Sin et al. *Advances in Difference Equations* (2017) 2017:306
DOI 10.1186/s13662-017-1369-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Analytical solutions to multi-term time-space Caputo-Riesz fractional diffusion equations on an infinite domain

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Abstract

The present paper deals with the Cauchy problem for the multi-term time-space fractional diffusion equation in one dimensional space. The time fractional derivatives are defined as Caputo fractional derivatives and the space fractional derivative is defined in the Riesz sense. Firstly the domain of the fractional Laplacian is extended to a Banach space. Then the analytical solutions are established by using the Luchko theorem and the multivariate Mittag-Leffler function.

MSC: Primary 26A33; secondary 35E15; 35K05; 35R11; 45K05

Keywords: analytical solution; Caputo fractional derivative; Riesz fractional derivative; multi-term fractional diffusion equation; multivariate Mittag-Leffler function

1 Introduction

The fractional calculus has already become a powerful tool which describes many nonlinear complex phenomena arising in fluid mechanics, thermodynamics, plasma dynamics, continuum mechanics, quantum mechanics, electrodynamics and biological systems [1, 2]. In particular, the fractional diffusion equations capture well the anomalous diffusion process with continuous time random walks [3, 4].

In this paper, we consider the following initial value problem for the multi-term time-space Caputo-Riesz fractional diffusion equation:

$$\sum_{j=0}^{n-1} a_j D^{\alpha_j} u(t,x) = -b(-\triangle)^{\beta} u(t,x), \tag{1.1}$$

$$u(0,x) = g(x), \tag{1.2}$$

where $n \geq 1$, $a_0 = 1, a_i > 0, \alpha_i > 0, \alpha_{n-1} < \cdots < \alpha_0 \leq 1, b > 0, 0 < \beta \leq 1, x \in R = (-\infty, \infty)$, $t \geq 0$, the symbol D^{α} denotes the Caputo-type fractional derivative defined by [5]

$$D^{\alpha} u(t) = \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_0^t (t-s)^{\lceil \alpha \rceil - \alpha - 1} u^{(\lceil \alpha \rceil)}(s)\, ds$$

and the symbol $(-\triangle)^{\beta}$ denotes the fractional Laplacian defined by [6]

$$(-\triangle)^{\beta}u(t) = F^{-1}\left\{|s|^{2\beta}Fu(s)\right\}(t), \tag{1.3}$$

where F means the Fourier transform.

In fractional calculus the most popular fractional derivatives are Caputo derivative and Riemann-Liouville derivative. Because of the convenience in handling initial conditions, the Caputo fractional derivative has been more widely used in practice [7]. However, the Caputo fractional derivative is usually defined for the continuously differentiable functions [5, 7]. In [8] the authors gave a new definition of the Caputo fractional derivative on a bounded interval in the fractional Sobolev space and proved the maximal regularity of solutions of time fractional diffusion equations. The fractional Laplacian is also a well-known nonlocal operator which plays an important role in the potential theory [9]. The authors of [10] considered the relation between fractional Laplacian and fractional Sovolev space. The fractional Laplacian operator on a bounded interval is defined in terms of the eigenvalues and eigenfunctions of the Laplacian operator [8, 11]. The fractional Laplacian on a unbounded interval is usually defined in the Schwartz space which is too narrow for many important applications. Thus in [12] the solution space of analytical solutions of fractional time-space Caputo-Riesz diffusion equations on an infinite domain was not illustrated and the authors [13] established mild solutions by deriving an equivalent integral equation.

Since multi-term fractional diffusion equations are more flexible than single-term fractional diffusion equations in modeling the anomalous diffusion phenomena, they have often appeared in recent publications [11, 14–17]. By establishing the maximum principle for multi-term time fractional diffusion equations with Caputo derivatives and proving some properties of multivariate Mittag-Leffler functions, the authors [14, 15] studied the well-posedness and the long-time asymptotic behavior. In [17] the authors proved the maximum principle for multi-term time-space Caputo-Riesz fractional diffusion equations and derived the uniqueness and continuous dependence of the solution. The authors of [11] used the Luchko theorem to obtain the analytical solutions for multi-term time-space Caputo-Riesz fractional advection-diffusion equations on a bounded interval. However, to the best of our knowledge, multi-term time-space Caputo-Riesz fractional diffusion equations on an infinite domain have not been considered in the literature yet.

In the present paper, by extending the domain of the fractional Laplacian to a Banach space and using the multivariate Mittag-Leffler function, the analytical solutions of the multi-term fractional diffusion equation (1.1)-(1.2) are obtained. Especially the meaning of the analytical solutions is found.

2 Extension of domain of fractional Laplacian

In this section the domain of the fractional Laplacian operator (1.3) is extended to a Banach space. Firstly we recall the concepts of Lebesgue space and Schwartz space.

Definition 2.1 ([18], p.110) *The space L^2 means the set of all measurable functions $u : R \to R$ such that $\|u\|_{L^2} < \infty$, where*

$$\|u\|_{L^2} = \int_R \left|u(x)\right|^2 dx.$$

Definition 2.2 ([18], p.214) *The space S means the set of all C^∞ functions $u : R \to R$ such that $\|u\|_{r,q} < \infty$ for all $r, q = 0, 1, \dots,$ where*

$$\|u\|_{r,q} = \sup_{x\in R}\left(1 + |x|^r\right)\sum_{m=0}^{q}\left|u^{(m)}(x)\right|.$$

Definition 2.3 *By M_β we mean the completion of the Schwartz space S over R with the norm $\|\cdot\|_{M_\beta}$ defined by*

$$\|f\|_{M_\beta} = \left\| |t|^{2\beta} Ff(t)\right\|_{L^2}, \quad f \in S. \tag{2.1}$$

For any $f \in M_\beta$, there exists a sequence $\{f_m \in S\}$ such that $\|f\|_{M_\beta} = \lim_{m\to\infty}\|f_m\|_{M_\beta}$ and $\|f_m - f_r\|_{M_\beta} \to 0$ as $m, r \to \infty$.

Theorem 2.4 *The fractional Laplacian $(-\triangle)^\beta$ is extended to the Banach space M_β.*

Proof By using the extension principle, we can easily prove the result. □

Theorem 2.5 $H_\beta = \{f \in L^2 : |t|^{2\beta} Ff(t) \in L^2\} \subset M_\beta$.

Proof Let suppose that $f \in H_\beta$ and $\epsilon > 0$. Then there exists a real number $r_\epsilon > 0$ such that

$$\int_{|t|>r_\epsilon} |t|^{4\beta}(Ff)^2(t)\,dt < \epsilon^2.$$

There exists a function $g_\epsilon \in C_0^\infty([-r_\epsilon, r_\epsilon])$ such that

$$\int_{|t|<r_\epsilon} (Ff - g_\epsilon)^2(t)\,dt < \frac{\epsilon^2}{r_\epsilon^{4\beta}}.$$

Let

$$g_\epsilon^*(t) := \begin{cases} g_\epsilon & \text{for } t \in [-r_\epsilon, r_\epsilon], \\ 0 & \text{else,} \end{cases}$$

and $f_\epsilon := F^{-1}(g_\epsilon^*)$. Then $f_\epsilon \in S$. We have

$$\begin{aligned} \left\||t|^{2\beta}\left(Ff(t) - Ff_\epsilon(t)\right)\right\|_{L^2}^2 &= \int_{|t|>r_\epsilon} |t|^{4\beta}(Ff)^2(t)\,dt + \int_{|t|<r_\epsilon} |t|^{4\beta}(Ff - Ff_\epsilon)^2(t)\,dt \\ &\le \epsilon^2 + r_\epsilon^{4\beta}\int_{|t|<r_\epsilon}(Ff - Ff_\epsilon)^2(t)\,dt \le 2\epsilon^2. \end{aligned}$$

Then $\|f - f_{\frac{1}{m}}\|_{M_\beta} = \||t|^{2\beta}(Ff(t) - Ff_{\frac{1}{m}}(t))\|_{L^2} \to 0$ as $m \to \infty$, which implies that $f \in M_\beta$. □

3 Solution of the multi-term fractional diffusion equation

In this section the analytical solution to the initial value problem (1.1)-(1.2) is obtained by using the Luchko theorem.

Definition 3.1 ([19], p.3) *A real- or complex-valued function $f(x), x > 0$, is said to be in the space $C_\alpha, \alpha \in R$, if there exists a real number $p > \alpha$ such that $f(x) = x^p f_1(x)$, with a function $f_1(x) \in C[0,\infty)$.*

Definition 3.2 ([19], p.4) *A function $f(x), x > 0$, is said to be in the space $C_\alpha^m, m \in N \cup \{0\}$, if and only if $f^{(m)} \in C_\alpha$.*

Lemma 3.3 ([19], p.6) *Let $u \in C_{-1}^r, r \in N \cup \{0\}$. Then the Caputo fractional derivative $D^\alpha u, 0 \le \alpha \le r$, is well defined and the inclusion*

$$D^\alpha u \in \begin{cases} C_{-1}, & r-1 < \alpha \le r, \\ C^{r-1}[0,\infty) \subset C_{-1}, & r-k-1 < \alpha \le r-k, k = 1,\dots,r-1, \end{cases}$$

holds true.

The following is the well-known Luchko theorem (Theorem 4.1 in [19]).

Lemma 3.4 ([19], p.15) *Let $\gamma_0 > \cdots > \gamma_p \ge 0$ and $c_i \in R$. The initial value problem*

$$\begin{aligned} &D^{\gamma_0} v(t) - \sum_{j=1}^{p} c_j D^{\gamma_j} v(t) = G(t), \\ &v^{(j)}(0) = d_j, \quad j = 0,1,\dots,\lceil \gamma_0 \rceil - 1, \end{aligned} \tag{3.1}$$

where the function G is assumed to lie in C_{-1} if $\gamma_0 \in N$, in C_{-1}^1 if $\gamma_0 \notin N$, and the unknown function $v(t)$ is to be determined in the space $C_{-1}^{\lceil \gamma_0 \rceil}$, and it has a solution, unique in the space $C_{-1}^{\lceil \gamma_0 \rceil}$, of the form

$$v(t) = v_G(t) + \sum_{j=0}^{\lceil \gamma_0 \rceil - 1} d_j v_j(t), \quad t \ge 0.$$

Here

$$v_G(t) = \int_0^t s^{\gamma_0 - 1} E_{(\cdot),\gamma_0}(s) G(t-s)\, ds$$

is a solution of the problem (3.1) *with zero initial conditions, and the system of functions*

$$v_j(t) = \frac{t^j}{j!} + \sum_{l=l_j+1}^{p} c_l t^{j+\gamma_0-\gamma_l} E_{(\cdot),j+1+\gamma_0-\gamma_l}(t), \quad j = 0,\dots,\lceil \gamma_0 \rceil - 1,$$

fulfills the initial conditions $v_j^{(l)} = \delta_{jl}, j, l = 0,\dots,\lceil \gamma_0 \rceil - 1$. The function

$$E_{(\cdot),\beta}(t) = E_{(\gamma_0-\gamma_1,\dots,\gamma_0-\gamma_p),\beta}\left(c_1 t^{\gamma_0-\gamma_1}, \dots, c_p t^{\gamma_0-\gamma_p}\right)$$

is a particular case of the multivariate Mittag-Leffler function

$$E_{(x_1,\ldots,x_p),y}(z_1,\ldots,z_p)=\sum_{k=0}^{\infty}\sum_{\substack{l_1+\cdots+l_p=k\\ l_1\geq 0,\ldots,l_p\geq 0}}\frac{k!}{l_1!\cdots l_p!}\frac{\prod_{j=1}^{p}z_j^{l_j}}{\Gamma(y+\sum_{j=1}^{p}x_jl_j)}. \tag{3.2}$$

The natural numbers l_j are determined from the condition

$$\begin{cases}\lceil\gamma_{l_j}\rceil\geq j+1,\\ \lceil\gamma_{l_j+1}\rceil\leq j.\end{cases}$$

In the case $\lceil\gamma_r\rceil\leq j$ for any $r=1,\ldots,p$, we set $l_j=0$ and, if $\lceil\gamma_r\rceil\geq j+1$ for any $r=1,\ldots,p$, then $l_j=p$.

The Mittag-Leffer type functions are very crucial in the theory of fractional differential equations [7, 20–22]. Now we prove a property of the multivariate Mittag-Leffer function which appears in the analytical solution of the initial value problem (1.1)-(1.2).

Lemma 3.5 *Let $0\leq x_p<\cdots<x_0\leq 1, c_0,\ldots,c_p>0$. Then the function*

$$\left|t^{x_0}E_{(x_0-x_1,\ldots,x_0-x_p,x_0),1+x_0}\left(-c_1t^{x_0-x_1},\ldots,-c_pt^{x_0-x_p},-c_0t^{x_0}\right)\right|$$

is bounded for all $t\geq 0$.

Proof The multivariate Mittag-Leffer function can be rewritten by using the Hankel integral representation of $1/\Gamma(z)$ [5],

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_{Ha(\epsilon+)}e^ss^{-z}\,ds,$$

where $r>0, Ha(\epsilon+)=\{z\in C:|z|=\epsilon,0\leq|\arg(z)|\leq\pi\}\cup\{z\in C:|z|>\epsilon,|\arg(z)|=\pi\}$. For any $t>0$, there exists a $r_t>0$ such that

$$r_t>\max\left\{t,t\left(\sum_{i=0}^{p}|c_i|\right)^{1/(x_0-x_1)}\right\}.$$

Then we have, for $r>r_t$,

$$\begin{aligned}&t^{x_0}E_{(x_0-x_1,\ldots,x_0-x_p,x_0),1+x_0}\left(-c_1t^{x_0-x_1},\ldots,-c_pt^{x_0-x_p},-c_0t^{x_0}\right)\\&=\frac{t^{x_0}}{2\pi i}\int_{Ha(r+)}\sum_{k=0}^{\infty}\sum_{\substack{l_0+\cdots+l_p=k\\ l_0\geq 0,\ldots,l_p\geq 0}}\frac{(-1)^kk!}{l_0!\cdots l_p!}\prod_{j=0}^{p}c_j^{l_j}t^{x_0l_0+\sum_{j=1}^{p}(x_0-x_j)l_j}\frac{e^s}{s^{1+x_0+x_0l_0+\sum_{j=1}^{p}(x_0-x_j)l_j}}\,ds\\&=\frac{t^{x_0}}{2\pi i}\int_{Ha(r+)}\sum_{k=0}^{\infty}(-1)^k\sum_{\substack{l_0+\cdots+l_p=k\\ l_0\geq 0,\ldots,l_p\geq 0}}\frac{k!}{l_0!\cdots l_p!}\prod_{j=0}^{p}c_j^{l_j}\left(\frac{t}{s}\right)^{x_0l_0+\sum_{j=1}^{p}(x_0-x_j)l_j}\frac{e^s}{s^{1+x_0}}\,ds\end{aligned}$$

$$= \frac{1}{2\pi i}\int_{Ha(r/t+)}\sum_{k=0}^{\infty}(-1)^k \sum_{\substack{l_0+\cdots+l_p=k\\ l_0\geq 0,\ldots,l_p\geq 0}} \frac{k!}{l_0!\cdots l_p!}\prod_{j=0}^{p} c_j^{l_j}\xi^{-x_0 l_0-\sum_{j=1}^{p}(x_0-x_j)l_j}\frac{e^{\xi t}}{\xi^{1+x_0}}\,d\xi$$

$$= \frac{1}{2\pi i}\int_{Ha(r/t+)}\sum_{k=0}^{\infty}(-1)^k\left(c_0\xi^{-x_0}+\sum_{j=1}^{p}c_j\xi^{x_j-x_0}\right)^k\frac{e^{\xi t}}{\xi^{1+x_0}}\,d\xi$$

$$= \frac{1}{2\pi i}\int_{Ha(r/t+)}\frac{1}{1+c_0\xi^{-x_0}+\sum_{j=1}^{p}c_j\xi^{x_j-x_0}}\frac{e^{\xi t}}{\xi^{1+x_0}}\,d\xi$$

$$= \frac{1}{2\pi i}\int_{Ha(r/t+)}\frac{1}{\xi^{x_0}+\sum_{j=1}^{p}c_j\xi^{x_j}+c_0}\frac{e^{\xi t}}{\xi}\,d\xi.$$

Let $r_0 > r_t$ be a sufficiently large real number that satisfies the condition: all zeros of the function $\xi^{x_0}+\sum_{j=1}^{p}c_j\xi^{x_j}+c_0$ are contained in the circle $O(r_0)=\{z\in C:|s|=r_0, 0\leq|\arg(z)|\leq\pi\}$. Let $L(r_0,\phi)=\{z\in C:|z|>r_0,|\arg(z)|=\pi\}$. For simplicity, we denote

$$h(\xi):=\frac{1}{\xi^{x_0}+\sum_{j=1}^{p}c_j\xi^{x_j}+c_0}\frac{e^{\xi t}}{\xi}.$$

Then we have

$$\int_{Ha(r_0+)}h(\xi)\,d\xi=\int_{L(r_0,\phi)+O(r_0)}h(\xi)\,d\xi=K_1+K_2,$$

where

$$K_1=\int_{L(r_0,\phi)}h(\xi)\,d\xi,\qquad K_2=\int_{O(r_0)}h(\xi)\,d\xi,$$

$$K_1=\int_{r_0}^{\infty}\left(\frac{e^{rt\cos\pi}e^{irt\sin\pi}}{r^{x_0}e^{i\pi x_0}+\sum_{j=1}^{p}c_jr^{x_j}e^{i\pi x_j}+c_0}-\frac{e^{rt\cos\pi}e^{-irt\sin\pi}}{r^{x_0}e^{-i\pi x_0}+\sum_{j=1}^{p}c_jr^{x_j}e^{-i\pi x_j}+c_0}\right),$$

$$\frac{dr}{r}\leq\int_{r_0}^{\infty}\frac{2e^{-rt}}{|r^{x_0}-\sum_{j=1}^{p}|c_j|r^{x_j}-|c_0||}\frac{dr}{r}\to 0,\quad r_0\to\infty.$$

If $x_0,\ldots,x_p$ are all rational numbers, then the function $\xi(\xi^{x_0}+\sum_{j=1}^{p}c_j\xi^{x_j}+c_0)$ has finitely many zeros. Then by Cauchy's residue theorem, we have

$$K_2=2\pi i\sum_{i=1}^{k}\operatorname{Res}(h,z_i),$$

where z_i is a zero of the function $\xi(\xi^{x_0}+\sum_{j=1}^{p}c_j\xi^{x_j}+c_0)$ and $\operatorname{Res}(h,z_i)$ is the residue of $h(\xi)$ at z_i. If z_i is a pole of order m, then the residue of $h(\xi)$ at z_i is obtained by the formula

$$\operatorname{Res}(h,z_i)=\frac{1}{(m-1)!}\lim_{z\to z_i}\frac{d^{m-1}}{dz^{m-1}}\bigl((z-z_i)^m h(z)\bigr).$$

Then there exists a function h_i such that

$$\operatorname{Res}(h,z_i)=h_i(z_i)e^{z_it}=h_i(z_i)e^{|z_i|t\cos\arg(z_i)}e^{it\sin\arg(z_i)}.$$

It follows from $c_i > 0$ for any i that, if $|\arg(\xi)| \le \pi/2$, then $0 < |\arg(\xi^{x_0} + \sum_{j=1}^{p} c_j \xi^{x_j} + c_0)| \le \pi/2$. Therefore $|\arg(z_i)| > \pi/2$ and $|\operatorname{Res}(h, z_i)| \le |h_i(z_i)|$. Thus we have

$$|K_2| \le 2\pi \sum_{i=1}^{k} \bigl|h_i(z_i)\bigr|,$$

which implies that

$$\left| \int_{Ha(r_0+)} h(\xi)\, d\xi \right| \le |K_1| + |K_2| \le 2\pi \sum_{i=1}^{k} \bigl|h_i(z_i)\bigr|.$$

If $x_0, \ldots, x_p$ are all real numbers, then, since the set of rational numbers is everywhere dense in the set of real numbers and the function

$$t^{x_0} E_{(x_0-x_1,\ldots,x_0-x_p,x_0),1+x_0}\bigl(-c_1 t^{x_0-x_1}, \ldots, -c_p t^{x_0-x_p}, -c_0 t^{x_0}\bigr)$$

is continuous with respect to $x_0, \ldots, x_p$, we can obtain the desired result. □

Lemma 3.6 *Let* $0 < x_p < \cdots < x_0 \le 1, y > 0$. *Let* $z_0, z_1, \ldots, z_p \in C$ *satisfy* $\mu \le |\arg z_0| \le \pi$ *and* $-l \le z_j \le 0$ $(j = 1, \ldots, p)$ *for some fixed* $\mu \in (x_0\pi/2, x_0\pi)$ *and* $l > 0$. *Then there exists a* $K > 0$ *depending only on* μ, l, x_j $(j = 0, \ldots, p)$ *and* y *such that*

$$\bigl|E_{(x_0-x_1,\ldots,x_0-x_p,x_0),y}(z_1, \ldots, z_p, z_0)\bigr| < \frac{K}{1 + |z_0|}.$$

Proof By (3.2), it is obvious that

$$E_{(x_0-x_1,\ldots,x_0-x_p,x_0),y}(z_1, \ldots, z_p, z_0) = E_{(x_0,x_0-x_1,\ldots,x_0-x_p),y}(z_0, z_1, \ldots, z_p).$$

Then, using Lemma 3.2 in [14], we can prove the result. □

Theorem 3.7 *Let* $g \in H_\beta$. *Then the Cauchy problem* (1.1)-(1.2) *has a unique solution in* $C^1_{-1}([0,\infty), M_\beta)$. *In particular, the solution is in* $C^1_{-1}([0,\infty), H_\beta)$ *and is given by*

$$\begin{aligned} u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} &\hat{g}(\xi)\bigl[1 - |\xi|^{2\beta} t^{\alpha_0} E_{(\alpha_0-\alpha_1,\ldots,\alpha_0-\alpha_{n-1},\alpha_0),1+\alpha_0} \\ &\bigl(-a_1 t^{\alpha_0-\alpha_1}, \ldots, -a_{n-1} t^{\alpha_0-\alpha_{n-1}}, -|\xi|^{2\beta} t^{\alpha_0}\bigr)\bigr] \cos(x\xi)\, d\xi, \end{aligned}$$

where $\hat{g}$ *means the Fourier transform of* g. *The solution* $u(t,x)$ *is bounded for all* $t \ge 0$ *and* $x \in R$.

Proof Applying the Fourier transform to equation (1.1) with respect to the space variable x, we have

$$\sum_{j=0}^{n-1} a_j D^{\alpha_j} \hat{u}(t,\xi) + |\xi|^{2\beta} \hat{u}(t,\xi) = 0,$$

$$\hat{u}(0,\xi) = \hat{g}(\xi).$$

By Lemma 3.4, we have

$$\hat{u}(t,\xi) = \hat{g}(\xi)\big[1 - |\xi|^{2\beta} t^{\alpha_0} E_{(\alpha_0-\alpha_1,\ldots,\alpha_0-\alpha_{n-1},\alpha_0),1+\alpha_0}\big(-a_1 t^{\alpha_0-\alpha_1}, \ldots, -a_{n-1} t^{\alpha_0-\alpha_{n-1}}, -|\xi|^{2\beta} t^{\alpha_0}\big)\big].$$

By Lemma 3.6, for any $t > 0$, there exists a $M_t > 0$ such that

$$\big||\xi|^{2\beta} t^{\alpha_0} E_{(\alpha_0-\alpha_1,\ldots,\alpha_0-\alpha_{n-1},\alpha_0),1+\alpha_0}\big(-a_1 t^{\alpha_0-\alpha_1}, \ldots, -a_{n-1} t^{\alpha_0-\alpha_{n-1}}, -|\xi|^{2\beta} t^{\alpha_0}\big)\big| < M_t$$

for any $\xi \in R$ and $|\hat{u}(t,\xi)| \leq (M_t + 1)|\hat{g}(\xi)|$. Then $u(t,\cdot) \in H_\beta$. Using the inverse Fourier transform with respect to ξ, we obtain

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\xi)\big[1 - |\xi|^{2\beta} t^{\alpha_0} E_{(\alpha_0-\alpha_1,\ldots,\alpha_0-\alpha_{n-1},\alpha_0),1+\alpha_0} \big(-a_1 t^{\alpha_0-\alpha_1}, \ldots, -a_{n-1} t^{\alpha_0-\alpha_{n-1}}, -|\xi|^{2\beta} t^{\alpha_0}\big)\big] \cos(x\xi)\, d\xi.$$

Then we have

$$\big|u(t,x)\big| \leq \frac{(M_t + 1)}{2\pi}\int_{-\infty}^{\infty} \big|\hat{g}(\xi)\big|\, d\xi.$$

Meanwhile, by Lemma 3.5, we obtain

$$\big|u(t,x)\big| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \big|\hat{g}(\xi)\big|\big(1 + K_\xi |\xi|^{2\beta}\big)\, d\xi,$$

where

$$K_\xi = \sup_{t>0}\big|t^{\alpha_0} E_{(\alpha_0-\alpha_1,\ldots,\alpha_0-\alpha_{n-1},\alpha_0),1+\alpha_0}\big(-a_1 t^{\alpha_0-\alpha_1}, \ldots, -a_{n-1} t^{\alpha_0-\alpha_{n-1}}, -|\xi|^{2\beta} t^{\alpha_0}\big)\big|.$$

From Lemma 3.6, there exists a $K > 0$ such that $K_\xi < K$ for any $\xi \in R$. Then we have

$$\big|u(t,x)\big| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \big|\hat{g}(\xi)\big|\big(1 + K|\xi|^{2\beta}\big)\, d\xi,$$

which implies that $u(t,x)$ is bounded. □

Acknowledgements
The authors would like to thank referees for their valuable advices for the improvement of this article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SCS, GIR and MCK participated in obtaining the main results of this manuscript and drafted the manuscript. All authors read and approved the final manuscript.

Publisher's Note

Received: 27 April 2017 Accepted: 20 September 2017 Published online: 02 October 2017

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