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Existence and blowup of solutions for the modified Klein-Gordon-Zakharov equations for plasmas with a quantum correction

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Abstract

This paper studies the existence and blowup of solutions for the modified Klein-Gordon-Zakharov equations for plasmas with a quantum correction, which describe the interaction between high frequency Langmuir waves and low frequency ion-acoustic waves in a plasma considering the quantum effects. Firstly the existence and uniqueness of the local smooth solutions are obtained by the a priori estimates and the Galerkin method. Secondly, and what is more, by introducing some auxiliary functionals and invariant manifolds, the authors study and derive a sharp threshold for the global existence and blowup of solutions by applying potential well argument and the concavity method. Furthermore, two more specific conditions of how small the initial data are for the solutions to exist globally are concluded by the dilation transformation.

Keywords: existence and blowup; modified Klein-Gordon-Zakharov equations; quantum correction; dilation transformation

1 Introduction

In this paper, we consider the existence and blowup for the following modified Klein-Gordon-Zakharov equations for plasmas with a quantum correction:

$$u_{tt} - \Delta u + u = -nu - |u|^2 u, \quad x \in \Omega, t > 0, \tag{1}$$

$$n_t + \nabla \cdot V = 0, \quad x \in \Omega, t > 0, \tag{2}$$

$$V_t + \nabla n + \nabla |u|^2 - H^2 \nabla \Delta n = 0, \quad x \in \Omega, t > 0, \tag{3}$$

with the periodic initial conditions

$$u(x + Le_i, t) = u(x,t), \qquad n(x + Le_i, t) = n(x,t), \qquad V(x + Le_i, t) = V(x,t), \tag{4}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad n(x,0) = n_0(x), \qquad V(x,0) = V_0(x), \tag{5}$$

where $x \in \Omega$, $t \geq 0$. The spatial domain Ω is a bounded domain in two dimensional real Euclidean space, and the time $t \geq 0$. The unknown complex vector-valued function $u = u(x,t)$ is the electric field, $V = V(x,t)$ is an unknown real vector-valued function and $n = n(x,t)$ is an unknown real scalar-valued function, which represents the density fluctuation of ions. System (1)-(3) describes the interaction between Langmuir waves and ion-acoustic waves in a plasma considering the quantum effects [1–4]. The quantum parameter H expresses the ratio between the ion plasmon energy and the electron thermal energy, which appears in the modified Zakharov equations for plasmas with a quantum correction [4]. $L > 0$ is the period and e_i $(i = 1,2)$ is the standard coordinate vector.

Combining (2) and (3) to eliminate the function $V(x,t)$, we find that system (1)-(3) is equivalent to

$$\begin{cases} u_{tt} - \Delta u + u + nu + |u|^2 u = 0, \\ n_{tt} - \Delta n - \Delta(|u|^2) + H^2 \Delta^2 n = 0, \end{cases} \tag{6}$$

which combines the Klein-Gordon equation and the modified Zakharov equations for plasmas with a quantum correction. The latter equation describes the interaction between high frequency Langmuir waves and low frequency ion-acoustic waves considering the quantum effects [4]. The importance of quantum effects in ultrasmall electronic devices, in dense astrophysical plasma systems and in laser plasmas has produced an increasing interest in the investigation of the quantum counterpart of some of the classical plasma physics phenomena [5]. That is why we call system (1)-(3) the modified Klein-Gordon-Zakharov equations for plasmas with a quantum correction. Thus it is an interesting topic to study the coupled system (1)-(3) mathematically. For the modified Zakharov equations with a quantum correction, a series of works have been devoted to the mathematical analysis. For example, we studied the exact solutions and obtained kinds of exact traveling wave solutions [6]; and we also investigated the existence of weak solutions for another form of Zakharov equations under Dirichlet boundary condition [7]. Long time behavior of the solutions for the dissipative modified Zakharov equations is also studied in our previous work [8, 9]. For some other classical results, see [4, 10] and the references therein.

System (1)-(3) with the absence of $H^2\nabla\Delta n$ has been studied in recent years. Guo and Yuan [11] proved the existence and uniqueness of global smooth solutions via the so-called continuous method and delicate a priori estimates and studied the asymptotic behavior of the solutions. Gan et al. discussed the instability of standing wave, global existence and blowup for the Klein-Gordon-Zakharov system [12]. Some exact solutions were also obtained by various methods, see [13, 14] and the references therein. Furthermore, when system (1)-(3) with the absence of $H^2\nabla\Delta n$ and $-|u|^2 u$, it is more classic and describes the interaction between Langmuir waves and ion sound waves in a plasma [1, 3]. Some existence and uniqueness of global solutions under different conditions were investigated by many researchers [2, 15–17], as well as the exact explicit traveling wave solutions [18, 19].

However, for the modified Klein-Gordon-Zakharov equations (1)-(3) with $H^2\nabla\Delta n$ and $-|u|^2 u$, which is more suitable to describe the interaction between high frequency Langmuir waves and low frequency ion-acoustic waves in a plasma considering the quantum effects, no more research results in mathematical studies have been obtained as we know, even the study on the well-posedness for the equations. Thus, in the present paper, we are going to do the research tentatively and investigate the existence and blowup for the modified Klein-Gordon-Zakharov equations (1)-(5). More specifically, we adopt a priori estimates and the Galerkin method to study the existence and uniqueness of local smooth solutions for the periodic initial value problems (1)-(5) under the condition that the initial data are sufficiently regular. What follows and what is more, by introducing two auxiliary functionals and two invariant manifolds, we study and derive a sharp threshold for the global existence and blowup of the solutions by applying potential well argument [20] and the concavity method [21]. Moreover, two more specific conditions of how small the initial data are for the solutions to exist globally are investigated by some suitable dilation transformation.

The rest of paper is organized as follows. In Section 2, we briefly give some notations and preliminaries. In Section 3, we establish a prior estimates for the solutions of the periodic initial value problem (1)-(5) and obtain the existence and uniqueness for the local solutions. In Section 4, a sharp threshold for global existence and blowup of the solutions are derived by utilizing two invariant manifolds, applying potential well argument and the concavity method. In the last section we make some conclusions.

2 Notations and preliminaries

We shall use the following conventional notations throughout the paper. Let L^k_{per} and H^k_{per}, $k = 1, 2, \ldots$, denote the Hilbert and Sobolev spaces of L-periodic, complex-valued functions endowed with the usual L^2 inner product $(u, v) = \int_\Omega u(x)\overline{v}(x)\,dx$ and the norms

$$\|u\|_{L^2} = \sqrt{(u,u)}, \qquad \|u\|_{H^k} = \left(\sum_{|\alpha|\le k} \left\|D^\alpha u(x)\right\|\right)^{\frac{1}{2}}.$$

Here $\overline{v}$ denotes the complex conjugate of v. For brevity, we write $\|u\| = \|u\|_{L^2}$ and denote the L^p-norm by $\|u\|_p = (\int_\Omega |u|^p\,dx)^{1/p}$. Without any ambiguity, we denote a generic positive constant by C which may vary from line to line.

In the following sections, we frequently use the following inequalities.

Lemma 1 (Gagliardo-Nirenberg inequality [22]) *Let Ω be a bounded domain with $\partial\Omega$ in C^m, and let u be any function in $W^{m,r}(\Omega) \cap L^q(\Omega)$, $1 \le q, r \le \infty$. For any integer j, $0 \le j < m$, and for any number a in the interval $j/m \le a \le 1$, set*

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{r} - \frac{m}{N}\right) + (1-a)\frac{1}{q}.$$

If $m - j - N/r$ is not a nonnegative integer, then

$$\left\|D^j u\right\|_{L^p} \le C\|u\|^a_{W^{m,r}}\|u\|^{1-a}_{L^q}. \tag{7}$$

If $m - j - N/r$ is a nonnegative integer, then (7) *holds for $a = j/m$. The constant C depends only on Ω, r, q, j, a.*

As the specific cases for $N = 2$, there holds

$$\left\|D^j u\right\|_{L^\infty} \le C\|u\|^a_{H^m}\|u\|^{1-a}, \quad ma = j+1, \tag{8}$$

$$\left\|D^j u\right\|_{L^2} \le C\|u\|^a_{H^m}\|u\|^{1-a}, \quad ma = j, \tag{9}$$

$$\left\|D^j u\right\|_{L^4} \le C\|u\|^a_{H^m}\|u\|^{1-a}, \quad ma = j+1/2. \tag{10}$$

3 The local existence and uniqueness of solutions

In this section, we will obtain existence and uniqueness of local solutions for problem (1)-(5). Firstly we derive some a priori estimates for the solutions.

Lemma 2 *Assume* $u_0(x) \in H^1_{\mathrm{per}}(\Omega)$, $u_1(x) \in L^2_{\mathrm{per}}(\Omega)$, $n_0(x) \in H^1_{\mathrm{per}}(\Omega)$, *and* $V_0(x) \in L^2_{\mathrm{per}}(\Omega)$. *Then, for the solutions of problem* (1)-(5), *we have*

$$\|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2 + \|u\|^4_{L^4} + \|n\|^2 + \|\nabla n\|^2 + \|V\|^2 \le C, \tag{11}$$

where C is a constant depending only on $\|u_0\|_{H^1_{\mathrm{per}}(\Omega)}$, $\|u_1\|_{L^2_{\mathrm{per}}(\Omega)}$, $\|n_0\|_{H^1_{\mathrm{per}}(\Omega)}$ *and* $\|V_0\|_{L^2_{\mathrm{per}}(\Omega)}$.

Proof Multiplying (1) by $\overline{u}_t$, integrating with respect to x over Ω and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|u_t\|^2 + \|\nabla u\|^2 + \|u\|^2\big) + \mathrm{Re}\int_\Omega nu\overline{u}_t\,dx + \mathrm{Re}\int_\Omega |u|^2 u\overline{u}_t\,dx = 0. \tag{12}$$

Using equations (2) and (3) repeatedly and noticing the periodicity, we have

$$\begin{aligned}
&\mathrm{Re}\int_\Omega nu\overline{u}_t\,dx \\
&= \frac{1}{2}\int_\Omega n(u\overline{u}_t + \overline{u}u_t)\,dx = \frac{1}{2}\int_\Omega n\frac{d}{dt}\big(|u|^2\big)\,dx \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega n|u|^2\,dx - \frac{1}{2}\int_\Omega n_t|u|^2\,dx \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega n|u|^2\,dx + \frac{1}{2}\int_\Omega \nabla V|u|^2\,dx \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega n|u|^2\,dx - \frac{1}{2}\int_\Omega V\nabla\big(|u|^2\big)\,dx \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega n|u|^2\,dx - \frac{1}{2}\int_\Omega V\big(-V_t - \nabla n + H^2\nabla\Delta n\big)\,dx \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega n|u|^2\,dx + \frac{1}{4}\frac{d}{dt}\|V\|^2 - \frac{1}{2}\int_\Omega \nabla Vn\,dx + \frac{H^2}{2}\int_\Omega \nabla V\Delta n\,dx \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega n|u|^2\,dx + \frac{1}{4}\frac{d}{dt}\|V\|^2 + \frac{1}{2}\int_\Omega n_t n\,dx - \frac{H^2}{2}\int_\Omega n_t\Delta n\,dx \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega n|u|^2\,dx + \frac{1}{4}\frac{d}{dt}\|V\|^2 + \frac{1}{4}\frac{d}{dt}\|n\|^2 + \frac{H^2}{4}\frac{d}{dt}\|\nabla n\|^2
\end{aligned} \tag{13}$$

and

$$\mathrm{Re}\int_\Omega |u|^2 u\overline{u}_t\,dx = \frac{1}{2}\int_\Omega |u|^2(u\overline{u}_t + \overline{u}u_t)\,dx = \frac{1}{4}\frac{d}{dt}\|u\|^4_{L^4}. \tag{14}$$

Combining (12),(13) and (14), we have

$$\begin{aligned}
&\frac{d}{dt}\bigg(\|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2 + \frac{1}{2}\|u\|^4_{L^4} + \frac{1}{2}\|n\|^2 \\
&\quad + \frac{H^2}{2}\|\nabla n\|^2 + \frac{1}{2}\|V\|^2 + \int_\Omega n|u|^2\,dx\bigg) = 0,
\end{aligned} \tag{15}$$

which implies

$$\begin{aligned}
&\|u_t\|^2 + \|u\|_{H^1}^2 + \frac{1}{2}\|u\|_{L^4}^4 + \frac{1}{2}\|n\|^2 + \frac{H^2}{2}\|\nabla n\|^2 + \frac{1}{2}\|V\|^2 + \int_\Omega n|u|^2\,dx \\
&\quad = \|u_1\|^2 + \|u_0\|_{H^1}^2 + \frac{1}{2}\|u_0\|_{L^4}^4 + \frac{1}{2}\|n_0\|^2 \\
&\qquad + \frac{H^2}{2}\|\nabla n_0\|^2 + \frac{1}{2}\|V_0\|^2 + \int_\Omega n_0|u_0|^2\,dx \\
&\quad = E_0(u_0,u_1,n_0,V_0),
\end{aligned} \tag{16}$$

where E_0 is a positive constant depending only on $\|u_0\|_{H^1_{\mathrm{per}}(\Omega)}$, $\|u_1\|_{L^2_{\mathrm{per}}(\Omega)}$, $\|n_0\|_{H^1_{\mathrm{per}}(\Omega)}$ and $\|V_0\|_{L^2_{\mathrm{per}}(\Omega)}$. Since

$$\left|\int_\Omega n|u|^2\,dx\right| \le \int_\Omega \frac{1}{2}\left(n^2 + |u|^4\right)dx = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u\|_{L^4}^4. \tag{17}$$

Thus we have

$$\|u_t\|^2 + \frac{1}{2}\|u\|^2 + \|\nabla u\|^2 + \frac{1}{2}\|n\|^2 + \frac{H^2}{2}\|\nabla n\|^2 + \frac{1}{2}\|V\|^2 \le E_0. \tag{18}$$

By the Gagliardo-Nirenberg inequality (10) and (18), we have

$$\|u\|_{L^4}^4 \le C\|u\|_{H^1}^2\|u\|^2 \le C. \tag{19}$$

Thus Lemma 2 is completed from (18) and (19). □

Lemma 3 *Assume* $u_0(x) \in H^2_{\mathrm{per}}(\Omega)$, $u_1(x) \in H^1_{\mathrm{per}}(\Omega)$, $n_0(x) \in H^2_{\mathrm{per}}(\Omega)$, *and* $V_0(x) \in H^1_{\mathrm{per}}(\Omega)$. *Then, for the solutions of problem* (1)-(5), *there holds*

$$\|\nabla u_t\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2 + \|\nabla n\|^2 + \|\Delta n\|^2 + \|\nabla V\|^2 \le C, \tag{20}$$

where C is a constant depending only on $\|u_0\|_{H^2_{\mathrm{per}}(\Omega)}$, $\|u_1\|_{H^1_{\mathrm{per}}(\Omega)}$, $\|n_0\|_{H^2_{\mathrm{per}}(\Omega)}$, $\|V_0\|_{H^1_{\mathrm{per}}(\Omega)}$ *and* T.

Proof Taking the inner product of (1) with $-\Delta\overline{u}_t$ in Ω and using the integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u_t\|^2 + \|\Delta u\|^2 + \|\nabla u\|^2\right) + \operatorname{Re}\int_\Omega \nabla(nu)\nabla\overline{u}_t\,dx \\
&\quad + \operatorname{Re}\int_\Omega \nabla\left(|u|^2u\right)\nabla\overline{u}_t\,dx = 0.
\end{aligned} \tag{21}$$

From Gagliardo-Nirenberg inequality (8) and Lemma 2, we have

$$\begin{aligned}
\left|\operatorname{Re}\int_\Omega \nabla(nu)\nabla\overline{u}_t\,dx\right| &= \int_\Omega |\nabla n||u||\nabla u_t|\,dx + \int_\Omega |n||\nabla u||\nabla u_t|\,dx \\
&\le \|u\|_{L^\infty}\|\nabla n\|\|\nabla u_t\| + \|n\|_{L^\infty}\|\nabla u\|\|\nabla u_t\|
\end{aligned}$$

$$\begin{aligned}&\le C\|u\|_{H^2}^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|\nabla u_t\| + C\|n\|_{H^2}^{\frac{1}{2}}\|n\|^{\frac{1}{2}}\|\nabla u_t\|\\&\le C\|\Delta u\|^2 + C\|\Delta n\|^2 + C\|\nabla u_t\|^2 \qquad (22)\end{aligned}$$

and

$$\begin{aligned}\left|\mathrm{Re}\int_\Omega \nabla\big(|u|^2 u\big)\nabla\overline{u}_t\,dx\right| &\le 3\|u\|_{L^\infty}^2\|\nabla u\|\|\nabla u_t\|\\&\le C\|u\|_{H^2}\|u\|\|\nabla u\|\|\nabla u_t\|\\&\le C\|\Delta u\|^2 + C\|\nabla u_t\|^2. \qquad (23)\end{aligned}$$

Combining (21),(22) and (23), we get

$$\frac{d}{dt}\big(\|\nabla u_t\|^2 + \|\Delta u\|^2 + \|\nabla u\|^2\big) \le C\big(\|\Delta u\|^2 + \|\Delta n\|^2 + \|\nabla u_t\|^2\big), \qquad (24)$$

where C is a positive constant.

On the other hand, equations (2) and (3) are equivalent to

$$n_{tt} - \Delta n + H^2\Delta^2 n = \Delta\big(|u|^2\big). \qquad (25)$$

Then one multiplies (25) by n_t and integrates with respect to x over Ω, and combines the result with (2) to have

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla V\|^2 + \|\nabla n\|^2 + H^2\|\Delta n\|^2\big) = \int_\Omega \Delta\big(|u|^2\big)(-\nabla V)\,dx. \qquad (26)$$

By the Gagliardo-Nirenberg inequality (8) and Lemma 2, we can obtain

$$\begin{aligned}\left|\int_\Omega \Delta\big(|u|^2\big)(-\nabla V)\,dx\right| &\le 2\int_\Omega |\nabla u|^2|\nabla V|\,dx + 2\int_\Omega |u\Delta u||\nabla V|\,dx\\&\le 2\|\nabla u\|_{L^4}^2\|\nabla V\| + 2\|u\|_{L^\infty}\|\Delta u\|\|\nabla V\|\\&\le C\|u\|_{H^2}^{\frac{3}{2}}\|u\|^{\frac{1}{2}}\|\nabla V\| + C\|u\|_{H^2}^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|\Delta u\|\|\nabla V\|\\&\le C\|\Delta u\|^2 + C\|\nabla V\|^2. \qquad (27)\end{aligned}$$

From (24),(26) and (27), it follows

$$\begin{aligned}&\frac{d}{dt}\big(\|\nabla u_t\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2 + \|\nabla n\|^2 + H^2\|\Delta n\|^2 + \|\nabla V\|^2\big)\\&\quad\le C\big(\|\Delta u\|^2 + \|\Delta n\|^2 + \|\nabla u_t\|^2 + \|\nabla V\|^2\big)\\&\quad\le C\big(\|\nabla u_t\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2 + \|\nabla n\|^2 + H^2\|\Delta n\|^2 + \|\nabla V\|^2\big). \qquad (28)\end{aligned}$$

By Gronwall's inequality, we can complete the proof of Lemma 3. □

Lemma 4 *Assume* $u_0(x) \in H^3_{\mathrm{per}}(\Omega)$, $u_1(x) \in H^2_{\mathrm{per}}(\Omega)$, $n_0(x) \in H^3_{\mathrm{per}}(\Omega)$, *and* $V_0(x) \in H^2_{\mathrm{per}}(\Omega)$. *Then, for the solutions of problem* (1)-(5), *there holds*

$$\|\Delta u_t\|^2 + \|\Delta u\|^2 + \|\nabla\Delta u\|^2 + \|\Delta n\|^2 + \|\nabla\Delta n\|^2 + \|\Delta V\|^2 \le C, \qquad (29)$$

where C is a constant depending only on $\|u_0\|_{H^3_{\mathrm{per}}(\Omega)}$, $\|u_1\|_{H^2_{\mathrm{per}}(\Omega)}$, $\|n_0\|_{H^3_{\mathrm{per}}(\Omega)}$, $\|V_0\|_{H^2_{\mathrm{per}}(\Omega)}$ *and* T.

Proof Multiplying (1) by $\Delta^2\overline{u}_t$, integrating with respect to x over Ω and taking the real part, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta u_t\|^2+\|\nabla\Delta u\|^2+\|\Delta u\|^2\right)+\mathrm{Re}\int_\Omega nu\Delta^2\overline{u}_t\,dx + \mathrm{Re}\int_\Omega |u|^2u\Delta^2\overline{u}_t\,dx=0. \tag{30}$$

Meanwhile, one takes the inner product of (25) with $-\Delta n_t$ in Ω, and combines it with (2) to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta V\|^2+\|\Delta n\|^2+H^2\|\nabla\Delta n\|^2\right)=\int_\Omega \Delta\left(|u|^2\right)\nabla\Delta V\,dx. \tag{31}$$

Adding (30) and (31) together, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Delta u_t\|^2+\|\nabla\Delta u\|^2+\|\Delta u\|^2+\|\Delta V\|^2+\|\Delta n\|^2+H^2\|\nabla\Delta n\|^2\right)\\&\quad=-\mathrm{Re}\int_\Omega nu\Delta^2\overline{u}_t\,dx-\mathrm{Re}\int_\Omega |u|^2u\Delta^2\overline{u}_t\,dx+\int_\Omega \Delta\left(|u|^2\right)\nabla\Delta V\,dx.\end{aligned} \tag{32}$$

Now we need some estimates for the terms on the right-hand side of (32). First from Gagliardo-Nirenberg inequality (8), Lemma 2 and Lemma 3, we have

$$\begin{aligned}&\left|-\mathrm{Re}\int_\Omega nu\Delta^2\overline{u}_t\,dx\right|\\&\quad\le\int_\Omega\left(|\Delta n||u|+2|\nabla n||\nabla u|+|n||\Delta u|\right)|\Delta u_t|\,dx\\&\quad\le\left(\|u\|_{L^\infty}\|\Delta n\|+2\|\nabla n\|_{L^\infty}\|\nabla u\|+\|n\|_{L^\infty}\|\Delta u\|\right)\|\Delta u_t\|\\&\quad\le C\|\Delta n\|^2+C\|\Delta u\|^2+C\|\nabla n\|_{H^2}^{\frac{1}{2}}\|\nabla n\|^{\frac{1}{2}}\|\Delta u_t\|+C\|\Delta u_t\|^2\\&\quad\le C\|\Delta u\|^2+C\|\Delta n\|^2+C\|\nabla\Delta n\|^2+C\|\Delta u_t\|^2\end{aligned} \tag{33}$$

and

$$\begin{aligned}\left|-\mathrm{Re}\int_\Omega |u|^2u\Delta^2\overline{u}_t\,dx\right|&\le\int_\Omega\left(6|u||\nabla u|^2+3|u|^2|\Delta u|\right)|\Delta u_t|\,dx\\&\le\left(6\|\nabla u\|_{L^\infty}^2\|u\|+3\|u\|_{L^\infty}^2\|\Delta u\|\right)\|\Delta u_t\|\\&\le C\|\nabla u\|_{H^2}\|\nabla u\|\|\Delta u_t\|+C\|\Delta u\|^2+C\|\Delta u_t\|^2\\&\le C\|\nabla\Delta u\|^2+C\|\Delta u\|^2+C\|\Delta u_t\|^2.\end{aligned} \tag{34}$$

Similarly, we can compute that

$$\left|\int_\Omega \Delta\left(|u|^2\right)\nabla\Delta V\,dx\right|\le C\|\nabla\Delta u\|^2+C\|\Delta u\|^2+C\|\Delta V\|^2. \tag{35}$$

Combining (32),(33), (34),and (35), we get

$$\begin{aligned}&\frac{d}{dt}\big(\|\Delta u_t\|^2+\|\nabla\Delta u\|^2+\|\Delta u\|^2+\|\Delta n\|^2+H^2\|\nabla\Delta n\|^2+\|\Delta V\|^2\big)\\&\quad\le C\big(\|\Delta u_t\|^2+\|\nabla\Delta n\|^2+\|\Delta u\|^2+\|\Delta n\|^2+\|\Delta V\|^2\big)\\&\quad\le C\big(\|\Delta u_t\|^2+\|\nabla\Delta u\|^2+\|\Delta u\|^2+\|\Delta n\|^2+H^2\|\nabla\Delta n\|^2+\|\Delta V\|^2\big),\end{aligned}\tag{36}$$

where C is a positive constant. Applying Gronwall's inequality, the proof of Lemma 4 is completed. □

Generally, based on the results of the previous lemmas and the mathematical deduction, we have the following lemma for problem (1)-(5).

Lemma 5 *Assume* $u_0(x)\in H^{k+1}_{\mathrm{per}}(\Omega)$, $u_1(x)\in H^k_{\mathrm{per}}(\Omega)$, $n_0(x)\in H^{k+1}_{\mathrm{per}}(\Omega)$, *and* $V_0(x)\in H^k_{\mathrm{per}}(\Omega)$ $(k\ge 0)$. *Then, for the solutions of problem* (1)-(5), *we have the following estimates*:

$$\|u_t\|^2_{H^k}+\|u\|^2_{H^{k+1}}+\|n\|^2_{H^{k+1}}+\|V\|^2_{H^k}\le C,\tag{37}$$

where C *is a positive constant depending only on* $\|u_0\|_{H^{k+1}_{\mathrm{per}}(\Omega)}$, $\|u_1\|_{H^k_{\mathrm{per}}(\Omega)}$, $\|n_0\|_{H^{k+1}_{\mathrm{per}}(\Omega)}$, $\|V_0\|_{H^k_{\mathrm{per}}(\Omega)}$ *and* T.

Based on the estimates in Lemmas 2-5, we can employ the Galerkin method to obtain the existence of local smooth solutions for problem (1)-(5). Similarly, the uniqueness of the smooth solutions can also be obtained by the usual method of energy estimates. We omit the detailed proof here. Thus we have the following existence and uniqueness theorems for the local smooth solutions of problem (1)-(5).

Theorem 1 (Existence and uniqueness for local smooth solutions) *Suppose that* $u_0(x)\in H^{k+1}_{\mathrm{per}}(\Omega)$, $u_1(x)\in H^k_{\mathrm{per}}(\Omega)$, $n_0(x)\in H^{k+1}_{\mathrm{per}}(\Omega)$, *and* $V_0(x)\in H^k_{\mathrm{per}}(\Omega)$ $(k\ge 2)$. *Then there exists unique local smooth solutions* $u(x,t)$, $n(x,t)$ *and* $V(x,t)$, *which satisfy*

$$\begin{aligned}&u(x,t)\in L^\infty\big(0,T;H^{k+1}_{\mathrm{per}}(\Omega)\big),\quad u_{tt}(x,t)\in L^\infty\big(0,T;H^{k-1}_{\mathrm{per}}(\Omega)\big),\\&n(x,t)\in L^\infty\big(0,T;H^{k+1}_{\mathrm{per}}(\Omega)\big),\quad n_t(x,t)\in L^\infty\big(0,T;H^{k-1}_{\mathrm{per}}(\Omega)\big),\\&V(x,t)\in L^\infty\big(0,T;H^{k}_{\mathrm{per}}(\Omega)\big),\quad V_t(x,t)\in L^\infty\big(0,T;H^{k-1}_{\mathrm{per}}(\Omega)\big),\end{aligned}$$

for the periodic initial value problem (1)-(5).

4 Global existence and blowup

In the previous section, when we employ the Gronwall's inequality to deduce the estimates for higher regularity, it holds only with $C=C(T)$. That is why we just obtain the local existence for the solutions. However, we found in the proof of Lemma 2 that the boundedness of the estimates is independent of the time T. Thus we need to ask, if the initial values $(u_0,u_1,n_0,V_0)\in H^1_{\mathrm{per}}(\Omega)\times L^2_{\mathrm{per}}(\Omega)\times H^1_{\mathrm{per}}(\Omega)\times L^2_{\mathrm{per}}(\Omega)$, does problem (1)-(5) admit a global unique solution? If it does, what are the conditions? In this section, we will answer these questions.

First, from the results in Lemma 2 and previous section theorem, we know that for any $(u_0, u_1, n_0, V_0) \in H^1_{\mathrm{per}}(\Omega) \times L^2_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega) \times L^2_{\mathrm{per}}(\Omega)$, problem (1)-(5) admits a unique solution

$$(u, u_t, n, V) \in C\big([0, T_{\max}); H^1_{\mathrm{per}}(\Omega) \times L^2_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega) \times L^2_{\mathrm{per}}(\Omega)\big), \tag{38}$$

where $t \in [0, T_{\max})$ $(0 < T_{\max} < \infty)$. And from (16), the following conserved energy holds:

$$E(u, u_t, n, V) = E(u_0, u_1, n_0, V_0), \tag{39}$$

where $E(u, u_t, n, V)$ is defined as

$$\begin{aligned} E(u, u_t, n, V) = {}& \|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2 + \frac{1}{2}\|u\|^4_{L^4} + \frac{1}{2}\|n\|^2 \\ & + \frac{H^2}{2}\|\nabla n\|^2 + \frac{1}{2}\|V\|^2 + \int_\Omega n|u|^2\,dx. \end{aligned} \tag{40}$$

As we have noticed, system (1)-(3) includes derivative nonlinearity and different-degree nonlinearities, thus we need to imply some proper techniques to handle these terms. To deal with the derivative nonlinearity, we first introduce a homogeneous Sobolev space $\dot{H}^{-1}_{\mathrm{per}}(\Omega)$ defined by

$$\begin{aligned} \dot{H}^{-1}_{\mathrm{per}}(\Omega) = \big\{ & n | \exists \nu : \Omega \to \Omega \text{ such that } n = -\nabla \cdot \nu, \nu \in L^2_{\mathrm{per}}(\Omega) \\ & \text{and } \|n\|_{\dot{H}^{-1}_{\mathrm{per}}(\Omega)} = \|\nu\|_{L^2_{\mathrm{per}}(\Omega)} \big\}. \end{aligned} \tag{41}$$

Second, we make the assumption that there exists a real vector-valued function $g(x,t) \in L^2_{\mathrm{per}}(\Omega)$ such that

$$g_t(x,t) = V(x,t). \tag{42}$$

For the different-degree nonlinearities, we introduce some functionals and manifolds to handle it. That is to say, for any $(\phi, \psi) \in H^1_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega)$, we define

$$F(\phi, \psi) := \|\phi\|^2 + \|\nabla\phi\|^2 + \frac{1}{2}\|\phi\|^4_{L^4} + \frac{1}{2}\|\psi\|^2 + \frac{H^2}{2}\|\nabla\psi\|^2 + \int_\Omega \psi|\phi|^2\,dx, \tag{43}$$

$$G(\phi, \psi) := 2\|\phi\|^2 + 2\|\nabla\phi\|^2 + 2\|\phi\|^4_{L^4} + \|\psi\|^2 + H^2\|\nabla\psi\|^2 + 3\int_\Omega \psi|\phi|^2\,dx, \tag{44}$$

$$P(\phi, \psi) := F(\phi, \psi) - \frac{1}{\lambda + 1} G(\phi, \psi), \tag{45}$$

and

$$\Phi := \big\{ (\phi, \psi) \in H^1_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega) | G(\phi, \psi) = 0, (\phi, \psi) \neq (0,0) \big\}, \tag{46}$$

$$\Phi^- := \big\{ (\phi, \psi) \in H^1_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega) | G(\phi, \psi) \leq 0, (\phi, \psi) \neq (0,0) \big\}, \tag{47}$$

where $\lambda > 1$ is a constant.

Additionally, we define two constrained variational problems

$$d_\Phi := \inf_{(\phi,\psi)\in\Phi} F(\phi,\psi), \qquad d_{\Phi^-} := \inf_{(\phi,\psi)\in\Phi^-} P(\phi,\psi). \tag{48}$$

According to the definitions, it is easy to find that the energy functional E can be rewritten as

$$E(u,u_t,n,V) = \|u_t\|^2 + \frac{1}{2}\|V\|^2 + F(u,n), \tag{49}$$

or

$$E(u,u_t,n,V) = \|u_t\|^2 + \frac{1}{2}\|V\|^2 - \frac{1}{2}\|u\|_{L^4}^4 - \frac{1}{2}\int_\Omega n|u|^2\,dx + \frac{1}{2}G(u,n). \tag{50}$$

For the properties of $F(\phi,\psi)$ and d_Φ, we have the following results.

Proposition 4.1 *$F(\phi,\psi)$ is bounded below on Φ, $F(\phi,\psi) > 0$ for all $(\phi,\psi)\in\Phi$ and $d_\Phi > 0$.*

Proof First, from (43) and (44), we have

$$3F(\phi,\psi) - G(\phi,\psi) = \|\phi\|^2 + \|\nabla\phi\|^2 - \frac{1}{2}\|\phi\|_{L^4}^4 + \frac{1}{2}\|\psi\|^2 + \frac{H^2}{2}\|\nabla\psi\|^2. \tag{51}$$

Thus on Φ there holds $G(\phi,\psi) = 0$ and

$$F(\phi,\psi) = \frac{1}{3}\|\phi\|^2 + \frac{1}{3}\|\nabla\phi\|^2 + \frac{1}{6}\|\psi\|^2 + \frac{H^2}{6}\|\nabla\psi\|^2 - \frac{1}{6}\|\phi\|_{L^4}^4. \tag{52}$$

On the other hand, from (44) we have

$$\begin{aligned}2\|\phi\|^2 + 2\|\nabla\phi\|^2 + 2\|\phi\|_{L^4}^4 + \|\psi\|^2 + H^2\|\nabla\psi\|^2 &= -3\int_\Omega \psi|\phi|^2\,dx\\ &\le \frac{3}{2}\|\psi\|^2 + \frac{3}{2}\|\phi\|_{L^4}^4,\end{aligned} \tag{53}$$

which implies

$$0 \le 2\|\phi\|^2 + 2\|\nabla\phi\|^2 + H^2\|\nabla\psi\|^2 \le \frac{1}{2}\|\psi\|^2 - \frac{1}{2}\|\phi\|_{L^4}^4. \tag{54}$$

That is also to say

$$\|\psi\|^2 > \|\phi\|_{L^4}^4 \quad \text{on } \Phi. \tag{55}$$

Combining (52) and (55), we can complete the proof of Proposition 4.1. □

For the functional $P(\phi,\psi)$ and d_{Φ^-}, we have the following results.

Proposition 4.2 *If $G(\phi,\psi) \le 0$ and $(\phi,\psi) \ne (0,0)$, then $P(\theta^{\frac{1}{2}}\phi,\theta\psi)$ is an increasing function of $\theta\in(0,\infty)$. And*

$$d_\Phi = \inf_{(\phi,\psi)\in\Phi} F(\phi,\psi) = d_{\Phi^-} = \inf_{(\phi,\psi)\in\Phi^-} P(\phi,\psi). \tag{56}$$

Furthermore, if $G(\phi,\psi) < 0$, *then*

$$P(\phi,\psi) > d_\Phi. \tag{57}$$

Proof The first result that $P(\theta^{\frac{1}{2}}\phi,\theta\psi)$ is an increasing function of θ can be proved directly by computing the derivation with respect to θ. Here we prove (56).

First, from definition (44), we have

$$\begin{aligned} G\bigl(\theta^{\frac{1}{2}}\phi,\theta\psi\bigr) &= 2\theta\|\phi\|^2 + 2\theta\|\nabla\phi\|^2 + 2\theta^2\|\phi\|_{L^4}^4 + \theta^2\|\psi\|^2 \\ &\quad + H^2\theta^2\|\nabla\psi\|^2 + 3\theta^2\int_\Omega \psi|\phi|^2\,dx. \end{aligned} \tag{58}$$

Then, for $\theta = 1$, $G(\phi,\psi) \le 0$, and for $\theta > 0$ close to zero, $G(\theta^{\frac{1}{2}}\phi,\theta\psi) > 0$. Thus from the continuity there exists $\theta_0 \in (0,1)$ such that $G(\theta_0^{\frac{1}{2}}\phi,\theta_0\psi) = 0$, which also implies $(\theta_0^{\frac{1}{2}}\phi,\theta_0\psi) \in \Phi$. Noticing $\Phi \subset \Phi^-$, and from (48), there holds

$$\begin{aligned} d_\Phi &\le \inf_{(\theta_0^{\frac{1}{2}}\phi,\theta_0\psi)\in\Phi} F\bigl(\theta_0^{\frac{1}{2}}\phi,\theta_0\psi\bigr) \\ &\le \inf_{(\theta_0^{\frac{1}{2}}\phi,\theta_0\psi)\in\Phi^-} \left(P\bigl(\theta_0^{\frac{1}{2}}\phi,\theta_0\psi\bigr) + \frac{1}{\lambda+1}G\bigl(\theta_0^{\frac{1}{2}}\phi,\theta_0\psi\bigr)\right) \\ &= \inf_{(\theta_0^{\frac{1}{2}}\phi,\theta_0\psi)\in\Phi^-} P\bigl(\theta_0^{\frac{1}{2}}\phi,\theta_0\psi\bigr) \le \inf_{(\phi,\psi)\in\Phi^-} P(\phi,\psi) \\ &= d_{\Phi^-}. \end{aligned} \tag{59}$$

Meanwhile, we have

$$\begin{aligned} d_\Phi &= \inf_{(\phi,\psi)\in\Phi} F(\phi,\psi) = \inf_{(\phi,\psi)\in\Phi}\left(P(\phi,\psi) + \frac{1}{\lambda+1}G(\phi,\psi)\right) \\ &= \inf_{(\phi,\psi)\in\Phi} P(\phi,\psi) \ge \inf_{(\phi,\psi)\in\Phi^-} P(\phi,\psi) = d_{\Phi^-}. \end{aligned} \tag{60}$$

Thus one combines (59) and (60) to get (56).

Finally, since $G(\phi,\psi) < 0$, then there also exists $\theta \in (0,1)$ such that $G(\theta^{\frac{1}{2}}\phi,\theta\psi) = 0$ and $(\phi,\psi) \neq (0,0)$. Thus there holds

$$P(\phi,\psi) > P\bigl(\theta^{\frac{1}{2}}\phi,\theta\psi\bigr) = F\bigl(\theta^{\frac{1}{2}}\phi,\theta\psi\bigr) - \frac{1}{\lambda+1}G\bigl(\theta^{\frac{1}{2}}\phi,\theta\psi\bigr) = F\bigl(\theta^{\frac{1}{2}}\phi,\theta\psi\bigr) \ge d_\Phi, \tag{61}$$

which leads to (57). The proof of Proposition 4.2 is completed. □

Since $d_\Phi > 0$, then we define a set $\mathcal{S}$ as

$$\begin{aligned} \mathcal{S} := \bigl\{&(\phi_1,\phi_2,\psi_1,\psi_2) \in H^1_{\mathrm{per}}(\Omega)\times L^2_{\mathrm{per}}(\Omega)\times H^1_{\mathrm{per}}(\Omega)\times L^2_{\mathrm{per}}(\Omega) : \\ &E(\phi_1,\phi_2,\psi_1,\psi_2) < d_\Phi\bigr\}, \end{aligned}$$

and introduce two invariant sets as

$$\mathcal{S}_1 := \{(\phi_1,\phi_2,\psi_1,\psi_2)\in\mathcal{S}|G(\phi_1,\psi_1)>0\}\cup\{(0,\phi_2,0,\psi_2)\in\mathcal{S}\},$$
$$\mathcal{S}_2 := \{(\phi_1,\phi_2,\psi_1,\psi_2)\in\mathcal{S}|G(\phi_1,\psi_1)<0\}.$$

For the sets $\mathcal{S}_1$ and $\mathcal{S}_2$, we have the following proposition.

Proposition 4.3 *$\mathcal{S}_1$ and $\mathcal{S}_2$ are invariant sets under the solution flow generated by the periodic initial value problem* (1)-(5).

Proof We prove the proposition by contradiction. First we prove that $\mathcal{S}_1$ is an invariant set. Let $(u_0,u_1,n_0,V_0)\in\mathcal{S}_1$, and suppose that there exists a time t_1 such that $(u(t_1),u_t(t_1),n(t_1),V(t_1))\notin\mathcal{S}_1$, then $G(u(t_1),n(t_1))\le 0$ and $(u(t_1),n(t_1))\ne(0,0)$, which implies $(u(t_1),n(t_1))\in\Phi^-$. Set

$$s=\inf\{0\le t\le t_1|(u(t),u_t(t),n(t),V(t))\notin\mathcal{S}_1\}, \tag{62}$$

then $G(u(t),n(t))\ge 0$ for all $0\le t<s$. Let $\{s_k\}$ be the minimizing sequence for problem (62), then $(u(s_k),n(s_k))\in\Phi^-$. By the weak lower semi-continuity of $G(u(\cdot),n(\cdot))$, we have

$$G(u(s),n(s))\le\lim_{k\to\infty}\inf G(u(s_k),n(s_k))\le 0,\quad (u(s),n(s))\ne(0,0). \tag{63}$$

On the other hand, from (45), (49) and (56) in Proposition 4.2, we obtain

$$\begin{aligned}P(u(s),n(s)) &= \lim_{t\to s^-}\inf P(u(t),n(t))\\ &\le\lim_{t\to s^-}\inf\left(P(u(t),n(t))+\frac{1}{\lambda+1}G(u(t),n(t))\right)\\ &=\lim_{t\to s^-}\inf F(u(t),n(t))\\ &\le\lim_{t\to s^-}\inf E(u(t),u_t(t),n(t),V(t))\\ &< d_\Phi = d_{\Phi^-},\end{aligned} \tag{64}$$

which contradicts definition (48). So $\mathcal{S}_1$ is invariant.

Now we turn to proving that $\mathcal{S}_2$ is also invariant. Similarly, let $(u_0,u_1,n_0,V_0)\in\mathcal{S}_2$, and assume that there exists t_2 such that $(u(t_2),u_t(t_2),n(t_2),V(t_2))\notin\mathcal{S}_2$, which implies $G(u(t_2),n(t_2))\ge 0$. From (45) and (49), we have

$$\begin{aligned}P(u(t_2),n(t_2)) &= F(u(t_2),n(t_2))-\frac{1}{\lambda+1}G(u(t_2),n(t_2))\\ &\le F(u(t_2),n(t_2))<d_\Phi.\end{aligned} \tag{65}$$

Let

$$s=\inf\{0\le t\le t_2|(u(t),u_t(t),n(t),V(t))\notin\mathcal{S}_2\}, \tag{66}$$

then $G(u(t),n(t)) > d_\Phi$ for all $0 \le t < s$. However, from (45), there holds

$$\begin{aligned} G\bigl(u(s),n(s)\bigr) &= \lim_{t\to s^-}\inf(\lambda+1)\bigl[F\bigl(u(t),n(t)\bigr) - P\bigl(u(t),n(t)\bigr)\bigr] \\ &\le \lim_{t\to s^-}\inf(\lambda+1)\bigl[E\bigl(u(t),u_t(t),n(t),V(t)\bigr) - d_\Phi\bigr] \\ &\le (\lambda+1)\bigl[E(u_0,u_1,n_0,V_0) - d_\Phi\bigr] \\ &< 0. \end{aligned} \tag{67}$$

From Proposition 4.2, we know that if $G(u(s),n(s)) < 0$, then $G(u(s),n(s)) > d_\Phi$, which makes contradiction. Thus $\mathcal{S}_2$ is also invariant. The proof of Proposition 4.3 is completed. □

Based on the results from previous propositions, we can derive a sharp threshold of global existence and blowup for the solution $(u(x,t),n(x,t),V(x,t))$ to the periodic initial value problem (1)-(5) in terms of the relationship between the initial energy $E(u_0,u_1,n_0,V_0)$ and $d_\Phi > 0$. Here we state the main results as follows.

Theorem 2 (Global existence and blowup) *Suppose that* $(u_0,u_1,n_0,V_0) \in H^1_{\mathrm{per}}(\Omega) \times L^2_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega) \times L^2_{\mathrm{per}}(\Omega)$ *and satisfy*

$$E(u_0,u_1,n_0,V_0) < d_\Phi. \tag{68}$$

Then

(1) *If*

$$G(u_0,n_0) < 0, \tag{69}$$

then the solution $(u(x,t),n(x,t),V(x,t))$ *of the periodic initial value problem* (1)-(5) *blows up in finite time. That is, there exists* $T > 0$ *such that*

$$\lim_{t\to T}\bigl(\|u\|^2_{L^2_{\mathrm{per}}(\Omega)} + \|n\|^2_{\dot{H}^{-1}_{\mathrm{per}}(\Omega)}\bigr) = \infty.$$

(2) *If*

$$G(u_0,n_0) > 0, \tag{70}$$

then the solution $(u(x,t),n(x,t),V(x,t))$ *of the periodic initial value problem* (1)-(5) *exists globally on* $t \in [0,\infty)$ *and satisfies*

$$\|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2 + \frac{1}{2}\|u\|^4_{L^4} + \frac{1}{2}\|n\|^2 + \frac{H^2}{2}\|\nabla n\|^2 + \frac{1}{2}\|V\|^2 < d_\Phi, \tag{71}$$

or

$$\|u_t\|^2 + \frac{4}{3}\|u\|^2 + \frac{4}{3}\|\nabla u\|^2 + \frac{1}{6}\|n\|^2 + \frac{2}{3}H^2\|\nabla n\|^2 + \frac{1}{2}\|V\|^2 \le 2d_\Phi + \frac{1}{6}C^\star d_\Phi^2, \tag{72}$$

where $C^\star$ *is a positive constant which satisfies the Gagliardo-Nirenberg inequality.*

Proof First we prove (1) of Theorem 2. From (39), (49) and condition (68), we have

$$F(u_0,n_0) \le E(u_0,u_1,n_0,V_0) < d_\Phi. \tag{73}$$

(1) When $G(u_0,n_0) < 0$, as the initial values $(u_0,u_1,n_0,V_0) \in \mathcal{S}_2$, and it follows that $(u(x,t),u_t(x,t),n(x,t),V(x,t)) \in \mathcal{S}_2$ by Proposition 4.3. Thus

$$G\big(u(x,t),n(x,t)\big) < 0 \text{ for } t \in [0,T). \tag{74}$$

And from (49) and (73), there holds

$$F\big(u(x,t),n(x,t)\big) < d_\Phi. \tag{75}$$

On the other hand, since $(u(x,t),u_t(x,t),n(x,t),V(x,t))$ is a solution of periodic initial value problem (1)-(5), under assumption (42), we set

$$Y(t) = 2\big\|u(x,t)\big\|^2 + \big\|g(x,t)\big\|^2. \tag{76}$$

Thus we get

$$Y'(t) = 2\int_\Omega (u_t\overline{u} + u\overline{u}_t)\,dx + 2\int_\Omega gg_t\,dx, \tag{77}$$

where $\overline{u}$ is the complex conjugate of u. By computing $F''(t)$, one can obtain

$$\begin{aligned}
Y''(t) &= 2\int_\Omega (u_{tt}\overline{u} + u_t\overline{u}_t + u_t\overline{u}_t + u\overline{u}_{tt})\,dx + 2\int_\Omega (g_tg_t + gg_{tt})\,dx \\
&= 4\|u_t\|^2 + 4\int_\Omega \operatorname{Re}(u_{tt}\overline{u})\,dx + 2\|g_t\|^2 + 2\int_\Omega gg_{tt}\,dx \\
&= 4\|u_t\|^2 + 4\int_\Omega \operatorname{Re}\big((\Delta u - u - nu - |u|^2u)\overline{u}\big)\,dx + 2\|g_t\|^2 \\
&\quad + 2\int_\Omega g\big(-\nabla n - \nabla|u|^2 + H^2\nabla\Delta n\big)\,dx \\
&= 4\|u_t\|^2 - 4\|u\|^2 - 4\|\nabla u\|^2 - 4\|u\|_{L^4}^4 - 4\int_\Omega n|u|^2\,dx + 2\|V\|^2 \\
&\quad + 2\int_\Omega (\nabla\cdot g)\big(n + |u|^2 - H^2\Delta n\big)\,dx \\
&= 4\|u_t\|^2 + 2\|V\|^2 - 4\|u\|^2 - 4\|\nabla u\|^2 - 4\|u\|_{L^4}^4 - 4\int_\Omega n|u|^2\,dx \\
&\quad - 2\|n\|^2 - 2\int_\Omega n|u|^2\,dx - 2H^2\|\nabla n\|^2 \\
&= 2\big(2\|u_t\|^2 + \|V\|^2\big) - 4\|u\|^2 - 4\|\nabla u\|^2 - 4\|u\|_{L^4}^4 - 2\|n\|^2 \\
&\quad - 2H^2\|\nabla n\|^2 - 6\int_\Omega n|u|^2\,dx \\
&= 2\big(2\|u_t\|^2 + \|V\|^2\big) - 2G(u,n).
\end{aligned} \tag{78}$$

Combining (74), it yields that

$$Y''(t) > 0, \quad \text{for } t \in [0, T). \tag{79}$$

Before continuing to complete the proof, we give some property about the function $Y(t)$ defined in (76).

Proposition 4.4 *The function $Y'(t)$ in* (77) *is positive for some t. That is, there exists $t_1 > 0$ such that $Y'(t) > 0$ for all $t > t_1$.*

Proof We prove it by contradiction. Suppose that, for all $t > 0$, there holds

$$Y'(t) \le 0. \tag{80}$$

Combining (79), $Y(t)$ must tend to a finite and nonnegative limit Y_0 as $t \to \infty$. From Proposition 4.3, it concludes $Y_0 > 0$. And as $t \to \infty$, there holds $Y(t) \to Y_0$, $Y'(t) \to 0$ and $Y''(t) \to 0$. Then, from (78) and $G(u, n) < 0$, it follows that

$$\lim_{t\to\infty} 2\bigl(2\|u_t\|^2 + \|V\|^2\bigr) = 0 \tag{81}$$

and

$$\lim_{t\to\infty} G(u, n) = 0. \tag{82}$$

Now, for any fixed $t > 0$, and since $G(u, n) < 0$, there exists $0 < \theta < 1$ such that $G(\theta^{\frac{1}{2}}u, \theta n) = 0$ and $(\theta^{\frac{1}{2}}u, \theta n) \neq 0$. Thus from (44) we can get

$$-\frac{3}{2}\theta^2 \int_\Omega n|u|^2\,dx = \theta\bigl(\|u\|^2 + \|\nabla u\|^2\bigr) + \theta^2\|u\|_{L^4}^4 + \frac{1}{2}\theta^2\|n\|^2 + \frac{1}{2}H^2\theta^2\|\nabla n\|^2 \tag{83}$$

and

$$\int_\Omega n|u|^2\,dx < 0. \tag{84}$$

Thus by (48) we get

$$F\bigl(\theta^{\frac{1}{2}}u, \theta n\bigr) \ge d_\Phi. \tag{85}$$

From (43) and (83) it follows that

$$\begin{aligned} F\bigl(\theta^{\frac{1}{2}}u, \theta n\bigr) &= \theta\bigl(\|u\|^2 + \|\nabla u\|^2\bigr) + \frac{1}{2}\theta^2\|u\|_{L^4}^4 + \frac{1}{2}\theta^2\|n\|^2 \\ &\quad + \frac{H^2}{2}\theta^2\|\nabla u\|^2 + \theta^2\int_\Omega n|u|^2\,dx \\ &= -\frac{3}{2}\theta^2\int_\Omega n|u|^2\,dx - \frac{1}{2}\theta^2\|u\|_{L^4}^4 + \theta^2\int_\Omega n|u|^2\,dx \\ &= -\frac{1}{2}\theta^2\int_\Omega n|u|^2\,dx - \frac{1}{2}\theta^2\|u\|_{L^4}^4. \end{aligned} \tag{86}$$

Thus

$$
\begin{aligned}
F(u,n)-F\bigl(\theta^{\frac{1}{2}}u,\theta n\bigr) &= \|u\|^2+\|\nabla u\|^2+\frac{1}{2}\|u\|_{L^4}^4+\frac{1}{2}\|n\|^2+\frac{H^2}{2}\|\nabla n\|^2 \\
&\quad +\int_\Omega n|u|^2\,dx+\frac{1}{2}\theta^2\int_\Omega n|u|^2\,dx+\frac{1}{2}\theta^2\|u\|_{L^4}^4 \\
&= \|u\|^2+\|\nabla u\|^2+\|u\|_{L^4}^4+\frac{1}{2}\|n\|^2+\frac{H^2}{2}\|\nabla n\|^2 \\
&\quad +\int_\Omega n|u|^2\,dx+\frac{1}{2}\theta^2\int_\Omega n|u|^2\,dx+\frac{1}{2}\bigl(\theta^2-1\bigr)\|u\|_{L^4}^4. \qquad (87)
\end{aligned}
$$

On the other hand, from definition (44) and (82), we can obtain as $t\to\infty$,

$$
\|n\|^2+2\|u\|_{L^4}^4+3\int_\Omega n|u|^2\,dx\le 0. \qquad (88)
$$

Together with

$$
-\frac{3}{2}\|n\|^2-\frac{3}{2}\int_\Omega |u|^4\,dx\le 3\int_\Omega n|u|^2\,dx\le \frac{3}{2}\|n\|^2+\frac{3}{2}\int_\Omega |u|^4\,dx, \qquad (89)
$$

there holds

$$
\int_\Omega |u|^4\,dx\le \|n\|^2, \qquad (90)
$$

and

$$
\int_\Omega |u|^4\,dx\le \|n\|^2\le -2\|u\|_{L^4}^4-3\int_\Omega n|u|^2\,dx, \qquad (91)
$$

which shows

$$
\int_\Omega |u|^4\,dx\le -\int_\Omega n|u|^2\,dx. \qquad (92)
$$

Since $0<\theta<1$, we have as $t\to\infty$,

$$
\frac{1}{2}\bigl(\theta^2-1\bigr)\int_\Omega |u|^4\,dx\ge \frac{1}{2}\bigl(\theta^2-1\bigr)\biggl(-\int_\Omega n|u|^2\,dx\biggr). \qquad (93)
$$

Combining (87) and (93), we get

$$
\begin{aligned}
F(u,n)-F\bigl(\theta^{\frac{1}{2}}u,\theta n\bigr) &\ge \|u\|^2+\|\nabla u\|^2+\|u\|_{L^4}^4+\frac{1}{2}\|n\|^2 \\
&\quad +\frac{H^2}{2}\|\nabla n\|^2+\int_\Omega n|u|^2\,dx+\frac{1}{2}\int_\Omega n|u|^2\,dx \\
&= \|u\|^2+\|\nabla u\|^2+\|u\|_{L^4}^4+\frac{1}{2}\|n\|^2 \\
&\quad +\frac{H^2}{2}\|\nabla n\|^2+\frac{3}{2}\int_\Omega n|u|^2\,dx \\
&= \frac{1}{2}G(u,n). \qquad (94)
\end{aligned}
$$

In all, from (82), (85) and (94), we can conclude that as $t \to \infty$,

$$F(u,n) \geq F\bigl(\theta^{\frac{1}{2}}u, \theta n\bigr) \geq d_\Phi, \tag{95}$$

which contradicts $F(u,n) < d_\Phi$ from (75). So supposition (80) is not true. Thus $Y'(t) > 0$ for some t. Thus the proof of Proposition 4.4 is completed. □

Corollary 1 *Under the conditions of Proposition* 4.4, *the function* $Y(t)$ *defined in* (76) *and* $Z(t) = \|u(x,t)\|^2$ *are both increasing for all* $t > t_1$.

Proof We can compute that

$$Z'(t) = \int_\Omega (u_t\overline{u} + u\overline{u}_t)\,dx$$

and

$$\begin{aligned} Z''(t) &= 2\|u_t\|^2 - 2\|u\|^2 - 2\|\nabla u\|^2 - 2\|u\|_{L^4}^4 - 2\int_\Omega n|u|^2\,dx \\ &= 2\|u_t\|^2 + \|n\|^2 + H^2\|\nabla n\|^2 - \biggl(-\int_\Omega n|u|^2\,dx\biggr) - G(u,n) \\ &\geq 2\|u_t\|^2 + \|n\|^2 + H^2\|\nabla n\|^2 - \biggl(\frac{1}{2}\|n\|^2 + \frac{1}{2}\|u\|_{L^4}^4\biggr) - G(u,n) \\ &\geq 2\|u_t\|^2 + \frac{1}{2}\|n\|^2 + H^2\|\nabla n\|^2 - \frac{1}{2}\|u\|_{L^4}^4 - G(u,n). \end{aligned} \tag{96}$$

Since $G(u,n) < 0$, we have

$$2\|u\|^2 + 2\|\nabla u\|^2 + 2\|u\|_{L^4}^4 + \|n\|^2 + H^2\|\nabla n\|^2 \leq -3\int_\Omega n|u|^2\,dx \leq \frac{3}{2}\|u\|_{L^4}^4 + \frac{3}{2}\|n\|^2,$$

which implies that

$$0 < 2\|u\|^2 + 2\|\nabla u\|^2 + 2\|u\|_{L^4}^4 + H^2\|\nabla n\|^2 \leq \frac{1}{2}\|n\|^2 - \frac{1}{2}\|u\|_{L^4}^4. \tag{97}$$

Combining $G(u,n) < 0$, (96) and (97), we have $Z''(t) > 0$. So $Z'(t)$ is strictly increasing for all $t > 0$. Thus if we choose (u_0, u_1) properly such that $Z'(0) = \int_\Omega (u_0\overline{u_1} + u_1\overline{u_0})\,dx \geq 0$, then for all $t > 0$, $Z'(t) > 0$. Therefore, $Z(t) = \|u(x,t)\|^2$ is strictly increasing for all $t > 0$. Without loss of generality and for simplicity, we omit the condition in the present paper and assume that if $Y(t)$ is increasing for all $t > t_1$, then $\|u\|^2$ is increasing for all $t > t_1$. □

Now we go back to the proof. First from (39), (40) and (78), one gets

$$\begin{aligned} -6\int_\Omega n|u|^2\,dx = {}& -6E(u_0,u_1,n_0,V_0) + 6\|u_t\|^2 + 6\|u\|^2 + 6\|\nabla u\|^2 \\ & + 3\|u\|_{L^4}^4 + 3\|n\|^2 + 3H^2\|\nabla n\|^2 + 3\|V\|^2 \end{aligned}$$

and

$$\begin{aligned} Y''(t) &= 5\big(2\|u_t\|^2 + \|V\|^2\big) + 2\|u\|^2 + 2\|\nabla u\|^2 \\ &\quad + H^2\|\nabla n\|^2 + \|n\|^2 - \|u\|_{L^4}^4 - 6E(u_0, u_1, n_0, V_0). \end{aligned} \tag{98}$$

Since $E(u_0, u_1, n_0, V_0)$ is a fixed value, and by Corollary 1, the term $2\|u\|^2 - 6E(u_0, u_1, n_0, V_0)$ will eventually become positive and still remain positive thereafter. Meanwhile, combining (97), we know that the quantity

$$2\|u\|^2 + 2\|\nabla u\|^2 + H^2\|\nabla n\|^2 + \|n\|^2 - \|u\|_{L^4}^4 - 6E(u_0, u_1, n_0, V_0)$$

will eventually become positive and will remain positive thereafter. Thus

$$Y''(t) \geq 5\big(2\|u_t\|^2 + \|V\|^2\big). \tag{99}$$

From (76), (77) and (99), we obtain

$$Y'(t) = 2\int_\Omega (u_t\overline{u} + u\overline{u}_t)\,dx + 2\int_\Omega g g_t\,dx \leq 4\|u_t\|\|u\| + 2\|g\|\|V\| \tag{100}$$

and

$$\begin{aligned} \big(Y'(t)\big)^2 &\leq 16\|u_t\|^2\|u\|^2 + 16\|u_t\|\|u\|\|g\|\|V\| + 4\|g\|^2\|V\|^2 \\ &= 4\big(4\|u_t\|^2\|u\|^2 + \big(2\|u_t\|\|g\|\big)\cdot\big(2\|u\|\|V\|\big) + \|g\|^2\|V\|^2\big) \\ &\leq 4\big(4\|u_t\|^2\|u\|^2 + 2\|u_t\|^2\|g\|^2 + 2\|u\|^2\|V\|^2 + \|g\|^2\|V\|^2\big) \\ &= \frac{4}{5}\times 5\big(2\|u\|^2 + \|g\|^2\big)\big(2\|u_t\|^2 + \|V\|^2\big) \\ &\leq \frac{4}{5}Y(t)Y''(t), \end{aligned} \tag{101}$$

which implies

$$Y(t)Y''(t) - \frac{5}{4}\big(Y'(t)\big)^2 \geq 0. \tag{102}$$

On the other hand, we have

$$\big(Y^{-\frac{1}{4}}(t)\big)'' = -\frac{1}{4}Y^{-\frac{9}{4}}(t)\bigg(Y(t)Y''(t) - \frac{5}{4}\big(Y'(t)\big)^2\bigg). \tag{103}$$

Thus from (102) we have

$$\big(Y^{-\frac{1}{4}}(t)\big)'' \leq 0. \tag{104}$$

Therefore $F^{-\frac{1}{4}}(t)$ is convex for sufficiently large t, and $Y(t) \geq 0$, thus there exists a finite time $T^\star$ such that

$$\lim_{t\to T^\star} Y^{-\frac{1}{4}}(t) = 0,$$

which implies

$$\lim_{t\to T^\star} Y(t) = \lim_{t\to T^\star}\left(2\|u\|^2 + \|g\|^2\right) = \infty. \tag{105}$$

Thus one gets $T < \infty$ and

$$\lim_{t\to T}\left(\|u\|^2_{L^2_{\text{per}}(\Omega)} + \|n\|^2_{\dot{H}^{-1}_{\text{per}}(\Omega)}\right) = \infty. \tag{106}$$

The proof (1) of Theorem 2 is completed.

(2) Now we turn to proving (2) of Theorem 2. When $G(u_0, n_0) > 0$, (73) and Proposition 4.3 imply that $(u(x,t), u_t(x,t), n(x,t), V(x,t)) \in \mathcal{S}_1$ and $E(u, u_t, n, V) < d_\Phi$. There will be two cases to be discussed: $\int_\Omega n|u|^2\,dx \geq 0$ and $\int_\Omega n|u|^2\,dx < 0$, respectively.

For case (i) $\int_\Omega n|u|^2\,dx \geq 0$, from (39), (40) and (73) we have

$$\begin{aligned} &\|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2 + \frac{1}{2}\|u\|^4_{L^4} + \frac{1}{2}\|n\|^2 + \frac{H^2}{2}\|\nabla n\|^2 + \frac{1}{2}\|V\|^2 \\ &\quad \leq E(u_0, u_1, n_0, V_0) < d_\Phi. \end{aligned} \tag{107}$$

Thus we established the a priori estimates of u_t in $L^2(\Omega)$, u in $H^1(\Omega)$, n in $H^1(\Omega)$ and V in $L^2(\Omega)$ for $t \in [0, T)$. Thus it must be $T = \infty$. Then the solution (u, n, V) of the periodic initial value problem (1)-(5) exists globally on $t \in [0, \infty)$. Furthermore, (107) implies estimate (71).

For case (ii) $\int_\Omega n|u|^2\,dx < 0$. First, from Hölder's inequality, we have

$$-\int_\Omega n|u|^2\,dx \leq \frac{1}{2}\|n\|^2 + \frac{1}{2}\|u\|^4_{L^4}. \tag{108}$$

Thus we can get

$$\begin{aligned} E(u, u_t, n, V) &= \|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2 + \frac{1}{2}\|u\|^4_{L^4} + \frac{1}{2}\|n\|^2 \\ &\quad + \frac{H^2}{2}\|\nabla n\|^2 + \frac{1}{2}\|V\|^2 - \left(-\int_\Omega n|u|^2\,dx\right) \\ &\geq \|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2 + \frac{H^2}{2}\|\nabla n\|^2 + \frac{1}{2}\|V\|^2, \end{aligned} \tag{109}$$

which leads to

$$\begin{aligned} \|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2 + \frac{H^2}{2}\|\nabla n\|^2 + \frac{1}{2}\|V\|^2 &\leq E(u, u_t, n, V) \\ &= E(u_0, u_1, n_0, V_0) < d_\Phi. \end{aligned} \tag{110}$$

Meanwhile, from $G(u, n) > 0$ and $F(u, n) < d_\Phi$, we obtain

$$\int_\Omega n|u|^2\,dx > -\frac{2}{3}\|u\|^2 - \frac{2}{3}\|\nabla u\|^2 - \frac{2}{3}\|u\|^4_{L^4} - \frac{1}{3}\|n\|^2 - \frac{1}{3}H^2\|\nabla n\|^2 \tag{111}$$

and

$$\|u\|^2 + \|\nabla u\|^2 + \frac{1}{2}\|u\|_{L^4}^4 + \frac{1}{2}\|n\|^2 + \frac{H^2}{2}\|\nabla n\|^2 + \int_\Omega n|u|^2\,dx < d_\Phi. \tag{112}$$

Thus from (111) and (112) there holds

$$\frac{1}{3}\|u\|^2 + \frac{1}{3}\|\nabla u\|^2 + \frac{1}{6}\|n\|^2 + \frac{1}{6}H^2\|\nabla n\|^2 < d_\Phi + \frac{1}{6}\|u\|_{L^4}^4. \tag{113}$$

According to inequality (10) in Lemma 1 and (110), we have

$$\|u\|_{L^4}^4 \le C^\star\big(\|u\|^2 + \|\nabla u\|^2\big)^2 \le C^\star d_\Phi^2, \tag{114}$$

where $C^\star$ is a positive constant which satisfies the Gagliardo-Nirenberg inequality. Combining (113) and (114) yields

$$\frac{1}{3}\|u\|^2 + \frac{1}{3}\|\nabla u\|^2 + \frac{1}{6}\|n\|^2 + \frac{1}{6}H^2\|\nabla n\|^2 < d_\Phi + \frac{1}{6}C^\star d_\Phi^2. \tag{115}$$

Therefore, by (110) and (115), we have

$$\begin{aligned} &\|u_t\|^2 + \frac{4}{3}\|u\|^2 + \frac{4}{3}\|\nabla u\|^2 + \frac{1}{6}\|n\|^2 + \frac{2}{3}H^2\|\nabla n\|^2 + \frac{1}{2}\|V\|^2 \\ &\quad \le 2d_\Phi + \frac{1}{6}C^\star d_\Phi^2. \end{aligned} \tag{116}$$

Similarly, we have established the a priori estimates of u_t in $L^2(\Omega)$, u in $H^1(\Omega)$, n in $H^1(\Omega)$ and V in $L^2(\Omega)$ for $t \in [0,T)$. Thus it must be $T = \infty$. Then the solution (u,n,V) of the periodic initial value problem (1)-(5) exists globally on $t \in [0,\infty)$. Furthermore, (116) implies estimate (72).

From the discussions of case (i) and case (ii), we complete the proof of (2) of Theorem 2. In sum, the proof of Theorem 2 is completed. □

Based on the results in Theorem 2, we give two more specific conditions of how small the initial data are for the solutions to exist globally.

Theorem 3 (Small initial values criterions) *Suppose that* $(u_0,u_1,n_0,V_0) \in H^1_{\mathrm{per}}(\Omega) \times L^2_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega) \times L^2_{\mathrm{per}}(\Omega)$ *and satisfy*

$$(1)\quad \begin{cases} \int_\Omega n_0|u_0|^2 < 0, \\ \|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2 + \frac{1}{2}\|u_0\|_{L^4}^4 + \frac{1}{2}\|n_0\|^2 \\ \quad + \frac{H^2}{2}\|\nabla n_0\|^2 + \frac{1}{2}\|V_0\|^2 < d_\Phi, \end{cases} \tag{117}$$

or

$$(2)\quad \begin{cases} \int_\Omega n_0|u_0|^2 > 0, \\ \|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2 + \|u_0\|_{L^4}^4 + \|n_0\|^2 \\ \quad + \frac{H^2}{2}\|\nabla n_0\|^2 + \frac{1}{2}\|V_0\|^2 < d_\Phi. \end{cases} \tag{118}$$

Then the solution $(u(x,t), n(x,t), V(x,t))$ *of the periodic initial value problem* (1)-(5) *exists globally.*

Proof (1) If $\int_\Omega n_0|u_0|^2 < 0$, and from (117), we have

$$\begin{aligned} E(u_0,u_1,n_0,V_0) &= \|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2 + \frac{1}{2}\|u_0\|_{L^4}^4 + \frac{1}{2}\|n_0\|^2 \\ &\quad + \frac{H^2}{2}\|\nabla n_0\|^2 + \frac{1}{2}\|V_0\|^2 + \int_\Omega n_0|u_0|^2\,dx \\ &\le \|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2 + \frac{1}{2}\|u_0\|_{L^4}^4 + \frac{1}{2}\|n_0\|^2 \\ &\quad + \frac{H^2}{2}\|\nabla n_0\|^2 + \frac{1}{2}\|V_0\|^2 \\ &< d_\Phi. \end{aligned} \tag{119}$$

Next we will prove $G(u_0,n_0) > 0$. If it is not true, there holds $G(u_0,n_0) \le 0$. Similar to Proposition 4.2, there exists $0 < \theta \le 1$ such that $G(\theta^{\frac{1}{2}}u_0,\theta n_0) = 0$ and $(\theta^{\frac{1}{2}}u_0,\theta n_0) \neq 0$. Since $(u_0,n_0) \neq (0,0)$, so $(\theta^{\frac{1}{2}}u_0,\theta n_0) \in \Phi$ and

$$F\big(\theta^{\frac{1}{2}}u_0,\theta n_0\big) \ge d_\Phi. \tag{120}$$

Meanwhile, for $0 < \theta \le 1$, $(\theta^{\frac{1}{2}}u_0,u_1,\theta n_0,V_0)$ satisfy condition (117), so we arrive at

$$\begin{aligned} F\big(\theta^{\frac{1}{2}}u_0,\theta n_0\big) &= \theta\|u_1\|^2 + \theta\|u_0\|^2 + \theta\|\nabla u_0\|^2 + \frac{1}{2}\theta^2\|u_0\|_{L^4}^4 + \frac{1}{2}\theta\|n_0\|^2 \\ &\quad + \frac{H^2}{2}\theta\|\nabla n_0\|^2 + \frac{1}{2}\theta\|V_0\|^2 + \theta^2\int_\Omega n_0|u_0|^2\,dx \\ &\le \|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2 + \frac{1}{2}\|u_0\|_{L^4}^4 + \frac{1}{2}\|n_0\|^2 \\ &\quad + \frac{H^2}{2}\|\nabla n_0\|^2 + \frac{1}{2}\|V_0\|^2 \\ &< d_\Phi, \end{aligned} \tag{121}$$

which is contradictory to (120). So there must be $G(u_0,n_0) > 0$. Combining (119) and Theorem 2, we obtain the first result.

(2) If $\int_\Omega n_0|u_0|^2 > 0$, and from (118), we have

$$\begin{aligned} E(u_0,u_1,n_0,V_0) &= \|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2 + \frac{1}{2}\|u_0\|_{L^4}^4 + \frac{1}{2}\|n_0\|^2 \\ &\quad + \frac{H^2}{2}\|\nabla n_0\|^2 + \frac{1}{2}\|V_0\|^2 + \int_\Omega n_0|u_0|^2\,dx \\ &\le \|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2 + \frac{1}{2}\|u_0\|_{L^4}^4 + \frac{1}{2}\|n_0\|^2 \\ &\quad + \frac{H^2}{2}\|\nabla n_0\|^2 + \frac{1}{2}\|V_0\|^2 + \frac{1}{2}\|n_0\|^2 + \frac{1}{2}\|u_0\|_{L^4}^4 \\ &\le \|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2 + \|u_0\|_{L^4}^4 + \|n_0\|^2 \end{aligned}$$

$$\begin{aligned} &+ \frac{H^2}{2}\|\nabla n_0\|^2 + \frac{1}{2}\|V_0\|^2 \\ &< d_\Phi. \end{aligned} \tag{122}$$

While from (44), there holds

$$\begin{aligned} G(u_0, n_0) &= 2\|u_0\|^2 + 2\|\nabla u_0\|^2 + 2\|u_0\|_{L^4}^4 + \|n_0\|^2 \\ &\quad + H^2\|\nabla n_0\|^2 + 3\int_\Omega n_0|u_0|^2\,dx \\ &\geq 0. \end{aligned} \tag{123}$$

Thus combining (122), (123) and Theorem 2, we obtain the second result. In sum, the proof of Theorem 3 is completed. □

Remark 1 From Theorem 3, one could see that if $(u_0, u_1, n_0, V_0) \in H^1_{\text{per}}(\Omega) \times L^2_{\text{per}}(\Omega) \times H^1_{\text{per}}(\Omega) \times L^2_{\text{per}}(\Omega)$, no matter $\int_\Omega n_0|u_0|^2 < 0$ or $\int_\Omega n_0|u_0|^2 > 0$, the solution $(u(x,t), n(x,t), V(x,t))$ of the periodic initial value problem (1)-(5) still exists globally only if

$$\|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2 + \|u_0\|_{L^4}^4 + \|n_0\|^2 + \frac{H^2}{2}\|\nabla n_0\|^2 + \frac{1}{2}\|V_0\|^2 < d_\Phi. \tag{124}$$

5 Conclusions

The modified Klein-Gordon-Zakharov equations combine the classical Klein-Gordon equation and the modified Zakharov equations for plasmas with a quantum correction, which considers the quantum effects. Thus it is better to describe the interaction between high frequency Langmuir waves and low frequency ion-acoustic waves. In this paper, we mainly do the mathematical analysis for the modified Klein-Gordon-Zakharov equations with periodic initial conditions. First we obtained the existence and uniqueness for local smooth solutions for the periodic initial value problem (1)-(5) via the a priori estimates and the Galerkin method. Secondly, a sharp threshold for the global existence and blowup of the solutions was derived by introducing some auxiliary functionals and invariant manifolds and applying potential well argument and the concavity method. Furthermore, two more specific conditions of how small the initial data are were given out to ensure that the solutions of the periodic initial value problem (1)-(5) exist globally.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

CG conceived of the study, performed the theoretical derivation and wrote this paper. SF revised and edited the paper. Both of the authors contributed equally in preparing this paper and approved this version. All authors read and approved the final manuscript.

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