomes extinct exponentially if $\mathcal{R}^s < 0$, whereas if $\mathcal{R}^s > 0$ and $\beta(i) > \alpha(i)$, $i \in \mathbb{S}$, then system (1.2) has a stable stationary distribution. In some sense, $\mathcal{R}^s$ is the threshold of model (1.2).

The rest of this paper is organized as follows. In Section 2, we present a sufficient condition for the extinction of the disease. In Section 3, conditions for the existence of a stable stationary distribution are given, and detailed proofs are presented. Finally, we give a brief discussion.

2 Extinction of the disease

In this section, we present a sufficient condition for the extinction of the disease.

Theorem 2.1 *If $\mathcal{R}^s < 0$, then the disease $I(t)$ tends to zero exponentially,*

Proof By the generalized Itó's formula it follows from (1.2) that

$$\begin{aligned} &\frac{\ln I(t) - \ln I(0)}{t} \\ &= \frac{1}{t}\int_0^t \frac{\beta(r(s))S(s)}{S(s)+I(s)}\,ds - \frac{1}{t}\int_0^t \left(\alpha\bigl(r(s)\bigr) + \mu\bigl(r(s)\bigr) + \frac{\sigma^2(r(s))}{2}\right) ds + \frac{1}{t}\int_0^t \sigma\bigl(r(s)\bigr)\,dB(s) \\ &\le \frac{1}{t}\int_0^t \left(\beta\bigl(r(s)\bigr) - \alpha\bigl(r(s)\bigr) - \mu\bigl(r(s)\bigr) - \frac{\sigma^2(r(s))}{2}\right) ds + \frac{1}{t}\int_0^t \sigma\bigl(r(s)\bigr)\,dB(s). \end{aligned}$$

Taking the limit as $t \to \infty$, we have

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le \sum_{i=1}^N \pi_i \mathcal{R}_i = \mathcal{R}^s < 0.$$

The proof is complete. □

3 Existence of stationary distribution and its stability

Let $x(t) = \ln S(t)$ and $y(t) = \ln I(t)$. Then system (1.2) becomes

$$\begin{cases} dx(t) = (\Lambda(r(t))e^{-x(t)} - \frac{\beta(r(t))e^{y(t)}}{e^{x(t)}+e^{y(t)}} - c_1(r(t)))\,dt + \sigma(r(t))\,dB(t), \\ dy(t) = (\frac{\beta(r(t))e^{x(t)}}{e^{x(t)}+e^{y(t)}} - c_2(r(t)))\,dt + \sigma(r(t))\,dB(t), \end{cases} \tag{3.1}$$

where $c_1(i) := \mu(i) + \frac{\sigma^2(i)}{2}$ and $c_2(i) := \alpha(i) + \mu(i) + \frac{\sigma^2(i)}{2}$.

To investigate the existence of a stationary distribution of system (1.2) and its stability, it suffices to consider the corresponding property for system (3.1).

Theorem 3.1 *Let $(x(t), y(t))$ be a solution of system (3.1) with initial value $(x(0), y(0), r(0)) \in \mathbb{R}^2 \times \mathbb{S}$. Then, for every $t > 0$, the distribution of $(x(t), y(t), r(t))$ has a density $u(t, x, y, i)$. If $\mathcal{R}^s > 0$ and $\beta(i) > \alpha(i)$, $i \in \mathbb{S}$, then there exists a unique density $u_*(x, y, i)$ such that*

$$\lim_{t\to\infty} \sum_{i=1}^N \iint_{\mathbb{R}^2} \bigl|u(t,x,y,i) - u_*(x,y,i)\bigr|\,dx\,dy = 0.$$

Next, we prove this theorem by Lemmas 3.1-3.4.

Let $(x^{(i)}(t), y^{(i)}(t))$ be a solution of system

$$\begin{cases} dx^{(i)}(t) = (\Lambda(i)e^{-x^{(i)}(t)} - \frac{\beta(i)e^{y^{(i)}(t)}}{e^{x^{(i)}(t)}+e^{y^{(i)}(t)}} - c_1(i))\,dt + \sigma(i)\,dB(t), \\ dy^{(i)}(t) = (\frac{\beta(i)e^{x^{(i)}(t)}}{e^{x^{(i)}(t)}+e^{y^{(i)}(t)}} - c_2(i))\,dt + \sigma(i)\,dB(t). \end{cases} \tag{3.2}$$

Denote by $\mathcal{A}_i$ the differential operators

$$\mathcal{A}_i f = \frac{\sigma^2(i)}{2}\left[\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial y^2}\right] - \frac{\partial(h_i^1 f)}{\partial x} - \frac{\partial(h_i^2 f)}{\partial y}, \quad f \in L^1\bigl(\mathbb{R}^2, \mathcal{B}\bigl(\mathbb{R}^2\bigr), m\bigr),$$

where $\mathcal{B}(\mathbb{R}^2)$ is the σ-algebra of Borel subsets of $\mathbb{R}^2$, m is the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, and

$$h_i^1(x,y) = \Lambda(i)e^{-x} - \frac{\beta(i)e^y}{e^x + e^y} - c_1(i), \qquad h_i^2(x,y) = \frac{\beta(i)e^x}{e^x + e^y} - c_2(i).$$

Next, we show that, for any $i \in \mathbb{S}$, the operator $\mathcal{A}_i$ generates an integral Markov semigroup $\{\mathcal{T}_i(t)\}_{t\geq 0}$ on the space $L^1(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), m)$ and

$$\int_0^\infty \mathcal{T}_i(t)f\,dt > 0 \quad \text{a.e. on } \mathbb{R}^2.$$

Lemma 3.1 *The semigroup $\{\mathcal{T}_i(t)\}_{t\geq 0}$ is an integral Markov semigroup.*

Proof If $a(x)$ and $b(x)$ are vector fields on $\mathbb{R}^d$, then the Lie bracket $[a,b]$ is a vector field given by

$$[a,b]_j(x) = \sum_{k=1}^d \left(a_k \frac{\partial b_j}{\partial x_k}(x) - b_k \frac{\partial a_j}{\partial x_k}(x) \right), \quad j = 1, 2, \ldots, d.$$

Let

$$a(x,y) = \left(\Lambda(i)e^{-x} - \frac{\beta(i)e^y}{e^x + e^y} - c_1(i), \frac{\beta(i)e^x}{e^x + e^y} - c_2(i) \right)'$$

and $b(x,y) = (\sigma(i), \sigma(i))'$.

By direct calculation, $[a,b] = (\sigma(i)\Lambda(i)e^{-x}, 0)'$. Thus,

$$\left| [a,b] \quad b \right| = \begin{vmatrix} \sigma(i)\Lambda(i)e^{-x} & \sigma(i) \\ 0 & \sigma(i) \end{vmatrix} = \sigma^2(i)\Lambda(i)e^{-x} > 0,$$

which means that b and $[a,b]$ are linearly independent on $\mathbb{R}^2$.

Thus, for every $(x,y) \in \mathbb{R}^2$, the vectors $b(x,y)$ and $[a,b](x,y)$ span the space $\mathbb{R}^2$. By the Hörmander theorem [6] the transition probability function of $(x^{(i)}(t), y^{(i)}(t))$ has a smooth density $k_i \in C^\infty((0,\infty) \times \mathbb{R}^2 \times \mathbb{R}^2)$. Then, for every $f \in L^1(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), m)$ satisfying $f \geq 0$ and $\|f\| = 1$,

$$\mathcal{T}_i(t)f(x,y) = \iint_{\mathbb{R}^2} k_i(t,x,y;u,v)f(u,v)\,du\,dv.$$

Hence, the semigroup $\{\mathcal{T}_i(t)\}_{t\geq 0}$ is an integral Markov semigroup. □

Lemma 3.2 *If $\beta(i) > \alpha(i)$, $i \in \mathbb{S}$, then, for every $f \in D$,*

$$\int_0^\infty \mathcal{T}_i(t)f\,dt > 0 \quad \textit{a.e. on } \mathbb{R}^2,$$

where $D = \{f \in L^1(\mathbb{R}^2) : f \geq 0, \|f\| = 1\}$.

Proof For any $(x_0, y_0) \in \mathbb{R}^2$, consider the following control system:

$$\dot{x}_\phi(t) = \Lambda(i)e^{-x_\phi(t)} - \frac{\beta(i)e^{y_\phi(t)}}{e^{x_\phi(t)} + e^{y_\phi(t)}} - c_1(i) + \sigma(i)\phi(t), \tag{3.3}$$

$$\dot{y}_\phi(t) = \frac{\beta(i)e^{x_\phi(t)}}{e^{x_\phi(t)} + e^{y_\phi(t)}} - c_2(i) + \sigma(i)\phi(t), \tag{3.4}$$

with initial value $(x_\phi(0), y_\phi(0)) = (x_0, y_0)$.

Let $D_{x_0,y_0;\phi}$ be the Frechét derivative of the function $h \mapsto \mathbf{x}_{\phi+h}(T)$ from $L^2([0,T];\mathbb{R})$ to $\mathbb{R}^2$, where $\mathbf{x}_{\phi+h} = (x_{\phi+h}, y_{\phi+h})'$. The derivative $D_{x_0,y_0;\phi}$ can be given by

$$D_{x_0,y_0;\phi}h = \int_0^T Q(T,s)\mathbf{v}h(s)\,ds,$$

where $\mathbf{v} = (\sigma(i), \sigma(i))'$, and $Q(t, t_0)$ $(T \geq t \geq t_0 \geq 0)$ is a matrix function such that $Q(t_0, t_0) = I$, $\partial Q(t, t_0)/\partial t = \Delta(t)Q(t, t_0)$, and $\Delta(t) = \mathbf{f}'(x_\phi(t), y_\phi(t))$, where $\mathbf{f}'$ is the Jacobian of

$$\mathbf{f} = \begin{bmatrix} \Lambda(i)e^{-x} - \frac{\beta(i)e^y}{e^x+e^y} - c_1(i) \\ \frac{\beta(i)e^x}{e^x+e^y} - c_2(i) \end{bmatrix}.$$

Assume that the following two conditions are satisfied:

1. The derivative $D_{x_0,y_0;\phi}$ has rank 2;
2. For any $(x_0, y_0) \in \mathbb{R}^2$ and $(x_1, y_1) \in \mathbb{R}^2$, there exist $T > 0$ and a smooth control ϕ such that the solution of system (3.3), (3.4) satisfies $x(0) = x_0, y(0) = y_0, x(T) = x_1$, and $y(T) = y_1$.

Then $k_i(T, x_1, y_1; x_0, y_0) > 0$ (see [7–9]). If this is the case, we have

$$\int_0^\infty \mathcal{T}_i(t)f\,dt > 0 \quad \text{a.e. on } \mathbb{R}^2.$$

So the rest of the proof is checking conditions 1 and 2.

First, we check condition 1. Let $\varepsilon \in (0, T)$ and $h = \mathbf{1}_{[T-\varepsilon,T]}$. Since $Q(T,s) = I + \Delta(T)(T - s) + o(T - s)$, we obtain

$$D_{x_0,y_0;\phi}h = \varepsilon\mathbf{v} + \frac{1}{2}\varepsilon^2\Delta(T)\mathbf{v} + o(\varepsilon^2), \quad \mathbf{v} = \begin{bmatrix} \sigma(i) \\ \sigma(i) \end{bmatrix},$$

$$\Delta(T)\mathbf{v} = \begin{bmatrix} -\Lambda(i)e^{-x} + \frac{\beta(i)e^{x+y}}{(e^x+e^y)^2} & -\frac{\beta(i)e^{x+y}}{(e^x+e^y)^2} \\ \frac{\beta(i)e^{x+y}}{(e^x+e^y)^2} & -\frac{\beta(i)e^{x+y}}{(e^x+e^y)^2} \end{bmatrix} \begin{bmatrix} \sigma(i) \\ \sigma(i) \end{bmatrix} = \begin{bmatrix} \sigma(i)\Lambda(i)e^{-x} \\ 0 \end{bmatrix},$$

where $x = x_\phi(T), y = y_\phi(T)$. Obviously, $\mathbf{v}$ and $\Delta(T)\mathbf{v}$ are linearly independent for any $(x,y) \in \mathbb{R}^2$. This implies $D_{x_0,y_0;\phi}$ has rank 2 for every $(x,y) \in \mathbb{R}^2$.

Now we prove that condition 2 is satisfied. In view of (3.3) and (3.4), we obtain

$$\dot{x}(t) - \dot{y}(t) + \beta(i) + c_1(i) - c_2(i) = \Lambda(i)e^{-x(t)}, \quad t \in (0, \infty).$$

Integrating from 0 to t, we get

$$x(t) - x(0) - y(t) + y(0) + \big[\beta(i) - \alpha(i)\big]t = \int_0^t \Lambda(i)e^{-x(s)}\,ds, \quad t \in (0, \infty). \tag{3.5}$$

In particular,

$$x_1 - x_0 - y_1 + y_0 + \big[\beta(i) - \alpha(i)\big]T = \int_0^T \Lambda(i)e^{-x(s)}\,ds, \quad t \in (0,\infty). \tag{3.6}$$

Choose $T > 0$ large enough such that the left side of this equality is positive. Then take a smooth function $x(t)$ with $x(0) = x_0$ and $x(T) = x_1$ that satisfies (3.6). Consequently, by (3.5) we can determine a smooth function $y(t)$ with $y(0) = y_0$ and $y(T) = y_1$. Thus, we can determine a smooth control $\phi(t)$ from (3.4), which means that condition 2 is satisfied. □

Let $(x(t), y(t))$ be the unique solution of system (3.1) with $(x(0), y(0), r(0)) \in \mathbb{R}^2 \times \mathbb{S}$. Then $(x(t), y(t), r(t))$ constitutes a Markov process on $\mathbb{R}^2 \times \mathbb{S}$. By Lemma 5.5 in [10], for $t > 0$, the distribution of the process $(x(t), y(t), r(t))$ is absolutely continuous, and its density $u = (u_1, u_2, \dots, u_N)$ (where $u_i := u(t, x, y, i)$) satisfies the following master equation:

$$\frac{\partial u}{\partial t} = \Gamma' u + \mathcal{A}u, \tag{3.7}$$

where $\mathcal{A}u = (\mathcal{A}_1 u_1, \mathcal{A}_2 u_2, \dots, \mathcal{A}_N u_N)'$.

Let $X = \mathbb{R}^2 \times \mathbb{S}$, Σ be the σ-algebra of Borel subsets of X, and $\hat{m}$ be the product measure on (X, Σ) given by $\hat{m}(B \times i) = m(B)$ for all $B \in \mathcal{B}(\mathbb{R}^2)$ and $i \in \mathbb{S}$. Obviously, $\mathcal{A}u$ generates a Markov semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ on the space $L^1(X, \Sigma, \hat{m})$, which is given by

$$\mathcal{T}(t)f = \big(\mathcal{T}_1(t)f(x, y, 1), \dots, \mathcal{T}_N(t)f(x, y, N)\big)', \quad f \in L^1(X, \Sigma, \hat{m}).$$

Let λ be a constant such that $\lambda > \max_{1\leq i\leq N}\{-\gamma_{ii}\}$ and $Q = \lambda^{-1}\Gamma' + I$. Then (3.7) becomes

$$\frac{\partial u}{\partial t} = \lambda Q u - \lambda u + \mathcal{A}u. \tag{3.8}$$

Obviously, Q is also a Markov operator on $L^1(X, \Sigma, \hat{m})$.

From the Phillips perturbation theorem [11], (3.8) with the initial condition $u(0, x, y, k) = f(x, y, k)$ generates a Markov semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ on the space $L^1(X)$ given by

$$\mathcal{P}(t)f = e^{-\lambda t}\sum_{n=0}^{\infty} \lambda^n S^{(n)}(t)f, \tag{3.9}$$

where $S^{(0)}(t) = \mathcal{T}(t)$, and

$$S^{(n+1)}(t)f = \int_0^t S^{(0)}(t-s)QS^{(n)}(s)f\,ds, \quad n \geq 0. \tag{3.10}$$

Lemma 3.3 *If $\beta(i) > \alpha(i)$, $i \in \mathbb{S}$, then the semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is asymptotically stable or is sweeping with respect to compact sets.*

Proof By Lemma 3.1, $\{\mathcal{P}(t)\}_{t\geq 0}$ is a partially integral Markov semigroup. In view of Lemma 3.2, (3.10), and $Q_{ij} > 0$ $(i \neq j)$, we know that, for every nonnegative $f \in L^1(X)$ with $\|f\| = 1$,

$$\int_0^\infty \mathcal{P}(t)f\,dt > 0 \quad \text{a.e. on } X.$$

By similar arguments to Corollary 1 in [12] it follows that $\{\mathcal{P}(t)\}_{t\geq 0}$ is asymptotically stable or is sweeping with respect to compact sets. □

Remark 3.1 A density f_* is called invariant if $\mathcal{P}(t)f_* = f_*$ for each $t > 0$. The Markov semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is called asymptotically stable if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|\mathcal{P}(t)f - f_*\| = 0 \quad \text{for } f \in D,$$

where $D = \{f \in L^1(X) : f \geq 0, \|f\| = 1\}$.

A Markov semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is called sweeping with respect to a set $A \in \Sigma$ if, for every $f \in D$,

$$\lim_{t\to\infty} \int_A \mathcal{P}(t)f(x)m(dx) = 0.$$

Lemma 3.4 *If $\mathcal{R}^s > 0$ and $\beta(i) > \alpha(i)$, $i \in \mathbb{S}$, then the semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is asymptotically stable.*

Proof We will construct a nonnegative C^2-function V and a closed set $U \in \mathcal{B}(\mathbb{R}^2)$ (which lies entirely in $\mathbb{R}^2$) such that, for any $i \in \mathbb{S}$,

$$\sup_{(x,y)\in\mathbb{R}^2\setminus U} \mathscr{A}^* V(x,y,i) < 0,$$

where

$$\begin{aligned}\mathscr{A}^* V(x,y,i) = {} & \frac{\sigma^2(i)}{2}\left[\frac{\partial^2 V}{\partial x^2} + 2\frac{\partial^2 V}{\partial x\,\partial y} + \frac{\partial^2 V}{\partial y^2}\right] + h_i^1\frac{\partial V}{\partial x} + h_i^2\frac{\partial V}{\partial y} \\ & + \sum_{j\neq i, j\in\mathbb{S}} \gamma_{ij}\bigl(V(x,y,j) - V(x,y,i)\bigr)\end{aligned} \tag{3.11}$$

and

$$h_i^1(x,y) = \Lambda(i)e^{-x} - \frac{\beta(i)e^y}{e^x + e^y} - c_1(i), \qquad h_i^2(x,y) = \frac{\beta(i)e^x}{e^x + e^y} - c_2(i).$$

In fact, $\mathscr{A}^*$ is the adjoint operator of the infinitesimal generator of the semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$.

Since the matrix Γ is irreducible, there exists a solution $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_N)$ of the Poisson system (see [13], Lemma 2.3) such that

$$\Gamma\varpi - \mathcal{R} = -\sum_{i=1}^N \pi_i\mathcal{R}_i\mathbf{1},$$

where $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_N)'$ and $\mathbf{1} = (1, 1, \ldots, 1)'$, that is, for any $i \in \mathbb{S}$,

$$\sum_{j\neq i, j\in\mathbb{S}} \gamma_{ij}(\varpi_j - \varpi_i) - \mathcal{R}_i = -\sum_{i=1}^N \pi_i\mathcal{R}_i = -\mathcal{R}^s. \tag{3.12}$$

Take fixed $r > 0$ such that

$$K - r\mathcal{R}^s < -2,$$

where

$$K := \max_{i\in\mathbb{S}}\{\Lambda(i) + c_1(i)\}.$$

We define a C^2-function V as follows:

$$V(x,y,i) = e^x + e^y - x - ry + r(\varpi_i + |\varpi|), \quad (x,y) \in \mathbb{R}^2.$$

By direct calculation we obtain

$$\begin{aligned}
\mathscr{A}^* V &= \Lambda(i) - \mu(i)e^x - \bigl(\alpha(i) + \mu(i)\bigr)e^y - \Lambda(i)e^{-x} + \frac{\beta(i)e^y}{e^x + e^y} + c_1(i) \\
&\quad + r\left[-\frac{\beta(i)e^x}{e^x + e^y} + c_2(i)\right] + r\sum_{j\neq i, j\in\mathbb{S}} \gamma_{ij}(\varpi_j - \varpi_i) \\
&= \Lambda(i) + c_1(i) - \mu(i)e^x - \bigl(\alpha(i) + \mu(i)\bigr)e^y - \Lambda(i)e^{-x} + (r+1)\frac{\beta(i)e^y}{e^x + e^y} \\
&\quad + r\left[-\beta(i) + c_2(i) + \sum_{j\neq i, j\in\mathbb{S}} \gamma_{ij}(\varpi_j - \varpi_i)\right] \\
&= \Lambda(i) + c_1(i) - \mu(i)e^x - \bigl(\alpha(i) + \mu(i)\bigr)e^y - \Lambda(i)e^{-x} + (r+1)\frac{\beta(i)e^y}{e^x + e^y} - r\mathcal{R}^s,
\end{aligned}$$

where (3.12) is used. Since

$$\mathscr{A}^* V \le K - \hat{\mu}e^x - \hat{\Lambda}e^{-x} + (r+1)\check{\beta},$$

choose $\kappa > 0$ large enough such that

$$\mathscr{A}^* V < -1, \quad \text{on } \mathbb{R}^2 - [-\kappa,\kappa] \times \mathbb{R}.$$

In addition, for any $(x,y) \in [-\kappa,\kappa] \times \mathbb{R}$, we have

$$\mathscr{A}^* V \le \begin{cases} K + (r+1)\frac{\check{\beta}e^y}{e^{-\kappa} + e^y} - r\mathcal{R}^s \to K - r\mathcal{R}^s < -2 & \text{as } y \to -\infty; \\ K - (\hat{\alpha} + \hat{\mu})e^y + (r+1)\check{\beta} \to -\infty & \text{as } y \to +\infty. \end{cases}$$

Choose $\epsilon > 0$ large enough such that

$$\mathscr{A}^* V < -1, \quad \text{on } [-\kappa,\kappa] \times \mathbb{R} - \mathbb{R} \times [-\epsilon,\epsilon].$$

Obviously, $(\mathbb{R}^2 - [-\kappa,\kappa] \times \mathbb{R}) \cup ([-\kappa,\kappa] \times \mathbb{R} - \mathbb{R} \times [-\epsilon,\epsilon]) = \mathbb{R}^2 - [-\kappa,\kappa] \times [-\epsilon,\epsilon]$. Take $U := [-\kappa,\kappa] \times [-\epsilon,\epsilon]$. Then, for any $i \in \mathbb{S}$,

$$\sup_{(x,y)\in\mathbb{R}^2\setminus U} \mathscr{A}^* V(x,y,i) < -1.$$

Such a function V is called a Khasminskiĭ function. By using arguments similar to those in [14], the existence of a Khasminskiĭ function implies that the semigroup is not sweeping from the set U. According to Lemma 3.4, the semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is asymptotically stable, which completes the proof. □

4 Discussion

In this paper, we consider the existence of a stationary distribution of a stochastic SI epidemic model with regime switching and its stability. We prove that if $\mathcal{R}^s < 0$, then the disease becomes extinct exponentially, whereas if $\mathcal{R}^s > 0$ and $\beta(i) > \alpha(i), i \in \mathbb{S}$, then the densities of the distributions of the solution can converge in L^1 to an invariant density. Since the diffusion is degenerate, we employ the Markov semigroup theory to study the long-time behavior of system (1.2).

It is known that to obtain the ergodicity, the strong Feller property and irreducibility of a Markov process are needed. However, according to the proof of Lemma 3.2, system (1.2) is not irreducible. This is the reason why we employ the Markov semigroup theory given by Rudnicki [12, 15]. In addition, we assume that $\gamma_{ij} > 0, i \neq j$, which is a strong condition. This condition is used to ensure that $r(t)$ is irreducible. In fact, under the condition that $r(t)$ is irreducible, just spending a little more time, all results obtained in this paper can be reproved.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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