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Analysis of an *SIRS* epidemic model with time delay on heterogeneous network

Qiming Liu[1]*, Meici Sun[1] and Tao Li[1,2]

*Correspondence:
lqmmath@163.com
[1]Shijiazhuang Campus, Arm Engineering University, Shijiazhuang, 050003, China
Full list of author information is available at the end of the article

Abstract

We discuss a novel epidemic *SIRS* model with time delay on a scale-free network in this paper. We give an equation of the basic reproductive number R_0 for the model and prove that the disease-free equilibrium is globally attractive and that the disease dies out when $R_0 < 1$, while the disease is uniformly persistent when $R_0 > 1$. In addition, by using a suitable Lyapunov function, we establish a set of sufficient conditions on the global attractiveness of the endemic equilibrium of the system.

Keywords: epidemic spreading; scale-free network; basic reproductive number; global attractiveness; time delay

1 Introduction

Following both the seminal work on small-world network phenomena by Watts and Strogatz [1] and the scale-free network, in which the probability of $p(k)$ for any node with k links to other nodes is distributed according to the power law $p(k) = Ck^{-\gamma}$ $(2 < \gamma \leq 3)$, suggested by Barabási and Albert [2], the spreading of an epidemic disease on heterogeneous networks, *i.e.*, scale-free networks, has been studied by many researchers [3–22].

Realistic epidemic models should include some past states of the system, and time delay plays an important role in the spreading process of the epidemic. For instance, it can simulate the incubation period of the infectious disease, the infectious period of patients, and the immunity period of recovery of the disease with time delay [21, 23]. However, only little attention has been given to the epidemic models with time delays on heterogeneous networks. Liu and Xu presented a functional differential equation *SEIRS* epidemic model, in which time delay represents the latent period and the immune period [19]. Liu and Deng *et al.* discussed a functional differential equation *SIS* model, in which time delay represents the average infectious period [20], obtained the basic reproduction number, and discussed the persistence of the disease. Wang and Wang *et al.* presented a functional differential equation *SIR* model, in which time delay represents the incubation period, and discussed the global stability of equilibria of the system [21]. Kang and Fu also established a functional differential equation *SIS* model with an infective vector and analyzed the global stability of the endemic equilibrium, the disease-free equilibrium [22], and so on.

Considering the fact that the immune individual may become the susceptible individual, Chen and Sun discussed an *SIRS* epidemic model without delay [15]. In [21], Wang and Wang *et al.* presented a delayed *SIR* model, in which time delay represents the incubation period during which the infectious agents develop in the vector, and discussed the global stability of equilibria of the system. Motivated by the work of Chen [15] and Wang [21], we will present a novel functional differential equation *SIRS* model in which time delay represents the incubation period of an infective vector to investigate the epidemic spreading on a heterogeneous network.

Assuming the size of the network is a constant N during the period of epidemic spreading, we also suppose that the degree of each node is time-invariant and $p(k)$ denotes the degree distribution of the network. Let $S_k(t)$, $I_k(t)$, and $R_k(t)$ be the relative density of susceptible nodes, infected nodes, and recovered nodes of connectivity k at time t, respectively, where $k = m, m+1, \ldots, n$, in which m and n are the minimum and maximum degree in network topology, respectively. Since the number of total nodes with degree k is a constant $p(k)N$ during the period of epidemic spreading, the following normalization conditions hold:

$$S_k(t) + I_k(t) + R_k(t) = 1, \quad k = m, m+1, \ldots, n.$$

In propagation processes of the epidemic via a vector (such as the mosquito), when a susceptible vector is infected by an infected one, there is a delay τ during which the infectious agents develop in the vector and the infected vector becomes itself infectious after the delay. At the same time, the vector's usual activities are in a limited range, and the vector population size is large enough such that at any time t a number of the infectious vector population in the vicinity of the infected nodes with degree k ($k = m, m+1, \ldots, n$) at any time t is simply proportional to the number of the infected nodes with degree k. The dynamical equations for the density $S_k(t)$, $I_k(t)$, and $R_k(t)$, at the mean-field level, satisfy the following set of functional differential equations when $t > 0$ [20, 21, 24]:

$$\begin{aligned}
\dot{S}_k(t) &= -\lambda(k)S_k(t)\Theta(t-\tau) - \delta S_k(t) + \mu R_k(t), \\
\dot{I}_k(t) &= \lambda(k)S_k(t)\Theta(t-\tau) - rI_k(t), \\
\dot{R}_k(t) &= rI_k(t) - \mu R_k(t) + \delta S_k(t),
\end{aligned} \tag{1}$$

where $\lambda(k)$ is a transmission coefficient of the disease between the susceptible nodes and infected nodes and r is the recovery rate from the infected nodes to the recovered nodes because the infected nodes (patients) are cured. δ is the removal rate from the susceptible nodes to the recovered nodes because the susceptible nodes acquire temporary immunity. μ is the removal rate from the recovered nodes to the susceptible nodes because the recovered nodes lose temporary immunity. The dynamics of n groups of *SIRS* subsystems are coupled through the function $\Theta(t)$, which represents the probability that any given link points to an infected node at time t. Considering the uncorrelated network [3, 11], we have

$$\Theta(t) = \frac{1}{\langle k\rangle}\sum_{k=m}^{n}\varphi(k)p(k)I_k(t), \tag{2}$$

where $\langle k\rangle = \sum_{k=m}^{n} p(k)k$ stands for the average node degree. $\lambda(k)$ is the correlated (k-dependent) infection rate, such as λk, $\lambda c(k)$ [11], and so on. $\varphi(k)$ denotes an infected node, with degree k, occupied edges which can transmit the disease, while $\varphi(k) = ak^\alpha/(1 + bk^\alpha)$,

$0 \le \alpha < 1$, $a > 0$, $b \ge 0$ [7]. With different parameters, $\varphi(k)$ can be divided into different cases, such as $\varphi(k) = k$ [3] when $\alpha = 1$, $b = 0$, and $a = 1$; $\varphi(k) = A$ [5] when $a = A$, $\alpha = 0$, and $b = 0$; $\varphi(k) = k^\alpha$ [6] when $a = 1$ and $b = 0$. Specially, if $b \neq 0$, $\varphi(k)$ will become gradually saturated as degree k increases, *i.e.*, $\lim_{k\to\infty} \varphi(k) = b/a$.

The initial conditions of system (1) are the following:

$$
\begin{aligned}
&S_k(\theta) = \zeta_k(\theta), \qquad I_k(\theta) = \psi_k(\theta), \\
&R_k(\theta) = \phi_k(\theta), \quad \theta \in [-\tau, 0], k = m, m+1, \dots, n,
\end{aligned}
\tag{3}
$$

where $\omega = (\zeta_m(\theta), \psi_m(\theta), \phi_m(\theta), \dots, \zeta_n(\theta), \psi_n(\theta), \phi_n(\theta)) \in C$ are nonnegative continuous on $[-\tau, 0]$ and $\zeta_k(0) > 0$, $\psi_k(0) > 0$ for $\theta = 0$. C denotes the Banach space $C([-\tau, 0], R^{3(n-m+1)})$ with the norm

$$
\|\omega\| = \left(\sum_{i=m}^{n} \left(\left|\zeta_i(\theta)\right|_\tau^2 + \left|\psi_i(\theta)\right|_\tau^2 + \left|\phi_i(\theta)\right|_\tau^2 \right) \right)^{1/2},
$$

where

$$
\left|f(\theta)\right|_\tau = \sup_{-\tau \le \theta \le 0} \left|f(\theta)\right|.
$$

It is well known, by the fundamental theory of functional differential equations [25], that system (1) has a unique solution $(S_m(t), \dots, S_n(t), I_m(t), \dots, I_n(t), R_m(t), \dots, R_n(t))$ satisfying the initial conditions (3). It is easy to show that all solutions of system (1) with initial conditions (3) are defined on $[0, +\infty)$ due to the boundedness of S_k, I_k, and R_k. In addition, using similar arguments as in [26], it is easy to show that all solutions of system (1) with initial conditions (3) remain positive for all $t \ge 0$.

The rest of this paper is organized as follows. The dynamical behaviors of the *SIRS* model are discussed in Section 2. Numerical simulations and discussions are given to demonstrate the main results in Section 3. Finally, the main conclusions of this work are summarized in Section 4.

2 Dynamical behaviors of the model

Note that $S_k(t) + I_k(t) + R_k(t) = 1$. If we discuss the dynamical behaviors of system (1), we only discuss the following equivalent system of model (1):

$$
\begin{aligned}
&\dot{S}_k(t) = -\lambda(k) S_k(t) \Theta(t-\tau) - \delta S_k(t) + \mu\bigl(1 - S_k(t) - I_k(t)\bigr), \\
&\dot{I}_k(t) = \lambda(k) S_k(t) \Theta(t-\tau) - r I_k(t).
\end{aligned}
\tag{4}
$$

Denote

$$
R_0 = \frac{\mu}{(\mu+\delta) r} \frac{\langle \lambda(k)\varphi(k)\rangle}{\langle k\rangle},
\tag{5}
$$

where $\langle f(k)\rangle = \sum_k f(k) p(k)$, in which $f(k)$ is a function.

Note that we obtain from the first equation of system (4) the following equation:

$$
\dot{S}_k(t) \le \mu - (\mu+\delta) S_k(t).
$$

By the standard comparison theorem in the theory of differential equations, we have

$$\lim_{t\to+\infty}\sup S_k(t) \le \mu/(\mu+\delta). \tag{6}$$

Hence we know

$$D_0 = \left\{ (S_m, I_m, \ldots, S_n, I_n) \in R_+^{2(n-m+1)} \,\middle|\, 0 < S_k, I_k, S_k + I_k \le 1, 0 < S_k \le \frac{\mu}{\mu+\delta} \right\}$$

is positively invariant with respect to system (4), and every forward orbit in $R_+^{2(n-m+1)}$ eventually enters D_0.

Theorem 1 *System* (4) *always has a disease-free equilibrium* $E_0(\frac{\mu}{\mu+\delta}, \ldots, \frac{\mu}{\mu+\delta}, 0, \ldots, 0)$. *System* (4) *has a unique endemic equilibrium* $E_*(S_m^*, S_{m+1}^*, \ldots, S_n^*, I_m^*, I_{m+1}^*, \ldots, I_n^*)$ *when* $R_0 > 1$.

Proof Obviously, the disease-free equilibrium E_0 of system (4) always exists. Now we discuss the existence of the endemic equilibrium of system (4). Note that the equilibrium E_* should satisfy

$$\begin{aligned} &-\lambda(k)S_k^*\Theta^* - \delta S_k^* + \mu\left(1 - S_k^* - I_k^*\right) = 0, \\ &\lambda(k)S_k^*\Theta^* - rI_k^* = 0, \end{aligned} \tag{7}$$

where

$$\Theta^* = \frac{1}{\langle k \rangle}\sum_k \varphi(k)p(k)I_k^*. \tag{8}$$

From (7), we obtain

$$I_k^* = \frac{\lambda(k)\mu\Theta^*}{(\mu+\delta)r + \lambda(k)(r+\mu)\Theta^*}. \tag{9}$$

Substituting this into (8), we obtain the self-consistency equality

$$\Theta^* = \frac{1}{\langle k \rangle}\sum_k \varphi(k)p(k)\frac{\lambda(k)\mu\Theta^*}{(\mu+\delta)r + \lambda(k)(r+\mu)\Theta^*}, \tag{10}$$

similar to the proof of Theorem 1 in [20]. Equation (10) has a unique positive solution when $R_0 > 1$ and consequently, system (4) has a unique endemic equilibrium E_* since (7) and (9) hold. □

Remark 1 We know that the basic reproductive number for system (1) is R_0. If $\varphi(k) = k$ and $\tau = 0$, system (1) reduces to the *SIRS* model (1.2) without delay in [15], and R_0 in this paper consists/coincides with one of the model in [15].

Theorem 2 *If* $R_0 < 1$, *the disease-free equilibrium* E_0 *of system* (4) *is globally attractive*.

Proof Obviously, we need only to discuss the global attractiveness of system (4) in D_0.

Consider the following Lyapunov function:

$$V(t) = \frac{1}{2}\Theta^2(t) + \eta \int_{t-\tau}^{t} \Theta^2(\mu)\,d\mu,$$

where η is a constant to be determined.

Calculating the derivative of $V(t)$ along the solution of (4), we get

$$\dot{V}(t)\big|_{(4)} = \Theta(t)\left[\frac{1}{\langle k\rangle}\sum_k \phi(k)p(k)\big(\lambda(k)S_k(t)\Theta(t-\tau) - rI_k(t)\big)\right] + \eta\Theta^2(t) - \eta\Theta^2(t-\tau).$$

Note that $0 < S_k(t) \le \mu/(\mu+\delta)$ according to the positive invariance of D_0. We have

$$\begin{aligned}\dot{V}(t)\big|_{(4)} &\le \Theta(t)\left[\frac{1}{\langle k\rangle}\sum_k \phi(k)p(k)\left(\lambda(k)\frac{\mu}{\mu+\delta}\Theta(t-\tau) - rI_k(t)\right)\right] + \eta\Theta^2(t) - \eta\Theta^2(t-\tau)\\ &= \Theta(t)\Theta(t-\tau)\frac{\mu}{\mu+\delta}\frac{\langle\lambda(k)\phi(k)\rangle}{\langle k\rangle} - r\Theta^2(t) + \eta\Theta^2(t) - \eta\Theta^2(t-\tau)\\ &\le \frac{1}{2}\big[\Theta^2(t) + \Theta^2(t-\tau)\big]\frac{\mu}{\mu+\delta}\frac{\langle\lambda(k)\phi(k)\rangle}{\langle k\rangle} - r\Theta^2(t) + \eta\Theta^2(t) - \eta\Theta^2(t-\tau).\end{aligned}$$

Let $\eta = \frac{1}{2}\frac{\mu}{\mu+\delta}\frac{\langle\lambda(k)\varphi(k)\rangle}{\langle k\rangle}$. We have

$$\dot{V}(t)\big|_{(4)} \le \left[\frac{\mu}{\mu+\delta}\frac{\langle\lambda(k)\phi(k)\rangle}{\langle k\rangle} - r\right]\Theta^2(t) = r(R_0 - 1)\Theta^2(t).$$

Thus $\dot{V}(t)|_{(4)} \le 0$ when $R_0 < 1$ and $\dot{V}(t)|_{(4)} = 0$ if and only if $\Theta = 0$. Note that the fact that $\Theta = 0$ means that $I_k = 0$. Moreover, $\lim_{t\to+\infty} S_k(t) = \mu/(\mu+\delta)$, the largest invariant set of $\dot{V}(t)|_{(1)} = 0$ is a singleton E_0. Therefore, by the LaSalle invariance principle (see [27, Chapter 2, Theorem 5.3]), the disease-free equilibrium E_0 is globally attractive. □

Lemma 1 ([28]) *Consider the following equation*:

$$\dot{x}(t) = a_1 x(t-\tau) - a_2 x(t),$$

where $a_1, a_2, \tau > 0$; $x(t) > 0$ *for* $-\tau \le t \le 0$. *We have*

(i) *if* $a_1 < a_2$, *then* $\lim_{t\to+\infty} x(t) = 0$,

(ii) *if* $a_1 > a_2$, *then* $\lim_{t\to+\infty} x(t) = +\infty$.

Theorem 3 *For system* (4), *if* $R_0 > 1$, *the disease-free equilibrium* E_0 *is unstable*, *and the disease is uniformly persistent*, i.e., *there exists a positive constant* ϵ *such that* $\lim_{t\to+\infty}\inf I_k(t) \ge \epsilon$, $k = m, m+1, \ldots, n$.

Proof Denote

$$\begin{aligned}X &= \big\{(\bar{S},\bar{\psi}) : \psi_k(\theta) \ge 0, \text{for all } \theta \in [-\tau,0], k = m, m+1, \ldots, n\big\},\\ X^0 &= \big\{(\bar{S},\bar{\psi}) : \psi_k(\theta) > 0, \text{for some } \theta \in [-\tau,0], k = m, m+1, \ldots, n\big\}.\end{aligned}$$

Consequently, we have

$$\partial X^0 = X/X^0 = \{(\bar{S}, \bar{\psi}) : \psi_i(\theta) = 0, \text{for all } \theta \in [-\tau, 0], i \in \{m, m+1, \dots, n\}\},$$

where $(\bar{S}, \bar{\psi}) = (S_m, S_{m+1}, \dots, S_n, \psi_m, \psi_{m+1}, \dots, \psi_n)$.

Let $(S_m(t), I_m(t), \dots, S_n, I_n(t)) = (S_m(t,\omega), I_m(t,\omega), \dots, S_n(t,\omega), I_n(t,\omega))$ be the solution of (4) with initial function $\omega = (\zeta_m(\theta), \psi_m(\theta), \dots, \psi_n(\theta), \phi_n(\theta))$ and

$$T(t)(\omega)(\theta) = (S_m(t+\theta,\omega), I_m(t+\theta,\omega), \dots, S_n(t+\theta,\omega), I_n(t+\theta,\omega)), \quad \theta \in [-\tau, 0].$$

Obviously, X and X^0 are positively invariant sets for $T(t)$. $T(t)$ is completely continuous for $t > 0$. Also, it follows from $0 < S_k(t), I_k(t) \leq 1$ for $t > 0$ that $T(t)$ is point-dissipative. E_0 is the unique equilibrium of system (4) on ∂X^0 and it is globally stable on ∂X^0, $\tilde{A_\partial} = \{E_0\}$, and E_0 is isolated and acyclic. Finally, the proof will be complete if we prove $W^s(E_0) \cap X^0 = \emptyset$, where $W^s(E_0)$ is the stable manifold of E_0. Suppose it is not true. Then there exists a solution $(\bar{S}, \bar{I})$ in X^0 such that

$$\lim_{t\to+\infty} (S_k(t), I_k(t)) = \left(\frac{\mu}{\mu+\delta}, 0\right), \quad k = m, m+1, \dots, n.$$

Since $R_0 > 1$, we may choose $0 < \eta < 1$ such that $\frac{\mu(1-\eta)}{(\lambda(k)\eta+\mu+\delta)r} \frac{\langle\lambda(k)\varphi(k)\rangle}{\langle k\rangle} > 1$. At the same time, there exists a $T_2 > 0$ such that $0 \leq I_k(t) < \eta$ for $t > T_2$ due to $\lim_{t\to+\infty} \inf I_k(t) = 0$.

When $t > T_2$, we obtain from the first equation of system (4)

$$\begin{aligned}\dot{S}_k(t) &> -\lambda(k)S_k\eta + \mu(1 - S_k(t) - \eta) - \delta S_k(t) \\ &= \mu(1-\eta) - (\lambda(k)\eta + \mu + \delta)S_k(t).\end{aligned}$$

Hence there exists a $T_3 > T_2$ such that the following equality holds when $t > T_2$:

$$S_k(t) \geq \frac{\mu(1-\eta)}{\lambda(k)\eta + \mu + \delta}. \tag{11}$$

For $t > T_2$, we have from (2) and (10)

$$\begin{aligned}\dot{\Theta}(t) &= \frac{1}{\langle k\rangle} \sum_k \varphi(k)p(k)\dot{I}_k(t) \\ &= \frac{1}{\langle k\rangle} \sum_k \varphi(k)p(k)[\lambda(k)S_k(t)\Theta(t-\tau) - rI_k(t)] \\ &\geq \frac{\mu(1-\eta)}{\lambda(k)\eta+\mu+\delta} \frac{\langle\lambda(k)\varphi(k)\rangle}{\langle k\rangle} \Theta(t-\tau) - r\Theta(t).\end{aligned}$$

Note that $\frac{\mu(1-\eta)}{(\lambda(k)\eta+\mu+\delta)r} \frac{\langle\lambda(k)\varphi(k)\rangle}{\langle k\rangle} > 1$. It follows that $\frac{\mu(1-\eta)}{\lambda(k)\eta+\mu+\delta} \frac{\langle\lambda(k)\varphi(k)\rangle}{\langle k\rangle} > r$. Hence we deduce from (10) that $\lim_{t\to+\infty} \Theta(t) = +\infty$ according to Lemma 1, contradicting $\lim_{t\to+\infty} \Theta(t) = 0$ due to $\lim_{t\to+\infty} I_k(t) = 0$. Hence $\lim_{t\to+\infty} \inf \Theta(t) \neq 0$. Moreover, there exists a $k_0 \in \{m, m+1, \dots, n\}$ such that $\lim_{t\to+\infty} \inf I_{k_0}(t) \neq 0$, contradicting $\lim_{t\to+\infty} \inf I_k(t) = 0$, $k = m, m+1, \dots, n$.

Hence, the infection is uniformly persistent according to Theorem 2.4 in [27, Chapter 8], *i.e.*, there exists an ϵ, being a positive constant, such that $\lim_{t\to+\infty} \inf I_k(t) \geq \epsilon$ and

$\lim_{t\to+\infty}\inf S_k(t) \geq \epsilon$. The disease-free equilibrium E_0 is unstable accordingly. This completes the proof. □

Furthermore, we obtain the following Theorem 4 about the global attractiveness of the endemic equilibrium E_* of system (4) by constructing a suitable Lyapunov function.

Theorem 4 *If* $R_0 > 1$, $\delta < r$, *and* $I_k^* < \mu/(\mu+\delta)(\delta/r)$, $k = m, m+1, \dots, n$, *then the endemic equilibrium* E_* *of system* (4) *is globally attractive.*

Proof For convenience, we still discuss system (1).

In accordance with (6) and $S_k(t) + I_k(t) + R_k(t) = 1$, $k = m, m+1, \dots, n$, we know

$$\tilde{D}_0 = \left\{ (S_m, I_m, R_m, \dots, S_n, I_n, R_n) \in R_+^{3(n-m+1)} \middle| \right. \\ \left. 0 < S_k, I_k, R_k, S_k + I_k + R_n = 1, 0 < S_k \leq \frac{\mu}{\mu+\delta} \right\} \tag{12}$$

is positively invariant with respect to system (4), and every forward orbit in $R_+^{3(n-m+1)}$ eventually enters $\tilde{D}_0$.

Thus we just need to discuss the global attractiveness of system (1) in $\tilde{D}_0$.

Denote $R_k^* = 1 - S_k^* - I_k^*$. Then $E^*(S_m^*, I_m^*, R_m^*, S_{m+1}^*, I_{m+1}^*, R_{m+1}^*, \dots, S_n^*, I_n^*, R_n^*,)$ is the endemic equilibrium of system (1). System (1) may be rewritten as the following system:

$$\begin{aligned} \dot{S}_k(t) &= -\sum_{l=m}^{n} \beta_{kl} S_k(t) I_l(t-\tau) - \delta S_k(t) + \mu R_k(t), \\ \dot{I}_k(t) &= \sum_{l=m}^{n} \beta_{kl} S_k(t) I_l(t-\tau) - r I_k(t), \\ \dot{R}_k(t) &= r - (\mu + r) R_k(t) - (r - \delta) S_k(t), \end{aligned} \tag{13}$$

where $\beta_{kl} = \frac{\lambda(l)\varphi(l)p(l)}{\langle k \rangle}$, $l = m, m+1, \dots, n$.

Note that the endemic equilibrium of system (1) satisfies

$$\begin{aligned} &-\sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* - \delta S_k^* + \mu R_k^* = 0, \\ &\sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* = r I_k^*, \\ &(\mu + r) R_k^* + (\delta - r) S_k^* = r. \end{aligned} \tag{14}$$

We have from (13) and (14) that

$$\begin{aligned} \dot{S}_k(t) &= -\sum_{l=m}^{n} \beta_{kl} S_k(t) I_l(t-\tau) + \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* - \delta (S_k - S_k^*) + \mu \bigl(R_k - R_k^*\bigr), \\ \dot{I}_k(t) &= \sum_{l=m}^{n} \beta_{kl} S_k(t) I_l(t-\tau) - \sum_{l=m}^{n} \beta_{kl} \frac{S_k^* I_l^*}{I_k^*} I_k(t), \\ \dot{R}_k(t) &= -(\mu + r)\bigl(R_k - R_k^*\bigr) + (\delta - r)\bigl(S_k - S_k^*\bigr). \end{aligned} \tag{15}$$

Let us consider

$$\begin{aligned} V_k(t) &= S_k(t) - S_k^* - S_k^* \ln \frac{S_k(t)}{S_k^*} + I_k(t) - I_k^* - I_k^* \ln \frac{I_k(t)}{I_k^*} \\ &\quad + \sum_{l=m}^{n} \beta_{kl} S_k^* \int_{t-\tau}^{t} \left(I_l(s) - I_l^* - I_l^* \ln \frac{I_l(s)}{I_l^*} \right) ds + \frac{\mu}{S_k^*(r-\delta)} \frac{(R_k(t) - R_k^*)^2}{2}. \end{aligned} \tag{16}$$

Calculating the derivative of $V_k(t)$ along a solution of (15), we get

$$\begin{aligned} \dot{V}_k(t)\big|_{(15)} &= -\delta \left(1 - \frac{S_k^*}{S_k}\right)(S_k - S_k^*) + \mu \left(1 - \frac{S_k^*}{S_k}\right)(R_k - R_k^*) \\ &\quad + \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* \left(2 - \frac{S_k^*}{S_k} - \frac{I_k}{I_k^*} - \frac{S_k I_l(t-\tau) I_k^*}{S_k^* I_k I_l^*} - \ln \frac{I_l}{I_l^*} + \ln \frac{I_l(t-\tau)}{I_l^*} + \frac{I_l}{I_l^*} \right) \\ &\quad + \mu (R_k - R_k^*) \left(1 - \frac{S_k}{S_k^*}\right) - \frac{\mu(r+\mu)}{S_k^*(r-\delta)} (R_k - R_k^*)^2. \end{aligned} \tag{17}$$

Note that

$$\begin{aligned} &-\delta \left(1 - \frac{S_k^*}{S_k}\right)(S_k - S_k^*) = -\delta S_k^* \left(G\left(\frac{S_k}{S_k^*}\right) + G\left(\frac{S_k^*}{S_k}\right) \right), \\ &\mu \left(1 - \frac{S_k^*}{S_k}\right)(R_k - R_k^*) + \mu (R_k - R_k^*) \left(1 - \frac{S_k}{S_k^*}\right) \\ &\quad = -\mu (R_k - R_k^*) \left(G\left(\frac{S_k}{S_k^*}\right) + G\left(\frac{S_k^*}{S_k}\right) \right) \\ &\quad = -\mu R_k G\left(\frac{S_k}{S_k^*}\right) - \mu R_k G\left(\frac{S_k^*}{S_k}\right) + \mu R_k^* \left(G\left(\frac{S_k}{S_k^*}\right) + G\left(\frac{S_k^*}{S_k}\right) \right), \end{aligned} \tag{18}$$

in which $G(a) = a - 1 - \ln a \geq 0$ and

$$\begin{aligned} &\sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* \left(2 - \frac{S_k^*}{S_k} - \frac{I_k}{I_k^*} - \frac{S_k I_l(t-\tau) I_k^*}{S_k^* I_k I_l^*} - \ln \frac{I_l}{I_l^*} + \ln \frac{I_l(t-\tau)}{I_l^*} + \frac{I_l}{I_l^*} \right) \\ &\quad = \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* \left(H\left(\frac{I_k}{I_k^*}\right) - H\left(\frac{I_l}{I_l^*}\right) \right) - \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* \left(G\left(\frac{S_k^*}{S_k}\right) + G\left(\frac{S_k}{S_k^*}\right) \right) \\ &\qquad - \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* G\left(\frac{S_k I_l(t-\tau) I_k^*}{S_k^* I_k I_l *} \right) + \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* G\left(\frac{S_k}{S_k^*}\right), \end{aligned} \tag{19}$$

in which $H(a) = -a + \ln a$.

Thus we deduce from (14), (17), (18), and (19) that

$$\dot{V}_k(t)\big|_{(15)} \leq \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* \left(H\left(\frac{I_k}{I_k^*}\right) - H\left(\frac{I_l}{I_l^*}\right) \right) + (-\mu R_k + r I_k^*) G\left(\frac{S_k}{S_k^*}\right). \tag{20}$$

Furthermore, since $S_k(t) \leq \frac{\mu}{\mu+\delta}$, we deduce from the last equation of system (15) that

$$\dot{R}_k(t) \geq r - (\mu + r) R_k(t) - (r - \delta) \frac{\mu}{\mu + \delta} = (\mu + r) \left(\frac{\delta}{\mu + \delta} - R_k(t) \right). \tag{21}$$

Since $I_k^* < \delta/(\mu+\delta)(\mu/r)$, we take $\varepsilon = \delta/(\mu+\delta) - I_k^* r/\mu > 0$. It follows from (23) that there exists a $T_4 > T_1$ such that $R_k(t) \geq \delta/(\mu+\delta) - \varepsilon$ when $t > T_4$. Hence $-\mu R_k + rI_k^* \leq -\mu(\delta/(\mu+\delta) - \varepsilon) + rI_k^* = 0$ when $t > T_4$, *i.e.*,

$$\bigl(-\mu R_k + rI_k^*\bigr)G\left(\frac{S_k}{S_k^*}\right) \leq 0. \tag{22}$$

In addition, the matrix

$$\bigl(\beta_{kl}S_k^*I_l^*\bigr)_{(n-m+1)\times(n-m+1)} = \left(\frac{\lambda(l)\varphi(l)p(l)}{\langle k\rangle}S_k^*I_l^*\right)_{(n-m+1)\times(n-m+1)}$$

is irreducible, so the following matrix is irreducible:

$$B = \begin{pmatrix} \sum_{l\neq m}\beta_{ml}S_m^*I_l^* & -\beta_{m+1,m}S_{m+1}^*I_m^* & \cdots & -\beta_{nm}S_n^*I_m^* \\ -\beta_{m,m+1}S_m^*I_{m+1}^* & \sum_{l\neq m+1}\beta_{m+1,l}S_{m+1}^*I_l^* & \cdots & -\beta_{n,m+1}S_n^*I_{m+1}^* \\ \cdots & \cdots & \cdots & \cdots \\ -\beta_{mn}S_m^*I_n^* & -\beta_{m+1,n}S_{m+1}^*I_n^* & \cdots & \sum_{l\neq n}\beta_{nl}S_n^*I_l^* \end{pmatrix}.$$

Hence there exists a positive vector $C = (c_m, c_{m+1}, \ldots, c_n)$ such that $BC = 0$, in which c_k is the cofactor of the kth diagonal entry of B and $m \leq k \leq n$ [29, Lemma 2.1].

It follows from $BC = 0$ that

$$\sum_{l=m}^{n} c_l\beta_{lk}S_l^*I_k^* = c_k\sum_{l=1}^{n}\beta_{kl}S_k^*I_l^*, \quad k = m, m+1, \ldots, n.$$

Consequently, we have

$$\begin{aligned}\sum_{k=m}^{n} c_k\sum_{l=m}^{n}\beta_{kl}S_l^*I_l^*H\left(\frac{I_k}{I_k^*}\right) &= \sum_{k=m}^{n}\sum_{l=m}^{n} c_l\beta_{lk}S_l^*I_k^*H\left(\frac{I_k}{I_k^*}\right) \\ &= \sum_{k=m}^{n}\sum_{l=m}^{n} c_k\beta_{kl}S_k^*I_l^*H\left(\frac{I_l}{I_l^*}\right) = \sum_{k=m}^{n} c_k\sum_{l=m}^{n}\beta_{kl}S_k^*I_l^*H\left(\frac{I_l}{I_l^*}\right),\end{aligned}$$

i.e.,

$$\sum_{k=1}^{n} c_k\sum_{l=1}^{n}\beta_{kl}S_k^*I_l^*\left(H\left(\frac{I_k}{I_k^*}\right) - H\left(\frac{I_l}{I_l^*}\right)\right) = 0. \tag{23}$$

Define a Lyapunov function as follows:

$$V(t) = \sum_{k=1}^{n} c_k V_k \tag{24}$$

and we deduce from (20), (22), and (23) that

$$\dot{V}(t)\big|_{(15)} \leq 0.$$

Moreover, $\dot{V}(t)|_{(17)} = 0$ if and only if $S_k = S_k^*$, $I_k = I_k^*$, and $R_k = R_k^*$. Therefore, by the LaSalle invariance principle (see [27, Chapter 2, Theorem 5.3]), the endemic equilibrium E^* is globally attractive when $R_0 > 1$, $\delta < r$ and $I_k^* < \mu/(\mu + \delta)(\delta/r)$, $k = m, m+1, \ldots, n$. The proof is completed. □

3 Numerical simulations and discussions

The basic reproductive number for system (4) (or (1)) is

$$R_0 = \frac{\mu}{(\mu + \delta)r} \frac{\langle \lambda(k)\varphi(k)\rangle}{\langle k\rangle}. \tag{25}$$

The equilibrium E_0 is globally attractive and the infection eventually disappears when $R_0 < 1$. The infection will always exist when $R_0 > 1$. Note that R_0 is irrelative to τ.

Now we present numerical simulations to demonstrate the above mentioned theorems. Note that we obtain the equilibria from system (4). The simulations are based on system (4) and a scale-free network in which the degree distribution is $p(k) = Ck^{-\gamma}$ and C satisfies $\sum_{k=m}^{n} p(k) = 1$. Assuming the network is finite, the maximum connectivity n of any node is related to the network age, measured as the number of nodes N [5, 8]. We have

$$n = mN^{1/(\gamma - 1)}. \tag{26}$$

Let $n = 100$ and $m = 1$ be suitable assumptions. Let $\phi(k) = ak^{\alpha}/(1 + bk^{\alpha})$ in which $a = 0.5$, $\alpha = 0.75$, $b = 0.02$, and $\lambda(k) = \lambda k$. The initial functions are $I_k(s) = 0.45$, $k = 2, 3, 4, 5$, and $I_k = 0$, $k \neq 2, 3, 4, 5$ for $s \in [0, \tau]$.

Denote

$$I(t) = \sum_k p(k) I_k(t).$$

Obviously, $I(t)$ is the relative average density of the infected nodes.

Case 1: Let $\lambda = 0.03$, $r = 0.06$, $\delta = 0.03$, $\mu = 0.06$, $\gamma = 2.5$, and $\tau = 2$. We obtain from (25) that $R_0 = 0.5064 < 1$. Figure 1 shows the dynamic behaviors of system (4). The numerical simulation shows that $\lim_{t\to+\infty} I(t) = 0$. It follows that $\lim_{t\to+\infty} I_k(t) = 0$ and the infection eventually disappears. The numerical result is consistent with Theorem 2.

Case 2: Let $\lambda = 0.1$, $r = 0.06$, $\delta = 0.05$, $\mu = 0.06$, $\gamma = 2.5$, $\tau = 2$, and $\delta < r$. We obtain from (25) that $R_0 = 1.4596 > 1$ and $\max I_k^* = 0.2930 < \mu/(\mu + \delta)(\delta/r) = 0.4545$. Figure 2

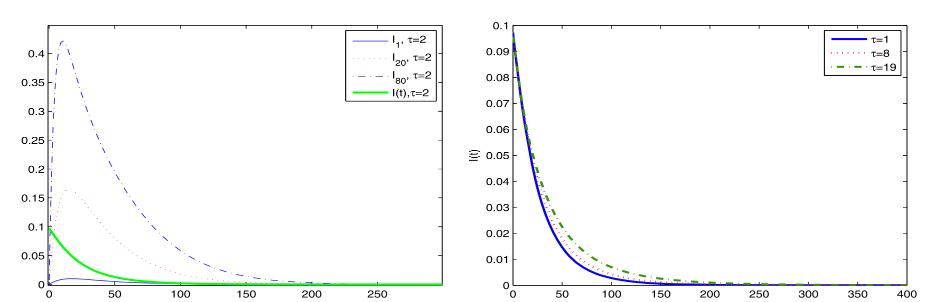

Figure 1 Dynamical behaviors of system (4) with $R_0 = 0.5046$.

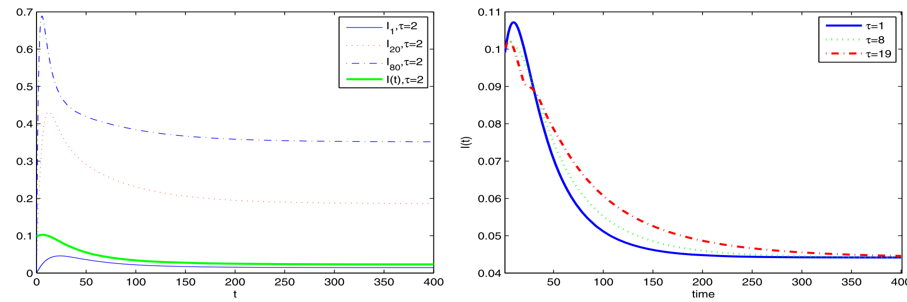

Figure 2 Dynamical behaviors of system (4) with R_0 = 1.4596.

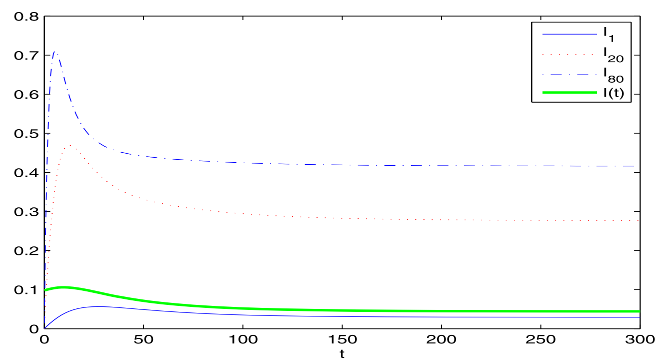

Figure 3 Dynamical behaviors of system (4) with τ = 2 and R_0 = 1.5793.

shows the dynamic behaviors of system (4). The average densities $I_k(t)$ and $I(t)$ converge to a positive constant as $t \to +\infty$ and the infection is uniformly persistent. The numerical result is consistent with Theorems 3 and 4.

Although the time delay τ has no effects on both the spreading threshold and the density of infected nodes at the stationary state according to (7) and (25), we find that the delay τ has much impact on the density $I(t)$ of the infected nodes; the slower the relative density of infected nodes converges to the stationary state, the larger τ gets. Thus the delay cannot be ignored. The numerical simulations in Figure 1 and Figure 2 verify it.

Moreover, let $\lambda = 0.1$, $r = 0.06$, $\delta = 0.03$, $\mu = 0.06$, $\gamma = 2.5$, $\tau = 2$, and $\delta < r$. We obtain from (25) that $R_0 = 1.7806 > 1$ but $I_k^* \geq \mu/(\mu + \delta)(\delta/r) = 0.3333$, $40 \leq k \leq 100$. Figure 3 shows the dynamic behaviors of system (4). The parameters of system (4) do not satisfy Theorem 4, but the relative density $I_k(t)$ and the relative average density $I(t)$ still converge to a positive constant as $t \to +\infty$. At the same time, no Hopf bifurcation occurs, so Theorem 4 has room for improvement.

4 Conclusions

An *SIRS* model with time delay on a scale-free network has been proposed, in which time delay describes the incubation period of the infective vector. We obtained the basic reproduction number R_0, which is irrelative to τ. The disease-free equilibrium is globally attractive and infection may disappear when $R_0 < 1$, while the infection is uniformly persistent when $R_0 > 1$. Moreover, the endemic equilibrium is globally attractive if $R_0 > 1$, $\delta < r$, and $I_k^* < \mu/(\mu + \delta)(\delta/r)$, $k = m, m+1, \ldots, n$. However, numerical simulations (Figure 3) show that the endemic equilibrium is still globally attractive even if $\delta < r$ and $I_k^* < \mu/(\mu+\delta)(\delta/r)$, $k = m, m+1, \ldots, n$ do not hold when $R_0 > 1$. Therefore, improvement of the sufficient condition in Theorem 4 on the global attractiveness of the endemic equilibrium of system (4) is an interesting but challenging problem.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to the expression of the model and the discussion of results. They read and approved the final manuscript.

Author details
[1]Shijiazhuang Campus, Arm Engineering University, Shijiazhuang, 050003, China. [2]The 260th Hospital of PLA, Shijiazhuang, 050041, China.

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