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Weak solutions for a coupled system of Pettis-Hadamard fractional differential equations

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Abstract
In this paper, by applying the technique of measure of weak noncompactness and Mönch's fixed point theorem, we investigate the existence of weak solutions under the Pettis integrability assumption for a coupled system of Hadamard fractional differential equations.

MSC: 26A33; 35H10; 35D30

Keywords: functional integral equation; coupled system; partial Pettis-Hadamard fractional integral; measure of weak noncompactness; weak solution

1 Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [1, 2]. There has been a significant development in fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [3, 4], Kilbas *et al.* [5], and the papers [6–20].

The measure of weak noncompactness was introduced by De Blasi [21]. The strong measure of noncompactness was considered by Banaś and Goebel [22] and subsequently developed and used in many papers; see, for example, Akhmerov *et al.* [23], Alvàrez [24], Benchohra *et al.* [25], Guo *et al.* [26], and the references therein. In [25, 27] the authors considered some existence results by applying the techniques of the measure of noncompactness. Recently, several researchers obtained other results by application of the technique of measure of weak noncompactness; see [4, 28, 29] and the references therein.

In this paper, we discuss the existence of weak solutions to the following coupled system of Hadamard fractional differential equations:

$$\begin{cases} ({}^H D_1^r u)(t) = f_1(t, u(t), v(t)), \\ ({}^H D_1^\rho v)(t) = f_2(t, u(t), v(t)); \end{cases} \qquad t \in I := [1, T], \tag{1}$$

with the following initial conditions:

$$\begin{cases} ({}^H I_1^{1-r} u)(t)|_{t=1} = \phi, \\ ({}^H I_1^{1-\rho} v)(t)|_{t=1} = \psi, \end{cases} \tag{2}$$

where $T > 1$, $r, \rho \in (0,1]$, $\phi, \psi \in E$, $f_1, f_2 : I \times E \times E \to E$ are given continuous functions, E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual E^* such that E is the dual of a weakly compactly generated Banach space X, ${}^H I_1^r$ is the left-sided mixed Hadamard integral of order r, and ${}^H D_1^r$ is the Hadamard fractional derivative of order r.

2 Preliminaries

Let C be the Banach space of all continuous functions w from I into E with the supremum (uniform) norm

$$\|w\|_\infty := \sup_{t \in I} \|w(t)\|_E.$$

As usual, $\mathrm{AC}(I)$ denotes the space of absolutely continuous functions from I into E. By $C_{r,\ln}(I)$, we denote the weighted space of continuous functions defined by

$$C_{r,\ln}(I) = \{w(t) : (\ln t)^r w(t) \in C\},$$

with the norm

$$\|w\|_{C_{r,\ln}} := \sup_{t \in I} \|(\ln t)^r w(t)\|_E.$$

We denote $\|w\|_{C_{r,\ln}}$ by $\|w\|_{C_r}$. Also, by $\mathcal{C}_{r,\rho,\ln}(I) := C_{r,\ln}(I) \times C_{\rho,\ln}(I)$ we denote the product weighted space with the norm

$$\|(u,v)\|_{\mathcal{C}_{r,\rho,\ln}(I)} = \|u\|_{C_r} + \|v\|_{C_\rho}.$$

In the following we denote $\|(u,v)\|_{\mathcal{C}_{r,\rho,\ln}(I)}$ by $\|(u,v)\|_{\mathcal{C}}$.

Let $(E, w) = (E, \sigma(E, E^*))$ be the Banach space E with its weak topology.

Definition 2.1 A Banach space X is called weakly compactly generated (WCG for short) if it contains a weakly compact set whose linear span is dense in X.

Definition 2.2 A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E, w) then $h(u_n) \to h(u)$ in (E, w)).

Definition 2.3 ([30]) The function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))\,ds$ for all $\phi \in E^*$, where the integral on the right-hand side is assumed to exist in the sense of Lebesgue (by definition, $u_J = \int_J u(s)\,ds$).

Let $P(I,E)$ be the space of all E-valued Pettis integrable functions on I, and $L^1(I,E)$ be the Banach space of Lebesgue integrable functions $u: I \to E$. Define the class $P_1(I,E)$ by

$$P_1(I,E) = \{u \in P(I,E) : \varphi(u) \in L^1(I,E) \text{ for every } \varphi \in E^*\}.$$

The space $P_1(I,E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*, \|\varphi\| \leq 1} \int_1^T \bigl|\varphi\bigl(u(x)\bigr)\bigr| \, d\lambda x,$$

where λ stands for a Lebesgue measure on I.

The following result is due to Pettis (see [30], Theorem 3.4 and Corollary 3.41).

Proposition 2.4 ([30]) *If $u \in P_1(I,E)$ and h is a measurable and essentially bounded E-valued function, then $uh \in P_1(J,E)$.*

For all that follows, the symbol "$\int$" denotes the Pettis integral.

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [5, 31] for a more detailed analysis.

Definition 2.5 ([5, 31]) The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1(I,E)$ is defined as

$$\bigl({}^H I_1^q g\bigr)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} \, ds,$$

provided the integral exists, where $\Gamma(\cdot)$ is the (Euler's) gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} \, dt, \quad \xi > 0.$$

Example 2.6 Let $0 < q < 1$. Then

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q} \quad \text{for a.e. } t \in [0,e].$$

Remark 2.7 Let $g \in P_1([1,T],E)$. For every $\varphi \in E^*$, we have

$$\varphi\bigl({}^H I_1^q g\bigr)(x) = \bigl({}^H I_1^q \varphi g\bigr)(x) \quad \text{for a.e. } x \in I.$$

Analogous to the Riemann-Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way. Set

$$\delta = x\frac{d}{dx}, \qquad q > 0, \qquad n = [q] + 1,$$

where $[q]$ is the integer part of q, and

$$\mathrm{AC}_\delta^n := \bigl\{u : [1,T] \to E : \delta^{n-1}\bigl[u(x)\bigr] \in \mathrm{AC}(I)\bigr\}.$$

Definition 2.8 ([5, 31]) The Hadamard fractional derivative of order q applied to the function $w \in \mathrm{AC}^n_\delta$ is defined as

$$\left({}^H D^q_1 w\right)(x) = \delta^n \left({}^H I^{n-q}_1 w\right)(x).$$

Example 2.9 Let $0 < q < 1$. Then

$${}^H D^q_1 \ln t = \frac{1}{\Gamma(2-q)} (\ln t)^{1-q} \quad \text{for a.e. } t \in [0, e].$$

It has been proved (see, e.g., Kilbas [32], Theorem 4.8) that in the space $L^1(I, E)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.,

$$\left({}^H D^q_1\right)\left({}^H I^q_1 w\right)(x) = w(x).$$

From Theorem 2.3 of [5], we have

$$\left({}^H I^q_1\right)\left({}^H D^q_1 w\right)(x) = w(x) - \frac{({}^H I^{1-q}_1 w)(1)}{\Gamma(q)} (\ln x)^{q-1}.$$

Corollary 2.10 *Let $h : I \to E$ be a continuous function. A function $w \in L^1(I, E)$ is said to be a solution of the equation*

$$\left({}^H D^q_1 w\right)(t) = h(t)$$

if and only if u satisfies the following Hadamard integral equation:

$$w(t) = \frac{({}^H I^{1-q}_1 u)(1)}{\Gamma(q)} (\ln t)^{q-1} + \left({}^H I^q_1 h\right)(t).$$

Definition 2.11 ([21]) Let E be a Banach space, Ω_E be the bounded subsets of E, and B_1 be the unit ball of E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by

$$\beta(X) = \inf\{\epsilon > 0 : \text{there exists weakly compact } \Omega \subset E \text{ such that } X \subset \epsilon B_1 + \Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \beta(A) \le \beta(B)$,
(b) $\beta(A) = 0 \Leftrightarrow A$ is weakly relatively compact,
(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,
(d) $\beta(\overline{A}^\omega) = \beta(A)$ ($\overline{A}^\omega$ denotes the weak closure of A),
(e) $\beta(A + B) \le \beta(A) + \beta(B)$,
(f) $\beta(\lambda A) = |\lambda| \beta(A)$,
(g) $\beta(\mathrm{conv}(A)) = \beta(A)$,
(h) $\beta(\bigcup_{|\lambda| \le h} \lambda A) = h\beta(A)$.

The next result follows directly from the Hahn-Banach theorem.

Proposition 2.12 *Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : I \to E$, let us denote

$$V(t) = \{v(t) : v \in V\}, \quad t \in I,$$

and

$$V(I) = \{v(t) : v \in V, t \in I\}.$$

Lemma 2.13 ([26]) *Let $H \subset C$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t))$ is continuous on I, and*

$$\beta_C(H) = \max_{t\in I} \beta\bigl(H(t)\bigr),$$

and

$$\beta\left(\int_I u(s)\,ds\right) \leq \int_I \beta\bigl(H(s)\bigr)\,ds,$$

where $H(s) = \{u(s) : u \in H, s \in I\}$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

For our purpose, we will need the following fixed point theorem.

Theorem 2.14 ([33]) *Let Q be a nonempty, closed, convex, and equicontinuous subset of a metrizable locally convex vector space $C(J, E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{\mathrm{conv}}\bigl(\{0\} \cup T(V)\bigr) \Rightarrow V \text{ is relatively weakly compact} \tag{3}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Existence of weak solutions

Let us start by defining what we mean by a weak solution of the coupled system (1)-(2).

Definition 3.1 By a weak solution of the coupled system (1)-(2), we mean measurable coupled functions $(u, v) \in \mathcal{C}_{r,\rho,\ln}$ satisfying conditions (2) and equations (1) on I.

The following hypotheses will be used in the sequel.

(H_1) For a.e. $t \in I$, the functions $v \to f_i(t, v, \cdot)$, $i = 1, 2$, and $w \to f_i(t, \cdot, w)$, $i = 1, 2$, are weakly sequentially continuous;

(H_2) For each $v, w \in E$, the function $t \to f(t, v, w)$ is Pettis integrable a.e. on I;

(H_3) There exists $p_i \in C(I, [0, \infty))$, $i = 1, 2$, such that for all $\varphi \in E^*$, we have

$$\bigl|\varphi\bigl(f_i(t, u, v)\bigr)\bigr| \leq \frac{p_i(t)\|\varphi\|}{1 + \|\varphi\| + \|u\|_E + \|v\|_E} \quad \text{for a.e. } t \in I \text{ and each } u, v \in E;$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\beta\big(f_i(t,B,B)\big) \leq (\ln t)^{1-r} p_i(t)\beta(B), \quad i = 1,2.$$

Set

$$p_i^* = \sup_{t \in I} p_i(t), \quad i = 1,2.$$

Theorem 3.2 *Assume that hypotheses* (H_1)-(H_4) *hold. If*

$$L := \frac{p_1^* \ln T}{\Gamma(1+r)} + \frac{p_2^* \ln T}{\Gamma(1+\rho)} < 1, \tag{4}$$

then the coupled system (1)-(2) *has at least one weak solution defined on* I.

Proof Define the operators $N_1 : C_{r,\ln} \to C_{r,\ln}$ and $N_2 : C_{\rho,\ln} \to C_{\rho,\ln}$ by

$$(N_1 u)(t) = \frac{\phi}{\Gamma(r)} (\ln t)^{r-1} + \int_1^t \left(\ln \frac{t}{s}\right)^{r-1} \frac{f_1(s,u(s),v(s))}{s\Gamma(r)}\, ds \tag{5}$$

and

$$(N_2 v)(t) = \frac{\psi}{\Gamma(\rho)} (\ln t)^{\rho-1} + \int_1^t \left(\ln \frac{t}{s}\right)^{\rho-1} \frac{f_2(s,u(s),v(s))}{s\Gamma(\rho)}\, ds. \tag{6}$$

Consider the continuous operator $N : \mathcal{C}_{r,\rho,\ln} \to \mathcal{C}_{r,\rho,\ln}$ defined by

$$\big(N(u,v)\big)(t) = \big((N_1 u)(x,y),(N_2 v)(t)\big). \tag{7}$$

First notice that the hypotheses imply that the functions $t \mapsto (\ln \frac{t}{s})^{r-1} \frac{f_1(s,u(s),v(s))}{s}$ and $t \mapsto (\ln \frac{t}{s})^{\rho-1} \frac{f_2(s,u(s),v(s))}{s}$, for a.e. $t \in I$, are Pettis integrable, and for each $(u,v) \in \mathcal{C}_{r,\rho,\ln}$, the function $t \mapsto f(t,u(t),v(t))$ is Pettis integrable over I. Thus, the operator N is well defined. Let $R > 0$ be such that

$$R > \frac{p_1^* \ln T}{\Gamma(1+r)} + \frac{p_2^* \ln T}{\Gamma(1+\rho)},$$

and consider the set

$$\begin{aligned} Q = \bigg\{ &(u,v) \in \mathcal{C}_{r,\rho,\ln} : \big\|(u,v)\big\|_{\mathcal{C}} \leq R \text{ and } \big\|(\ln t_2)^{1-r} u(t_2) - (\ln t_1)^{1-r} u(t_1)\big\|_E \\ &+ \big\|(\ln t_2)^{1-\rho} (N_2 u)(t_2) - (\ln t_1)^{1-\rho} (N_2 u)(t_1)\big\|_E \leq \frac{p_1^*}{\Gamma(1+r)} (\ln T)^{1-r} \left|\ln \frac{t_2}{t_1}\right|^r \\ &+ \frac{p_1^*}{\Gamma(r)} \int_1^{t_1} \left|(\ln t_2)^{1-r} \left(\ln \frac{t_2}{s}\right)^{r-1} - (\ln t_1)^{1-r} \left(\ln \frac{t_1}{s}\right)^{r-1}\right| ds \\ &+ \frac{p_2^*}{\Gamma(1+\rho)} (\ln T)^{1-\rho} \left|\ln \frac{t_2}{t_1}\right|^\rho \\ &+ \frac{p_2^*}{\Gamma(\rho)} \int_1^{t_1} \left|(\ln t_2)^{1-\rho} \left(\ln \frac{t_2}{s}\right)^{\rho-1} - (\ln t_1)^{1-\rho} \left(\ln \frac{t_1}{s}\right)^{\rho-1}\right| ds \bigg\}. \end{aligned}$$

Clearly, the subset Q is closed, convex, and equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 2.14. The proof will be given in several steps.

Step 1. N maps Q into itself.

Let $(u,v)\in Q$, $t\in I$, and assume that $(N_iu)(t)\neq 0$, $i=1,2$. Then there exists $\varphi\in E^*$ such that $\|(\ln t)^{1-r}(N_iu)(t)\|_E=\varphi(|(\ln t)^{1-r}(N_iu)(t)|)$. Thus

$$\left\|(\ln t)^{1-r}(N_1u)(t)\right\|_E=\varphi\left(\frac{\phi}{\Gamma(r)}+\frac{(\ln t)^{1-r}}{\Gamma(r)}\int_1^t\left(\ln\frac{t}{s}\right)^{r-1}\frac{f_1(s,u(s),v(s))}{s}\,ds\right).$$

Then

$$\begin{aligned}\left\|(\ln t)^{1-r}(N_1u)(t)\right\|_E&\leq\frac{(\ln t)^{1-r}}{\Gamma(r)}\int_1^t\left(\ln\frac{t}{s}\right)^{r-1}\frac{|\varphi(f_1(s,u(s),v(s))|}{s}\,ds\\&\leq\frac{p_1^*(\ln T)^{1-r}}{\Gamma(r)}\int_1^t\left(\ln\frac{t}{s}\right)^{r-1}\frac{ds}{s}\\&\leq\frac{p_1^*\ln T}{\Gamma(1+r)}.\end{aligned}$$

Again, we get

$$\left\|(\ln t)^{1-\rho}(N_2v)(t)\right\|_E\leq\frac{p_2^*\ln T}{\Gamma(1+\rho)}.$$

Thus, we obtain

$$\left\|\big(N(u,v)\big)\right\|_C\leq\frac{p_1^*\ln T}{\Gamma(1+r)}+\frac{p_2^*\ln T}{\Gamma(1+\rho)}.$$

Next, let $t_1,t_2\in I$ such that $t_1<t_2$, and let $(u,v)\in Q$, with

$$(\ln t_2)^{1-r}(N_1u)(t_2)-(\ln t_1)^{1-r}(N_1u)(t_1)\neq 0,$$

and

$$(\ln t_2)^{1-\rho}(N_2v)(t_2)-(\ln t_1)^{1-\rho}(N_2v)(t_1)\neq 0.$$

Then there exists $\varphi\in E^*$ with $\|\varphi\|=1$ such that

$$\left\|(\ln t_2)^{1-r}(N_1u)(t_2)-(\ln t_1)^{1-r}(N_1u)(t_1)\right\|_E=\varphi\big((\ln t_2)^{1-r}(N_1u)(t_2)-(\ln t_1)^{1-r}(N_1u)(t_1)\big)$$

and

$$\left\|(\ln t_2)^{1-\rho}(N_2v)(t_2)-(\ln t_1)^{1-\rho}(N_2v)(t_1)\right\|_E=\varphi\big((\ln t_2)^{1-\rho}(N_2v)(t_2)-(\ln t_1)^{1-\rho}(N_2v)(t_1)\big).$$

Then

$$\begin{aligned}&\left\|(\ln t_2)^{1-r}(N_1u)(t_2)-(\ln t_1)^{1-r}(N_1u)(t_1)\right\|_E\\&\quad=\varphi\big((\ln t_2)^{1-r}(N_1u)(t_2)-(\ln t_1)^{1-r}(N_1u)(t_1)\big)\end{aligned}$$

$$
\begin{aligned}
&\le \varphi\left((\ln t_2)^{1-r}\int_1^{t_2}\left(\ln\frac{t_2}{s}\right)^{r-1}\frac{f_1(s,u(s),v(s))}{s\Gamma(r)}\,ds\right.\\
&\quad\left.-(\ln t_1)^{1-r}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{r-1}\frac{f_1(s,u(s),v(s))}{s\Gamma(r)}\,ds\right).
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\left\|(\ln t_2)^{1-r}(N_1u)(t_2)-(\ln t_1)^{1-r}(N_1u)(t_1)\right\|_E\\
&\quad\le(\ln t_2)^{1-r}\int_{t_1}^{t_2}\left|\ln\frac{t_2}{s}\right|^{r-1}\frac{|\varphi(f_1(s,u(s),v(s)))|}{s\Gamma(r)}\,ds\\
&\qquad+\int_1^{t_1}\left|(\ln t_2)^{1-r}\left(\ln\frac{t_2}{s}\right)^{r-1}-(\ln t_1)^{1-r}\left(\ln\frac{t_1}{s}\right)^{r-1}\right|\frac{|\varphi(f_1(s,u(s),v(s)))|}{s\Gamma(r)}\,ds\\
&\quad\le(\ln t_2)^{1-r}\int_{t_1}^{t_2}\left|\ln\frac{t_2}{s}\right|^{r-1}\frac{p_1(s)}{\Gamma(r)}\,ds\\
&\qquad+\int_1^{t_1}\left|(\ln t_2)^{1-r}\left(\ln\frac{t_2}{s}\right)^{r-1}-(\ln t_1)^{1-r}\left(\ln\frac{t_1}{s}\right)^{r-1}\right|\frac{p_1(s)}{\Gamma(r)}\,ds.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&\left\|(\ln t_2)^{1-r}(N_1u)(t_2)-(\ln t_1)^{1-r}(N_1u)(t_1)\right\|_E\\
&\quad\le\frac{p_1^*}{\Gamma(1+r)}(\ln T)^{1-r}\left|\ln\frac{t_2}{t_1}\right|^{r}\\
&\qquad+\frac{p_1^*}{\Gamma(r)}\int_1^{t_1}\left|(\ln t_2)^{1-r}\left(\ln\frac{t_2}{s}\right)^{r-1}-(\ln t_1)^{1-r}\left(\ln\frac{t_1}{s}\right)^{r-1}\right|\,ds.
\end{aligned}
$$

Also, we can obtain that

$$
\begin{aligned}
&\left\|(\ln t_2)^{1-\rho}(N_2u)(t_2)-(\ln t_1)^{1-\rho}(N_2u)(t_1)\right\|_E\\
&\quad\le\frac{p_2^*}{\Gamma(1+\rho)}(\ln T)^{1-\rho}\left|\ln\frac{t_2}{t_1}\right|^{\rho}\\
&\qquad+\frac{p_2^*}{\Gamma(\rho)}\int_1^{t_1}\left|(\ln t_2)^{1-\rho}\left(\ln\frac{t_2}{s}\right)^{\rho-1}-(\ln t_1)^{1-\rho}\left(\ln\frac{t_1}{s}\right)^{\rho-1}\right|\,ds.
\end{aligned}
$$

Hence $N(Q)\subset Q$.

Step 2. N is weakly-sequentially continuous.

Let (u_n,v_n) be a sequence in Q, and let $(u_n(t),v_n(t))\to(u(t),v(t))$ in $(E,\omega)\times(E,\omega)$ for each $t\in I$. Fix $t\in I$, since for any $i\in\{1,2\}$ the function f_i satisfies assumption (H_1), we have $f_i(t,u_n(t),v_n(t))$ converges weakly uniformly to $f(t,u(t),v(t))$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies that $((N_1u_n)(t),(N_2v_n)(t))$ converges weakly uniformly to $((N_1u)(t),(N_2v)(t))$ in $(E,\omega)\times(E,\omega)$ for each $t\in I$. Thus, $N(u_n,v_n)\to(N(u),N(v))$. Hence, $N:Q\to Q$ is weakly-sequentially continuous.

Step 3. Implication (3) holds.

Let V be a subset of Q such that $\overline{V}=\overline{\operatorname{conv}}(N(V)\cup\{0\})$. Obviously

$$
V(t)\subset\overline{\operatorname{conv}}\big((NV)(t)\cup\{0\}\big),\quad t\in I.
$$

Further, as V is bounded and equicontinuous, by Lemma 3 in [34] the function $t \to v(t) = \beta(V(t))$ is continuous on I. From (H_3), (H_4), Lemma 2.13 and the properties of the measure β, for any $t \in I$, we have

$$\begin{aligned}
(\ln t)^{1-r} v(t) &\le \beta\big((\ln t)^{1-r}(NV)(t) \cup \{0\}\big) \\
&\le \beta\big((\ln t)^{1-r}(NV)(t)\big) \\
&\le \frac{(\ln T)^{1-r}}{\Gamma(r)} \int_1^t \left|\ln \frac{t}{s}\right|^{r-1} \frac{p_1(s)\beta(V(s))}{s}\,ds \\
&\quad + \frac{(\ln T)^{1-\rho}}{\Gamma(\rho)} \int_1^t \left|\ln \frac{t}{s}\right|^{\rho-1} \frac{p_2(s)\beta(V(s))}{s}\,ds \\
&\le \frac{(\ln T)^{1-r}}{\Gamma(r)} \int_1^t \left|\ln \frac{t}{s}\right|^{r-1} \frac{(\ln s)^{1-r} p_1(s) v(s)}{s}\,ds \\
&\quad + \frac{(\ln T)^{1-\rho}}{\Gamma(\rho)} \int_1^t \left|\ln \frac{t}{s}\right|^{/rho-1} \frac{(\ln s)^{1-\rho} p_2(s) v(s)}{s}\,ds \\
&\le \frac{p_1^* \ln T}{\Gamma(1+r)} \|v\|_{\mathcal{C}} + \frac{p_2^* \ln T}{\Gamma(1+\rho)} \|v\|_{\mathcal{C}}.
\end{aligned}$$

Thus

$$\|v\|_{\mathcal{C}} \le L\|v\|_{\mathcal{C}}.$$

From (4), we get $\|v\|_{\mathcal{C}} = 0$, that is, $v(t) = \beta(V(t)) = 0$ for each $t \in I$. And then, by Theorem 2 in [35], V is weakly relatively compact in $[\mathcal{C}]_{r,\rho,\ln}$. Applying now Theorem 2.14, we conclude that N has a fixed point which is a weak solution of the coupled system (1)-(2). □

4 An example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

We consider the following coupled system of Hadamard fractional differential equations:

$$\begin{cases} ({}^H D_1^{\frac{1}{2}} u_n)(t) = f_n(t, u(t), v(t)), \\ ({}^H D_1^{\frac{1}{2}} v_n)(t) = g_n(t, u(t), v(t)), \end{cases} \qquad t \in [1, e], \tag{8}$$

with the initial conditions

$$\begin{cases} ({}^H I_1^{\frac{1}{2}} u)(t)|_{t=1} = 0, \\ ({}^H I_1^{\frac{1}{2}} v)(t)|_{t=1} = 0, \end{cases} \tag{9}$$

where

$$f_n\big(t,u(t),v(t)\big) = \frac{ct^2}{1+\|u(t)\|_E+\|v(t)\|_E}\left(e^{-7}+\frac{1}{e^{t+5}}\right)u_n(t), \quad t\in[1,e],$$

and

$$g_n\big(t,u(t),v(t)\big) = \frac{ct^2e^{-6}}{1+\|u(t)\|_E+\|v(t)\|_E}v_n(t), \quad t\in[1,e],$$

with

$$u=(u_1,u_2,\ldots,u_n,\ldots), \qquad v=(v_1,v_2,\ldots,v_n,\ldots), \quad \text{and} \quad c:=\frac{e^4}{8}\Gamma\left(\frac{1}{2}\right).$$

Set

$$f=(f_1,f_2,\ldots,f_n,\ldots) \quad \text{and} \quad g=(g_1,g_2,\ldots,g_n,\ldots).$$

Clearly, the functions f and g are continuous. For each $u,v\in E$ and $t\in[1,e]$, we have

$$\big\|f\big(t,u(t),v(t)\big)\big\|_E \le ct^2\left(e^{-7}+\frac{1}{e^{t+5}}\right)$$

and

$$\big\|g\big(t,u(t),v(t)\big)\big\|_E \le ct^2e^{-6}.$$

Hence, hypothesis (H_3) is satisfied with $p_1^*=p_2^*=ce^{-4}$. We shall show that condition (4) holds with $T=e$. Indeed,

$$\frac{p_1^*\ln T}{\Gamma(1+r)}+\frac{p_2^*\ln T}{\Gamma(1+\rho)}=\frac{2c}{e^4\Gamma(\frac{3}{2})}=\frac{1}{2}<1.$$

Simple computations show that all conditions of Theorem 3.2 are satisfied. It follows that the coupled system (8)-(9) has at least one solution on $[1,e]$.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SA and MB contributed to Sections 1, 2, and 3. YZ and AA contributed to Sections 1 and 4. All authors read and approved the final manuscript.

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